Збірник праць Ін-ту математики НАН України 2013, том 10, N 4–5, 185–189

UDC 517.9

Vladimir V. Kisil

(School of Mathematics, University of Leeds, UK)

Boundedness of Relative Convolutions on Nilpotent Lie Groups

kisil@maths.leeds.ac.uk

Dedicated to memory of Professor Promarz M. Tamrazov

We discuss some norm estimations for integrated representations. We use the covariant transform to extend Howe's method from the Heisenberg group to general nilpotent Lie groups.

1. Introduction. Let G be a locally compact group, a left-invariant (Haar) measure on G is denoted by dg. Let ρ be a bounded representation of the group G in a vector space V. The representation can be extended to a function $k \in L_1(G, dg)$ though integration:

$$\rho(k) = \int_G k(g)\,\rho(g)\,dg. \tag{1}$$

It is a homomorphism of the convolution algebra $L_1(G, dg)$ to an algebra of bounded operators on V.

There are many important classes of operators described by (1), notably pseudodifferential operators (PDO) and Toeplitz operators [1 — 4]. Thus, it is important to have various norm estimations of $\rho(k)$. We already mentioned a straightforward inequality $\|\rho(k)\| \leq C\,\|k\|_1$ for $k \in L_1(G, dg)$, however, other classes are of interest as well.

If G is the Heisenberg group and ρ is its Schrödinger representation, then $\rho(\hat{a})$ is a PDO $a(X, D)$ with the symbol a [1, 5, 4]. Here, $\hat{a}$ is the

On leave from Odessa University

Fourier transform of a, as usual. The Calderón–Vaillancourt theorem [6, Ch. XIII] estimates $\|a(X,D)\|$ by L_∞-norm of a finite number of partial derivatives of a.

In this paper we revise the method used in [1, § 3.1] to prove the Calderón–Vaillancourt estimations. It was described as "rather magical" in [5, § 2.5]. We hope, that a usage of the covariant transform dispel the mystery without undermining the power of the method.

2. Preliminaries. Through the paper G denotes an exponential Lie group. For a square integrable irreducible representation ρ of G in a Hilbert space V and a fixed admissible mother wavelet $\phi \in V$, the wavelet transform $\mathcal{W}_\phi : V \to C_b(G)$ is [7, 3, 4]:

$$[\mathcal{W}_\phi v](g) := \left\langle \rho(g^{-1})v, \phi \right\rangle = \left\langle v, \rho(g)\phi \right\rangle, \qquad g \in G,\ v \in V. \tag{2}$$

For an unimodular G, the left $\Lambda(g) : f(g') \mapsto f(g^{-1}g')$ and the right $R : f(g') \mapsto f(g'g)$ regular representations of G are unitary operators on $L_2(G, dg)$. The covariant transforms intertwines the left and right regular representations of G with the following actions of ρ:

$$\Lambda(g)\mathcal{W}_\phi = \mathcal{W}_\phi \rho(g) \quad \text{and} \quad R(g)\mathcal{W}_\phi = \mathcal{W}_{\rho(g)\phi} \quad \text{for all } g \in G. \tag{3}$$

For a fixed admissible vector $\psi \in V$, the integrated representation (1) produces the contravariant transform $\mathcal{M}_\psi : L_1(G) \to V$, cf. [3, 4]:

$$\mathcal{M}_\psi^\rho(k) = \rho(k)\psi, \qquad \text{where } k \in L_1(G). \tag{4}$$

The contravariant transform $\mathcal{M}_\psi^\rho$ intertwines the left regular representation Λ on $L_2(G)$ and ρ:

$$\mathcal{M}_\psi^\rho\, \Lambda(g) = \rho(g)\, \mathcal{M}_\psi^\rho. \tag{5}$$

Combining with (3), we see that the composition $\mathcal{M}_\psi^\rho \circ \mathcal{W}_\phi^\rho$ of the covariant and contravariant transform intertwines ρ with itself. For an irreducible square integrable ρ and suitably normalised admissible ϕ and ψ, we use the Schur's lemma [7, Lem. 4.3.1], [8, Thm. 8.2.1] to conclude that:

$$\mathcal{M}_\psi^\rho \circ \mathcal{W}_\phi^\rho = \langle \psi, \phi \rangle\, I. \tag{6}$$

