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Bounds for holomorphic functionals on Teichmüller spaces and univalent functions

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Dedicated to memory of Professor Promarz M. Tamrazov

We establish universal distortion bounds for arbitrary holomorphic functionals on some Teichmüller spaces and on general classes of univalent functions in quasidisks.

1. Introductory remarks and results.

1.1. Classes of functions. Let L be an oriented quasicircle on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with interior and exterior domains D and D^* so that D^* contains the infinite point and D is bounded. Denote by $\Sigma(D^*)$ the collection of univalent functions on D^* with expansions

$$f(z) = z + b_0 + b_1 z^{-1} + O(1/z^2) \quad \text{near} \quad z = \infty, \tag{1.1}$$

and let $\Sigma^0(D^*)$ be its subset, which consists of f admitting quasiconformal extensions across L (hence to $\widehat{\mathbb{C}}$).

Any $f \in \Sigma^0$ is a solution to the Schwarz equation $S_w := (w''/w')' - -(w''/w')^2/2 = \varphi$ with given holomorphic φ in D^*, and to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ with μ from the ball of Beltrami coefficients

$$\mathbf{Belt}(D)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu(z)|D^* = 0, \|\mu\|_\infty < 1\}.$$

The admissible Schwarzians S_f range over a bounded domain in the complex Banach space $\mathbf{B}(D^*)$ of holomorphic functions $\varphi : D^* \to \widehat{\mathbb{C}}$ with norm

$\|\varphi\| = \sup_{D^*} \lambda_{D^*}^{-2}|\varphi(z)|$, generated by the hyperbolic metric $\lambda_{D^*}(z)|dz|$ of curvature -4 on D^*. This domain models the universal Teichmüller space $\mathbf{T} = \mathbf{T}(D^*)$ with the base point D^*.

Both equations determine f up to a constant c, or equivalently, up to a translation $w \mapsto w + b$. We fix this constant by passing to

$$\widetilde{f}(z) = f(z) - f(0), \tag{1.2}$$

which means that f is normalized by (1.1) and $f(0) = 0$. All other admissible values of $b = f(0)$ are those for which $f(z) \neq 0$ on D^*. This ensures compactness of $\Sigma^0(D^*)$ in the topology of uniform convergence on closed sets in $\mathbb{C}$. For any $f^\mu \in \Sigma^0(D^*)$, the admissible translations are generated only by b running over the closure of the complementary domain to $\widetilde{f}^\mu(D^*)$.

One can vary the functions (1.2) by

$$\omega = H^\mu(z) = w - \frac{1}{\pi} \iint\limits_{f(D)} \mu(\zeta) g(w, \zeta) d\xi d\eta + O(\|\mu\|_\infty^2) \quad (\zeta = \xi + i\eta), \tag{1.3}$$

where $g(w, \zeta) = 1/(\zeta - w) - 1/\zeta$ and the ratio $O(\|\mu\|_\infty^2)/\|\mu\|_\infty^2$ remains uniformly bounded on $\widehat{\mathbb{C}}$ as $\|\mu\|_\infty \to 0$ (see, e.g. [Kr1]). A simple modification of (1.3) yields the variation for f^μ with small $\|\mu\|_\infty$ and another additional normalization, for example, $f(1) = 1$.

1.2. Holomorphic functionals. Consider on $\Sigma^0(D^*)$ a holomorphic (continuous and Gateaux $\mathbb{C}$-differentiable) functional $J(f)$, which means that for any $f \in \Sigma^0(D^*)$ and small $t \in \mathbb{C}$,

$$J(f + th) = J(f) + tJ'_f h + O(t^2), \quad t \to 0, \tag{1.4}$$

in the topology of locally uniform convergence on D^*; here $J'_f h$ is a $\mathbb{C}$-linear functional.

Any J is lifted to a holomorphic function $\widetilde{J}(S_f; b)$ on the Bers fiber space $\mathrm{Fib}(\mathbf{T})$ over $\mathbf{T}$ consisting of pairs $(\varphi, b) \subset B \times \mathbb{C}$, where $\varphi = S_f \in \mathbf{T}$ and $b = f(0)$. By Bers' Isomorphism Theorem [Be2], $\mathrm{Fib}(\mathbf{T})$ is biholomorphically equivalent to the Teichmüller space of the punctured disk $\Delta \setminus \{0\}$, hence, a Banach domain. Therefore, for each fixed b (or equivalently, b_0 in (1.1)), the functional J is lifted to a holomorphic function $\widetilde{J}_b(S_f) = \widetilde{J}(S_f; b)$ on the space $\mathbf{T}(D^*)$ and to a holomorphic function $J_b(f^\mu)$ on the ball $\mathbf{Belt}(D)_1$ related by $\widehat{J}_b(\cdot) = \widetilde{J}_b \circ \phi_{\mathbf{T}}(\cdot)$ where $\phi_{\mathbf{T}}$ is the canonical factorizing projection $\mathbf{Belt}(D)_1 \to \mathbf{T}(D^*)$ (see, e.g., [Be2], [GL], [Kr1]).

