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Subdomain geometry of hyperbolic type metrics

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Dedicated to memory of Professor Promarz M. Tamrazov

Given a domain $G \subsetneq \mathbb{R}^n$ we study the quasihyperbolic and the distance ratio metrics of G and their connection to the corresponding metrics of a subdomain $D \subset G$. In each case, distances in the subdomain are always larger than in the original domain. Our goal is to show that, in several cases, one can prove a stronger domain monotonicity statement. We also show that under special hypotheses we have inequalities in the opposite direction.

1. Introduction. Recently many authors have studied what we call "hyperbolic type metrics" of a domain $G \subset \mathbb{R}^n$ [$1-6$]. Some of the examples are the Apollonian metric, the Möbius invariant metric, the quasihyperbolic metric and the distance ratio metric. The term "hyperbolic type metric" is for us just a descriptive term, we do not define it. The term is justified by the fact that the metric is similar to the hyperbolic metric of the unit ball $\mathbb{B}^n$. In this paper we will study a hyperbolic type metric m_G with the following two properties:

1. if $D \subset G$ is a subdomain, then $m_D(x,y) \geq m_G(x,y)$ for all $x,y \in D$,
2. sensitivity to the boundary variation: if $x_0 \in G$ and $D = G \setminus \{x_0\}$, then the metrics m_G and m_D are quite different close to x_0 whereas "far away"from x_0 we might expect that they are nearly equal (see Remark 2.10 (2)).

In particular, we require that m_G is defined for every proper subdomain of $\mathbb{R}^n$. The purpose of this paper is to study the subdomain monotonicity property (1) and to prove conditions under which we have a quantitative refinement of (1).

For a subdomain $G \subsetneq \mathbb{R}^n$ and $x, y \in G$ the *distance ratio metric* j_G is defined by

$$j_G(x,y) = \log\left(1 + \frac{|x-y|}{\min\{\delta_G(x), \delta_G(y)\}}\right),$$

where $\delta_G(x)$ denotes the Euclidean distance from x to the boundary ∂G of G. Sometimes we abbreviate δ_G by writing just δ. The above form of the j_G metric, introduced in [7], is obtained by a slight modification of a metric that was studied in [8, 9]. The *quasihyperbolic metric* of G is defined by the quasihyperbolic length minimizing property

$$k_G(x,y) = \inf_{\gamma \in \Gamma(x,y)} \ell_k(\gamma), \quad \ell_k(\gamma) = \int_\gamma \frac{|dz|}{\delta_G(z)},$$

where $\Gamma(x,y)$ represents the family of all rectifiable paths joining x and y in G, and $\ell_k(\gamma)$ is the quasihyperbolic length of γ (cf. [9]). For a given pair of points $x, y \in G$, the infimum is always attained [8], i.e., there always exists a quasihyperbolic geodesic $J_G[x,y]$ which minimizes the above integral, $k_G(x,y) = \ell_k(J_G[x,y])$ and furthermore with the property that the distance is additive on the geodesic: $k_G(x,y) = k_G(x,z) + k_G(z,y)$ for all $z \in J_G[x,y]$. If the domain G is emphasized we call $J_G[x,y]$ a k_G-geodesic. In this paper, our main work is to refine some inequalities between k_G metric, j_G metric and the Euclidean metric. Both the distance ratio and the quasihyperbolic metric qualify as hyperbolic type metrics because

- both are defined for every proper subdomain of $\mathbb{R}^n$,
- for the case of the unit ball $\mathbb{B}^n$ both are comparable to the hyperbolic metric of $\mathbb{B}^n$, see Section 2 below,
- it is well-known that both metrics satisfy the above properties (1) and (2).

These metrics have recently been studied, e.g., in [1, 3, 5]. We mainly study the following three problems and our main results will be given in Section 2, Section 3 and Section 4, respectively.

Problem 1.1. For some special domains, can we obtain certain upper estimates for the quasihyperbolic metric in terms of the distance ratio metric?

Indeed, inequalities of this type were used to characterize so called φ-domains in [7].

Problem 1.2. Is there some relation between k metric and the Euclidean metric? The same question can be asked for j metric and the Euclidean metric?

Let G_1 and G_2 be proper subdomains of $\mathbb{R}^n$. It is well know that if $G_1 \subset G_2$ then for all $x, y \in G_1$,

$$j_{G_1}(x,y) \geq j_{G_2}(x,y)$$

and

$$k_{G_1}(x,y) \geq k_{G_2}(x,y).$$

We expect some better results to hold for some special class of domains. This motivates the following question.