Let H be a subgroup of G and $X = G/H$ be the respective homogeneous space (the space of right cosets) with a (quasi-)invariant measure

dx [8, § 9.1]. There is the natural projection $\mathsf{p} : G \to X$. We usually fix a continuous section $\mathsf{s} : X \to G$ [8, § 13.2], which is a right inverse to p. We also define an operator of relative convolution on V [2, 4], cf. (1):

$$\rho(k) = \int_X k(x)\, \rho(\mathsf{s}(x))\, dx, \tag{7}$$

with a kernel k defined on $X = G/H$.

3. Norm Estimations. We start from the following lemma, which has a transparent proof in terms of covariant transform, cf. [1, § 3.1] and [5, (2.75)]. For the rest of the paper we assume that ρ is an irreducible square integrable representation of an exponential Lie group G in V and mother wavelet $\phi, \psi \in V$ are admissible.

Lemma 1. *Let $\phi \in V$ be such that, for $\Phi = \mathcal{W}_\phi \phi$, the reciprocal Φ^{-1} is bounded on G or $X = G/H$. Then, for the integrated representation* (1) *or relative convolution* (7)*, we have the inequality:*

$$\|\rho(f)\| \leq \left\|\Lambda \otimes R(f\Phi^{-1})\right\|, \tag{8}$$

where $(\Lambda \otimes R)(g) : k(g') \mapsto k(g^{-1}g'g)$ acts on the image of $\mathcal{W}_\phi$.

Proof. We know from (6) that $\mathcal{M}_\phi \circ \mathcal{W}_{\rho(g)\phi} = \langle \phi, \rho(g)\phi \rangle\, I$ on V, thus:

$$\mathcal{M}_\phi \circ \mathcal{W}_{\rho(g)\phi} \circ \rho(g) = \langle \phi, \rho(g)\phi \rangle\, \rho(g) = \Phi(g)\rho(g).$$

On the other hand, the intertwining properties (3) of the wavelet transform imply:

$$\mathcal{M}_\phi \circ \mathcal{W}_{\rho(g)\phi} \circ \rho(g) = \mathcal{M}_\phi \circ (\Lambda \otimes R)(g) \circ \mathcal{W}_\phi.$$

Integrating the identity $\Phi(g)\rho(g) = \mathcal{M}_\phi \circ (\Lambda \otimes R)(g) \circ \mathcal{W}_\phi$ with the function $f\Phi^{-1}$ and use the partial isometries $\mathcal{W}_\phi$ and $\mathcal{M}_\phi$ we get the inequality.

The Lemma is most efficient if $\Lambda \otimes R$ act in a simple way. Thus, we give he following

Definition 1. We say that the subgroup H has the *complemented commutator property*, if there exists a continuous section $\mathsf{s} : X \to G$ such that:

$$\mathsf{p}(\mathsf{s}(x)^{-1} g \mathsf{s}(x)) = \mathsf{p}(g), \qquad \text{for all } x \in X = G/H,\ g \in G. \tag{9}$$

For a Lie group G with the Lie algebra $\mathfrak{g}$ define the Lie algebra $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$. The subgroup $H = \exp(\mathfrak{h})$ (as well as any larger subgroup) has the complemented commutator property (9). Of course, $X = G/H$ is non-trivial if $H \neq G$ and this happens, for example, for a nilpotent G. In particular, for the Heisenberg group, its centre has the complemented commutator property.

Note, that the complemented commutator property (9) implies:

$$\Lambda \otimes R(\mathsf{s}(x)) : g \mapsto gh, \qquad \text{for the unique } h = g^{-1}\mathsf{s}(x)^{-1}g\mathsf{s}(x) \in H. \tag{10}$$

For a character χ of the subgroup H, we introduce an integral transformation $\widetilde{\ }: L_1(X) \to C(G)$:

$$\widetilde{k}(g) = \int_X k(x)\, \chi(g^{-1}\mathsf{s}(x)^{-1}g\mathsf{s}(x))\, dx, \tag{11}$$

where $h(x,g) = g^{-1}\mathsf{s}(x)^{-1}g\mathsf{s}(x)$ is in H due to the relations (9). This transformation generalises the isotropic symbol defined for the Heisenberg group in [1, § 2.1].

Proposition 1. *Let a subgroup H of G has the complemented commutator property* (9) *and ρ_χ be an irreducible representation of G induced from a character χ of H, then*

$$\left\|\rho_\chi(f)\right\| \leq \left\|\widetilde{f\Phi^{-1}}\right\|_\infty, \tag{12}$$

with the sup*-norm of the function $\widetilde{f\Phi^{-1}}$ on the right.*

Proof. For an induced representation ρ_χ [8, § 13.2], the covariant transform $\mathcal{W}_\phi$ maps V to a space $L_2^\chi(G)$ of functions having the property $F(gh) = \chi(h)F(g)$ [4, § 3.1]. From (10), the restriction of $\Lambda \otimes R$ to the space $L_2^\chi(G)$ is, see:

$$\Lambda \otimes R(\mathsf{s}(x)) : \psi(g) \mapsto \psi(gh) = \chi(h(x,g))\psi(g).$$

In other words, $\Lambda \otimes R$ acts by multiplication on $L_2^\chi(G)$. Then, integrating the representation $\Lambda \otimes R$ over X with a function k we get an operator $(L \otimes R)(k)$, which reduces on the irreducible component to multiplication by the function $\widetilde{k}(g)$. Put $k = f\Phi^{-1}$ for $\Phi = \mathcal{W}_\phi\phi$. Then, from the inequality (8), the norm of operator $\rho_\chi(f)$ can be estimated by $\left\|\Lambda \otimes R(f\Phi^{-1})\right\| = \left\|\widetilde{f\Phi^{-1}}\right\|_\infty$.

For a nilpotent step 2 Lie group, the transformation (11) is almost the Fourier transform, cf. the case of the Heisenberg group in [1, § 2.1].

This allows to estimate $\left\|\widetilde{f\Phi^{-1}}\right\|_\infty$ through $\left\|\widetilde{f}\right\|_\infty$, where $\widetilde{f}$ is in the essence the symbol of the respective PDO. For other groups, the expression $g^{-1}\mathsf{s}(x)^{-1}g\mathsf{s}(x)$ in (11) contains non-linear terms and its analysis is more difficult. In some circumstance the integral Fourier operators [6, Ch. VIII] may be useful for this purpose.

References

[1] Howe R. *Quantum mechanics and partial differential equations* // J. Funct. Anal. — 1980. — **38**, № 2. — P. 188—254.

[2] Kisil V. V. *Relative convolutions. I. Properties and applications* // Adv. Math. — 1999. — **147**, № 1. — P. 35—73; E-print: `arXiv:funct-an/9410001`, On-line. Zbl933.43004.

[3] Kisil V. V. *Wavelets in Banach spaces* // Acta Appl. Math. — 1999. — **59**, № 1. — P. 79—109; E-print: `arXiv:math/9807141`, On-line.

[4] Kisil V. V. *Calculus of operators: Covariant transform and relative convolutions*, 2013. E-print: `arXiv:1304.2792`.

[5] Folland G. B. *Harmonic analysis in phase space* // Annals of Mathematics Studies. — **122**. — Princeton University Press, Princeton, NJ, 1989.

[6] Taylor M. E. *Pseudodifferential operators.* — Princeton Mathematical Series. — **34**. — Princeton University Press, Princeton, N.J., 1981.

[7] Ali S. T., Antoine J. P., Gazeau J. P. *Coherent States, Wavelets and Their Generalizations.* — Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 2000.

[8] Kirillov A. A. *Elements of the theory of representations.* — Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220. — Springer-Verlag, Berlin, 1976.