Problem 1.3. Let $G_1 \subset G_2$ be two proper subdomains in $\mathbb{R}^n$ such that $\partial G_1 \cap \partial G_2$ is either $\varnothing$ or a discrete set. Does there exist a constant $c > 1$ such that for all $x, y \in G_1$, the following holds:

$$m_{G_1}(x,y) \geq c m_{G_2}(x,y), \tag{1.4}$$

where $m_{G_i} \in \{j_{G_i}, k_{G_i}\}$ for $i = 1, 2$.

Our main results for Problem 1.1 are Theorems 2.5 and 2.9, for Problem 1.2 Theorems 3.3 and 3.4 and for Problem 1.3 Theorems 4.3 and 4.6. We also formulate some open problems and conjectures. Finally, it should be pointed out that there are many more metrics for which the above problems could be studied. For some of these metrics, see [10].

2. Results concerning Problem 1.1. In this section, we study Problem 1.1 and our main results are Theorems 2.5 and 2.9. The following proposition, which will be used in the proof of Theorems 2.5, gathers together several basic well-known properties of the metrics k_G and j_G, see for instance [9, 11]. The motivation comes from the well-known inequality

$$k_G(x,y) \geq \log\left(1 + \frac{L}{\min\{\delta(x), \delta(y)\}}\right) \geq j_G(x,y)\,, \tag{2.1}$$

for a domain $G \subsetneq \mathbb{R}^n, x, y \in G$, where $L = \inf\{\ell(\gamma) : \gamma \in \Gamma(x,y)\}$. One can ask: when do both the metrics j_G and k_G (or ρ_{B^n}) coincide?

Proposition 2.2.

1. *For* $x \in B^n$*, we have*

$$k_{B^n}(0,x) = j_{B^n}(0,x) = \log \frac{1}{1-|x|}\,.$$

2. *Moreover, for* $b \in S^{n-1}$ *and* $0 < r < s < 1$*, we have*

$$k_{B^n}(br,bs) = j_{B^n}(br,bs) = \log \frac{1-r}{1-s}\,.$$

3. *Let* $G \subsetneq \mathbb{R}^n$ *be a domain and* $z_0 \in G$*. Let* $z \in \partial G$ *be such that* $\delta(z_0) = |z_0 - z|$*. Then for all* $u, v \in [z_0, z]$*, we have*

$$k_G(u,v) = j_G(u,v) = \left|\log \frac{\delta(z_0) - |z_0 - u|}{\delta(z_0) - |z_0 - v|}\right| = \left|\log \frac{\delta(u)}{\delta(v)}\right|.$$

4. *For* $x, y \in B^n$ *we have*

$$j_{B^n}(x,y) \le \rho_{B^n}(x,y) \le 2 j_{B^n}(x,y)$$

with equality on the right hand side when $x = -y$*.*

Proof. (1) We see from (2.1) that

$$j_{B^n}(0,x) = \log \frac{1}{1-|x|} \le k_{B^n}(0,x) \le \int_{[0,x]} \frac{|dz|}{\delta(z)} = \log \frac{1}{1-|x|}$$

and hence $[0,x]$ is the k_{B^n}-geodesic between 0 and x.

The proof of (2) follows from (1) because the quasihyperbolic length is additive along a geodesic

$$k_{B^n}(0,bs) = k_{B^n}(0,br) + k_{B^n}(br,bs)\,.$$

The proof of (3) follows from (2).

The proof of (4) is given in [12, Lemma 7.56].

The hyperbolic geometry of B^n serves as model for the quasihyperbolic geometry and we will use below a few basic facts of the hyperbolic metric

ρ_{B^n} of B^n . These facts appear in standard textbooks of hyperbolic geometry and also in [11, Section 2]. For the case of B^n, we make use of an explicit formula [11, (2.18)] to the effect that for $x, y \in B^n$

$$\sinh \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|}{t}\,, \quad t = \sqrt{(1-|x|^2)(1-|y|^2)}\,. \tag{2.3}$$

It is readily seen that

$$\rho_{B^n} \le 2k_{B^n} \le 2\rho_{B^n}$$

and it is well-known by [12, Lemma 7.56] that a similar inequality also holds for j_{B^n}

$$j_{B^n} \le \rho_{B^n} \le 2j_{B^n}\,.$$

Remark 2.4. The proofs of Proposition 2.2 (1) and (2) show that the diameter $(-e, e), e \in S^{n-1}$, of B^n is a geodesic of k_{B^n} and hence the quasihyperbolic distance is additive on a diameter. At the same time we see that the j metric is additive on a radius of the unit ball but not on the full diameter because for $x \in B^n \setminus \{0\}$

$$j_{B^n}(-x,x) < j_{B^n}(-x,0) + j_{B^n}(0,x)\,.$$

In order to obtain certain upper estimates for the quasihyperbolic metric, in terms of the distance ratio metric, we present the following theorem.

Theorem 2.5.

1. *For $0 < s < 1$ and $x, y \in B^n(s)$, we have*

$$j_{B^n}(x,y) \le k_{B^n}(x,y) \le (1+s)\, j_{B^n}(x,y).$$

2. *Let $G \subsetneq \mathbb{R}^n$ be a domain, $w \in G$, and $w_0 \in (\partial G) \cap S^{n-1}(w, \delta(w))$. If $s \in (0,1)$ and $x, y \in B^n(w, s\delta(w)) \cap [w, w_0]$, then we have*

$$k_G(x,y) \le (1+s) j_G(x,y)\,.$$

Proof. (1) Fix $x, y \in B^n(s)$ and the geodesic γ of the hyperbolic metric joining them. Then $\gamma \subset B^n(s)$ and for all $w \in B^n(s)$ we have

$$\frac{1}{1-|w|} < \frac{1+s}{2} \frac{2}{1-|w|^2}\,.$$

Therefore, by Proposition 2.2 (4)

$$k_{B^n}(x,y) \le \int_\gamma \frac{|dw|}{1-|w|} \le \frac{1+s}{2} \int_\gamma \frac{2|dw|}{1-|w|^2} \le \frac{1+s}{2}\,\rho_{B^n}(x,y) \le$$

$$\le (1+s) j_{B^n}(x,y)$$

for $x, y \in B^n(s)$. The inequality $j_{B^n}(x,y) \le k_{B^n}(x,y)$ follows from (2.1).

For the proof of (2) set $B = B^n(w, \delta(w))$. Then by part (1)

$$k_G(x,y) \le k_B(x,y) \le (1+s) j_B(x,y) \le (1+s) j_G(x,y)\,.$$

This completes the proof of the theorem.

Remark 2.6. Theorem 2.5 (1) refines the well-known inequality in [13, Lemma 2.11] and [11, Lemma 3.7(2)] for the case of B^n. We have been unable to prove a similar statement for a general domain. However, a similar result for $\mathbb{R}^n \setminus \{0\}$ is obtained in the sequel (see Theorem 2.9). To obtain this, we collect some basic properties.

Martin and Osgood [14] proved the following explicit formula: for $x, y \in$ $\in \mathbb{R}^n \setminus \{0\}$

$$k_{\mathbb{R}^n\setminus\{0\}}(x,y) = \sqrt{\alpha^2 + \log^2(|x|/|y|)}\,, \tag{2.7}$$

where $\alpha = \measuredangle(x,0,y)$.

We here introduce a lemma which is a modification of [15, Lemma 4.9].

Lemma 2.8. *Let $z \in G = \mathbb{R}^n \setminus \{0\}$ and $k_G(x,z) = k_G(y,z)$ with $|z| \le |x|, |y|$. Then $\measuredangle(x,z,0) < \measuredangle(y,z,0)$ implies $|x-z| < |y-z|$.*

Proof. Let $k_G(x,z) = r$. By (2.7) the angle $\measuredangle(x,z,0)$ determines the point x uniquely up to a rotation about the line through 0 and z. By symmetry and similarity it is sufficient to consider only the case $n = 2$ and $z = e_1$. We will show that the function

$$f(s) = |x(s) - e_1|^2$$

is strictly increasing on $(0, \min\{r, \pi\})$, where

$$x(s) = (e^s \cos\phi(s), e^s \sin\phi(s)) \quad \text{ with } \varphi(s) = \sqrt{\min\{r,\pi\}^2 - s^2}\,.$$

For $s \in [0, \min\{r, \pi\}]$, a simple calculation gives

$$f(s) = |x(s)|^2 + 1 - 2|x(s)|\cos\phi(s) = e^{2s} + 1 - 2e^s \cos\phi(s)$$

and hence

$$f'(s) = 2e^s \left(e^s - \cos\phi(s) - \frac{s\sin\phi(s)}{\phi(s)} \right).$$

If $s \in (0, \min\{r, \pi\})$, then we see that

$$e^s - \cos\phi(s) - \frac{s\sin\phi(s)}{\phi(s)} \geq e^s - \cos\phi(s) - s \geq e^s - 1 - s > 0$$

and hence $f'(s) > 0$ implies the assertion.

Theorem 2.9. *Let $G = \mathbb{R}^n \setminus \{0\}$. Then*

1. *for $\alpha \in (0, \pi]$ and $x, y \in G$ with $\measuredangle(x, 0, y) \leq \alpha$*

$$k_G(x, y) \leq \frac{\alpha}{\log(1 + 2\sin(\alpha/2))}\, j_G(x, y) \leq (1 + \alpha)\, j_G(x, y).$$

2. *for $\varepsilon > 0$, $x \in G$ and $y \in B^n(|x|/t) \cup (\mathbb{R}^n \setminus \overline{B(t|x|)})$*

$$k_G(x, y) \leq (1 + \varepsilon)\, j_G(x, y),$$

where $t = \exp((1 + 1/\varepsilon)\log 3)$.

Proof. (1) We may assume that $|y| \geq |x|$. Fix $k_G(x, y) = c > 0$. Now $j_G(x, y) = \log(1 + |x - y|/|x|)$ and by Lemma 2.8 the quantity $k_G(x, y)/j_G(x, y)$ attains its maximum when α is maximal, which is equivalent to $|y| = |x|$. Thus,

$$\frac{k_G(x, y)}{j_G(x, y)} \leq \frac{\alpha}{\log\left(1 + \frac{2|x|\sin(\alpha/2)}{|x|}\right)} = \frac{\alpha}{\log\left(1 + 2\sin\frac{\alpha}{2}\right)}$$

and the first inequality follows.

Let us next prove the second inequality. We define the functions f and g by

$$f(x) = \log(1 + x) \quad \text{and} \quad g(x) = x/(1 + x/2).$$

Because

$$g'(x) = \frac{4}{(2 + x)^2} \leq \frac{1}{1 + x} = f'(x),$$

$g'(x) > 0$ for $x \geq 0$ and $f(0) = 0 = g(0)$, we have $g(x) \leq f(x)$. Thus,

$$\frac{\alpha}{\log(1 + 2\sin(\alpha/2))} \leq \frac{\alpha}{\frac{2\sin(\alpha/2)}{1+\sin(\alpha/2)}} = \frac{\alpha}{2}\left(1 + \frac{1}{\sin(\alpha/2)}\right).$$

The function $h(\alpha) = (\alpha/(2)\,(1+1/(\sin(\alpha/2)))$ is convex, since

$$h''(\alpha) = \frac{\alpha(3+\cos\alpha) - 4\sin\alpha}{16\sin^3(\alpha/2)} \geq 0.$$

Therefore, $h(\alpha) \leq \max\{h(0), h(\pi)\} = \pi$ on $[0,\pi]$ and $h(\alpha) \leq 1 + +(1-1/\pi)\,\alpha \leq 1+\alpha$ both imply the assertion.

(2) We prove that

$$k_G(x,-ux) \leq (1+\varepsilon) j_G(x,-ux),$$

where $u \in (0,1/t]$ or $u > t$. We may assume $x = e_1$. Now

$$\left(\frac{k_G(x,y)}{j_G(x,y)}\right)^2 = \frac{\pi^2 + \log^2(1/u)}{\log^2((|x|+u|x|+u)/u)} \geq \frac{\log^2(1/u)}{\log^2(3/u)} = A$$

and $A \geq 1+\varepsilon$ is equivalent to $u \leq 1/t$ or $u \geq t$. The assertion follows from (2.7).

Remark 2.10.

1. In Theorem 2.9 (1), the constant $h(\alpha) = \alpha/\log(1+2\sin(\alpha/2))$ appears with the bound $h(\alpha) \leq 1+\alpha$. This upper bound of $h(\alpha)$ is not sharp as can be seen from the proof. By computer simulations, we obtained that the sharp upper bounds are $h(\alpha) \leq 1 + +((1/\log 3) - (1/\pi))\alpha$ for $\alpha \in [0,\pi]$ and $h(\alpha) \leq 1 + \pi\alpha/(2\log(1+\sqrt{2}))$ for $\alpha \in [0,\pi/2]$. Lindén [4] proved the limiting case $\alpha = \pi$ of Theorem 2.9 (1) with the constant $c_0 \equiv \pi/\log(3)$. For $c \in (1, c_0)$, some of the level sets $L(c) = \{z : k_G(z,1)/j_G(z,1) = c\}$ are displayed in Figure 1.

2. Let $D \subset \mathbb{R}^n$ be a domain, and let $G = D \setminus \{x_0\}$ with $x_0 \in D$. For given $x, y \in G$ if there exists some constant $c \geq 1$ such that $\min\{d_D(x), d_D(y)\} \leq c\min\{|x-x_0|, |y-y_0|\}$, then by the definition of j-metric we have $j_G(x,y) \leq c j_D(x,y)$. We also see from [7, Lemma 2.53] that $k_G(x,y) \leq c_1(c) k_D(x,y)$ with $c_1(c)$ depending only on c.

3. Results concerning Problem 1.2. In this section, our main goal is to study Problem 1.2, that is, to compare the Euclidean metric and the quasihyperbolic metrics defined in a domain. Our main result is Theorem 3.3.

In the next lemma, we recall a sharp inequality for the hyperbolic metric of the unit ball proved in [11, (2.27)].

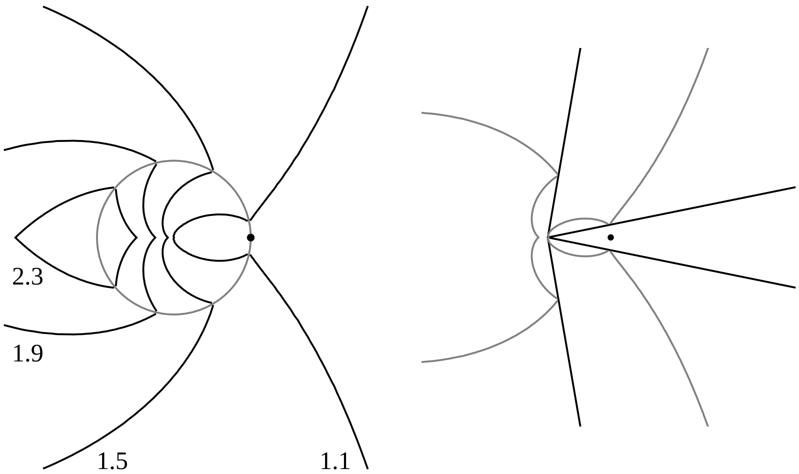

Figure 1: Left: Level sets $L(c) = \{z : k_G(z,1)/j_G(z,1) = c\}$ for $G = \mathbb{R}^n \setminus \{0\}$ and $c = 1.1$, 1.5, 1.9, 2.3. Right: Level sets $L(c)$ and angular domains as in Theorem 2.9 (1) for $c = 0.2$, 1.4.

Lemma 3.1. *For $x, y \in B^n$, let t be as in* (2.3). *Then*

$$\tanh^2 \frac{\rho_{B^n}(x,y)}{2} = \frac{|x-y|^2}{|x-y|^2 + t^2} ,$$

$$|x-y| \leqslant 2 \tanh \frac{\rho_{B^n}(x,y)}{4} = \frac{2|x-y|}{\sqrt{|x-y|^2+t^2} + t} ,$$

where equality holds for $x = -y$.

Earle and Harris [16] provided several applications of this inequality and extended this inequality to other metrics such as the Carathéodory metric. Notice that Lemma 3.1 gives a sharp bound for the modulus of continuity

$$id : (B^n, \rho_{B^n}) \to (B^n, |\cdot|) .$$

For a K-quasiconformal homeomorphism

$$f : (B^n, \rho_{B^n}) \to (B^n, \rho_{B^n})$$

an upper bound for the modulus of continuity is well-known, see [11, Theorem 11.2]. For $n = 2$ the result is sharp for each $K \geq 1$, see [17, p. 65 (3.6)]. The particular case $K = 1$ yields a classical Schwarz lemma.

As a preliminary step we record Jung's Theorem [18, Theorem 11.5.8] which gives a sharp bound for the radius of a Euclidean ball containing a given bounded domain.

Lemma 3.2. *Let $G \subset \mathbb{R}^n$ be a domain with* $\operatorname{diam} G < \infty$. *Then there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z,r)$, where $r \le \sqrt{n/(2n+2)}\operatorname{diam} G$.*

Theorem 3.3.

1. *If $x, y \in B^n$ are arbitrary and $w = |x-y|\, e_1/2$, then*

$$k_{B^n}(x,y) \ge k_{B^n}(-w,w) = 2\,k_{B^n}(0,w) = 2\,\log\frac{2}{2-|x-y|} \ge |x-y|\,,$$

where the first inequality becomes equality when $y = -x$. Moreover, the identity map $id : (B^n, k_{B^n}) \to (B^n, |.|)$ has the sharp modulus of continuity $\omega(t) = 2(1-e^{-t/2})$.

2. *Let $G \subsetneq \mathbb{R}^n$ be a domain with* $\operatorname{diam} G < \infty$ *and $r = \sqrt{n/(2n+2)}\operatorname{diam} G$. Then we have*

$$k_G(x,y) \ge 2\,\log\frac{2}{2-t} \ge t = |x-y|/r\,,$$

for all distinct $x, y \in G$ with equality in the first step when $G = B^n(z,r)$ and $z = (x+y)/2$. Moreover, the identity map $id : (G, k_G) \to (G, |.|)$ has the sharp modulus of continuity $\omega(t) = 2r(1-e^{-t/2})$.

Proof. (1) Without loss of generality, we may assume that $|x| \ge |y|$. We divide the proof into two cases.

Case I: The points x and y are both on a diameter of B^n.
If $0 \in [x,y]$, by Proposition 2.2 (1) we have

$$k_{B^n}(x,y) = k_{B^n}(x,0) + k_{B^n}(0,y) = \log\frac{1}{(1-|x|)(1-|y|)}\,,$$

and hence

$$k_{B^n}(-w,w) = 2\log\frac{1}{1-|w|}\,.$$

It is easy to verify that $k_{B^n}(x,y) \ge k_{B^n}(-w,w)$ is equivalent to $(|x|-|y|)^2 \ge 0\,.$

If $y \in [x,0]$, then the proof goes in a similar way. Indeed, in this situation

$$k_{B^n}(x,y) = \log\frac{1-|y|}{1-|x|} \geq k_{B^n}(-w,w)$$

is equivalent to

$$(|x|-|y|)\left(1-\frac{1}{1-|y|}\right) \leq \left(\frac{|x|-|y|}{2}\right)^2,$$

which is trivial as the left hand term is ≤ 0. Equality clearly holds if $y=-x$.

Case II: The points x and y are arbitrary in B^n.
Choose $y' \in B^n$ such that $|x-y| = |x-y'| = 2|w|$ with x and y' on a diameter of B^n. Then

$$k_{B^n}(x,y) \geq k_{B^n}(x,y') \geq k_{B^n}(-w,w)\,,$$

where the first inequality holds trivially and the second holds by *Case I.* The sharp modulus of continuity can be obtained by a trivial rearrangement of the first inequality from the statement.

(2) Since G is a bounded domain, by Lemma 3.2, there exists $z \in \mathbb{R}^n$ such that $G \subset B^n(z,r)$. Denote $B := B^n(z,r)$. Then the domain monotonicity property gives

$$k_G(x,y) \geq k_B(x,y)\,.$$

Without loss of generality we may now assume that $z=0$. Choose $u,v \in B$ in such a way that $u=-v$ and $|u-v| = 2|u| = |x-y|$. Hence by (1) we have

$$k_G(x,y) \geq k_B(x,y) \geq k_B(-u,u) = 2\,\log\frac{r}{r-|u|}\,.$$

This completes the proof.

A counterpart of Theorem 3.3 for the distance ratio metric j_G can be formulated in the following form (we omit the proofs, since they are very similar to the proofs of Theorem 3.3).

Theorem 3.4.

1. *If $x,y \in B^n$ are arbitrary and $w = |x-y|\,e_1/2$, then*

$$j_{B^n}(x,y) \geq j_{B^n}(-w,w) = \log\frac{2+t}{2-t} = 2\mathrm{artanh}(t/2) \geq t = |x-y|\,,$$

where the first inequality becomes equality when $y = -x$. Moreover, the identity map $id : (B^n, j_{B^n}) \to (B^n, |.|)$ *has the sharp modulus of continuity* $\omega(t) = 2\tanh(t/2)$.

2. *Let* $G \subsetneq \mathbb{R}^n$ *be a domain with* $\operatorname{diam} G < \infty$ *and* $r = \sqrt{n/(2n+2)}\operatorname{diam} G$. *Then we have*

$$j_G(x, y) \ge \log\frac{2+t}{2-t} \ge t = |x-y|/r\,,$$

for all distinct $x, y \in G$ *with equality in the first step when* $G = B^n(z, r)$ *and* $z = (x+y)/2$. *Moreover, the identity map* $id : (G, j_G) \to (G, |.|)$ *has the sharp modulus of continuity* $\omega(t) = 2r\tanh(t/2)$.

4. Results concerning Problem 1.3. In this final section we present our results on Problem 1.3.

Theorem 4.1. *Let* $G_1 = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ *and* $G_2 = \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 < 1\}$. *Then* (1.4) *holds for* k_G *metric with* $c = \sqrt{2}$ *but there is no constant* $c > 1$ *for which* (1.4) *holds for the* j_G *metric.*

Proof. Obviously, $\partial G_1 \cap \partial G_2 = \{e_1, -e_1, e_2, -e_2\}$ is a discrete set. For each $w \in G_1$, we prove that

$$\delta_{G_2}(w) \ge \sqrt{2}\delta_{G_1}(w). \tag{4.2}$$

Without loss of generality, we may assume that $\operatorname{Re}(w) \ge 0$ and $\operatorname{Im}(w) \ge 0$. Then $\operatorname{Re}(w) + \operatorname{Im}(w) \le 1$ and $\delta_{G_1}(w) = \frac{1}{\sqrt{2}}(1 - \operatorname{Re}(w) - \operatorname{Im}(w))$. Hence,

$$\delta_{G_2}(w) = 1 - \sqrt{\operatorname{Re}(w)^2 + \operatorname{Im}(w)^2} \ge 1 - \operatorname{Re}(w) - \operatorname{Im}(w) = \sqrt{2}\delta_{G_1}(w),$$

which proves inequality (4.2).

Given $z_1, z_2 \in G_1$, let γ be a quasihyperbolic geodesic joining z_1 and z_2 in G_1. Then by (4.2),

$$k_{G_2}(z_1, z_2) \le \int_\gamma \frac{|dw|}{\delta_{G_2}(w)} \le \int_\gamma \frac{|dw|}{\sqrt{2}\delta_{G_1}(w)} = \frac{1}{\sqrt{2}}\, k_{G_1}(z_1, z_2).$$

For the j_G metric case, let $x_0 = (1-\varepsilon, 0)$, $y_0 = (-1+\varepsilon, 0)$ where $\varepsilon \in (0, 1)$. Then

$$|x_0 - y_0| = 2 - 2\varepsilon$$

and

$$\delta_{G_2}(x_0) = \delta_{G_2}(y_0) = \varepsilon = \sqrt{2}\delta_{G_1}(x_0) = \sqrt{2}\delta_{G_1}(y_0).$$

Hence,

$$\frac{j_{G_1}(x_0, y_0)}{j_{G_2}(x_0, y_0)} = \frac{\log(1 + \frac{\sqrt{2}(2-2\varepsilon)}{\varepsilon})}{\log(1 + \frac{(2-2\varepsilon)}{\varepsilon})} \to 1, \quad \varepsilon \to 0.$$

Theorem 4.3. *For $0 < s < 1$, let $G_1 = \{(x, y) : |x|^s + |y|^s < 1\}$ and $G_2 = \{(x, y) : |x| + |y| < 1\}$. Then $k_{G_1}(z_1, z_2) \geq 2^{\frac{1}{s}-1} k_{G_2}(z_1, z_2)$ for all $z_1, z_2 \in G_1$.*

Proof. We first prove that for all $w \in G_1$, $\delta_{G_2}(w) \geq 2^{\frac{1}{s}-1}\delta_{G_1}(w)$. Let $w = (a, b) \in G_1$. By symmetry, we only need to consider the case $0 \leq b \leq a$. Denote $\gamma_s = \partial G_1 \cap \{(x, y) : x \geq 0, y \geq 0\}$, $\gamma_1 = \partial G_2 \cap \{(x, y) : x \geq 0, y \geq 0\}$. Let $y_1 \in \gamma_1$ be such that line ℓ_{0y_1}, which goes through 0 and y_1, is perpendicular to γ_1. Obviously, $\ell_{0y_1} \perp \gamma_s$, say at the point y_2. Let $y_3 \in \gamma_1$ be such that $[w, y_3] \perp \gamma_1$, y_4 be the intersection point of $[w, y_3]$ and γ_s and $w_1 \in \ell_{0y_1}$ be such that w_1, w and e_1 are collinear (see Figure 2).

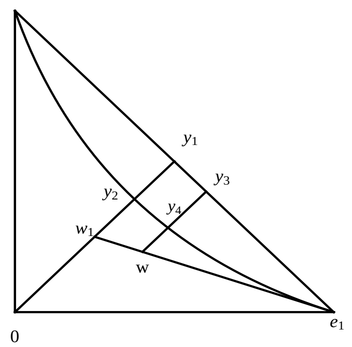

Figure 2: Points y_i, w and w_1 used in the proof of Theorem 4.3.

We observe first that

$$\frac{\delta_{G_2}(w)}{\delta_{G_1}(w)} \geq \frac{|w - y_3|}{|w - y_4|}. \tag{4.4}$$

By similar triangle property, we get

$$\frac{|w - y_3|}{|w_1 - y_1|} = \frac{|e_1 - w|}{|e_1 - w_1|} \geq \frac{|w - y_4|}{|w_1 - y_2|},$$

which, together with (4.4) and simple calculation, shows that

$$\frac{\delta_{G_2}(w)}{\delta_{G_1}(w)} \geq \frac{\delta_{G_2}(w_1)}{\delta_{G_1}(w_1)} \geq \frac{\delta_{G_2}(0)}{\delta_{G_1}(0)} = 2^{\frac{1}{s}-1}.$$

Given $z_1, z_2 \in G_1$, let β be a quasihyperbolic geodesic joining z_1 and z_2 in G_1. Then

$$k_{G_2}(z_1, z_2) \leq \int_\beta \frac{|dw|}{\delta_{G_2}(w)} \leq 2^{\frac{1}{s}-1} k_{G_1}(z_1, z_2).$$

We generalize the above two Theorems into the following conjecture.

Conjecture 4.5. For $0 < s < t$, let $G_1 = G_s = \{(x, y) : |x|^s + |y|^s \leq$ $\leq 1\}$, $G_2 = G_t = \{(x, y) : |x|^t + |y|^t \leq 1\}$. We conjecture that $k_{G_1}(z_1, z_2) \geq$ $\geq 2^{\frac{1}{s}-\frac{1}{t}} k_{G_2}(z_1, z_2)$ for all $z_1, z_2 \in G_1$.

The following result gives a solution to Problem 1.3.

Theorem 4.6. *Let G_1 be a bounded subdomain of the domain $G_2 \subsetneq \mathbb{R}^n$. Then for all $x, y \in G_1$*

$$m_{G_1}(x, y) \geq c m_{G_2}(x, y),$$

where $m_{G_i} \in \{j_{G_i}, k_{G_i}\}$ for $i = 1, 2$ and $c = 1 + \frac{2\,\mathrm{dist}(G_1, \partial G_2)}{\mathrm{diam}(G_1)}$.

Proof. We first prove the j_G metric case.

For each $x, y \in G_1$, we are going to prove

$$\log\left(1 + \frac{|x-y|}{\min\{\delta_{G_1}(x), \delta_{G_1}(y)\}}\right) \geq$$
$$\geq \left(1 + \frac{2\,\mathrm{dist}(G_1, \partial G_2)}{\mathrm{diam}(G_1)}\right) \log\left(1 + \frac{|x-y|}{\min\{\delta_{G_2}(x), \delta_{G_2}(y)\}}\right).$$

Since $\delta_{G_2}(w) \geq \delta_{G_1}(w) + \mathrm{dist}(G_1, \partial G_2)$ holds for all $w \in G_1$, then it suffices to prove

$$\mathrm{diam}(G_1) \log\left(1 + \frac{|x-y|}{\min\{\delta_{G_1}(x), \delta_{G_1}(y)\}}\right) \geq$$
$$\geq (\mathrm{diam}(G_1) + 2\,\mathrm{dist}(G_1, \partial G_2)) \times$$
$$\times \log\left(1 + \frac{|x-y|}{\min\{\delta_{G_1}(x) + \mathrm{dist}(G_1, \partial G_2), \delta_{G_1}(y) + \mathrm{dist}(G_1, \partial G_2)\}}\right).$$

Let
$$f(z) = (d+2z)\log(1+\frac{a}{b+z}),$$
where $d = \mathrm{diam}(G_1)$, $a = |x-y|$, $b = \delta_{G_1}(x)$ and $z \geq 0$.

Then
$$f'(z) = \log(1+\frac{a}{z+b}) - \frac{a(2z+d)}{(z+a+b)(z+b)}\,,$$
and
$$f''(z) = \frac{-2a(z+a+b)(z+b)+a(2z+d)(2z+2b+a)}{(z+a+b)^2(z+b)^2}\,.$$

Let
$$h(z) = -2a(z+a+b)(z+b)+a(2z+d)(2z+2b+a)\,.$$
It is easy to see that
$$h'(z) = a(2d-2b-a+2z) > 0,$$
which implies that $h(z) > h(0) > 0$.

Hence $f''(z) \geq 0$, which yields
$$f'(z) \leq f'(\infty) = 0$$
and so the function $f(z)$ is decreasing. Thus the assertion follows.

For the k_G metric case, we first prove that for each $w \in D_1$, the following inequality holds:
$$\delta_{G_2}(w) \geq (1+\frac{2\,\mathrm{dist}(G_1,\partial G_2)}{\mathrm{diam}(G_1)})\delta_{G_1}(w).$$

In fact, for each $w \in G_1$ we have
$$\begin{aligned}\mathrm{diam}(G_1)\delta_{G_2}(w) &\geq \mathrm{diam}(G_1)(\delta_{G_1}(w)+\mathrm{dist}(G_1,\partial G_2)) \geq \\ &\geq (\mathrm{diam}(G_1)+2\,\mathrm{dist}(G_1,\partial G_2))\delta_{G_1}(w).\end{aligned}$$

Given $x, y \in G_1$, let γ be a quasihyperbolic geodesic joining x and y in G_1. Then
$$k_{G_2}(x,y) \leq \int_\gamma \frac{|dw|}{\delta_{G_2}(w)} \leq \int_\gamma \frac{|dw|}{c\delta_{G_1}(w)} \leq \frac{1}{c}k_{G_1}(x,y),$$

where $c = 1 + \frac{2\,\mathrm{dist}(G_1, \partial G_2)}{\mathrm{diam}(G_1)}$.

Corollary 4.7. *Let* $0 < r < R$ *and* $G_1 = B^n(x, r)$, $G_2 = B^n(x, R)$. *Then for all* $x, y \in G_1$

$$m_{G_1}(x, y) \geq c m_{G_2}(x, y),$$

where $m_{G_i} \in \{j_{G_i}, k_{G_i}\}$ *for* $i = 1, 2$ *and for* $c = R/r$.

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