Using the well-known representation of J by a complex Borel measure on $\mathbb{C}$, one extends this functional to all holomorphic functions on D^* (cf. [Sch]). In particular, the value $J_{\mathrm{id}}(g(\mathrm{id}, z))$ of J on the identity map $\mathrm{id}(z) = z$ is well defined.

We assume that the functional derivative

$$\psi_0(z) = J'_{\mathrm{id}}(g(\mathrm{id}, z)) \tag{1.5}$$

is a meromorphic function on $\mathbb{C}$ without singularities in D, and ψ_0 belongs to the space $A_1(D)$ of integrable holomorphic functions on D. Put

$$\|J'_{\mathrm{id}}\| = \frac{1}{\pi} \iint_D |J'_{\mathrm{id}}(g(\mathrm{id}, z))| \, dxdy. \tag{1.6}$$

All this holds, for example, for the general distortion functionals of the form

$$J(f) := J(f(z_1), f'(z_1), \dots, f^{(\alpha_1)}(z_1); \dots; f(z_p), f'(z_p), \dots, f^{(\alpha_p)}(z_p)) \tag{1.7}$$

where z_j are distinguished distinct points in D^* with prescribed orders α_j (with $\widehat{J}(\mathbf{0}) = 0,\ \widehat{J}'(\mathbf{0}) \neq 0$). Another important example is given by

$$J(f) = g(S_f),$$

where g is a holomorphic function $\mathbf{T}(D^*) \to \mathbb{C}$.

1.3. Main theorems. The purpose of this paper is to establish some universal distortion estimates for arbitrary holomorphic functionals J.

Theorem 1.1. *Any functional (1.4) with $\widehat{J}(\mathbf{0}) = 0,\ \widehat{J}'(\mathbf{0}) \neq 0$ is estimated for all $0 < k < 1$ from below by*

$$\max_{\|\mu\|_\infty \le k} |J(f^\mu)| \ge \|J'_{\mathrm{id}}\| k. \tag{1.8}$$

This lower bound is sharp and it is attained for small k.

Theorem 1.2. *For each holomorphic functional J on $\Sigma^0(D^*)$ with $J(\mathrm{id}) = 0$ and holomorphic integrable derivative $J'_{\mathrm{id}}(g(\mathrm{id}, \cdot))$, there exists a holomorphic $J_0 : \ \Sigma^0(D^*) \to \mathbb{C}$ with $J_0(\mathrm{id}) = 0,\ \widehat{J}'_{0,\mathrm{id}}(g(\mathrm{id}, \cdot)) = \widehat{J}'_{\mathrm{id}}(g(\mathrm{id}, \cdot)) = \psi_0$, which satisfies*

$$\sup_{\Sigma^0(D^*)} |J_0(f)| = \|J'_{\mathrm{id}}\|. \tag{1.9}$$

The equality (1.9) *means that the corresponding function* $\widetilde{J}_0(S_f)/\|J'_{\mathrm{id}}\|$ *in the Teichmüller space* $\mathbf{T}(D)$ *is maximizing for the Carathéodory distance between any two points of the disk*

$$\Delta(J'_{\mathrm{id}}) = \{\phi_{\mathbf{T}}(t|J'_{\mathrm{id}}(g(\mathrm{id},\cdot))|/J'_{\mathrm{id}})(g(\mathrm{id},\cdot))\} \subset \mathbf{T}(D),$$

and this distance coincides with the Teichmüller–Kobayashi distance.

This theorem sheds light on the intrinsic connection between the extremals of many holomorphic functionals in geometric function theory and extremal functions for invariant distances in the universal Teichmüller space.

1.4. Remarks.

1. One can replace in Theorem 1.1 the assumption $f(z) \neq 0$ in D^*, for example, by $f(1) = 1$ (generically in the Carathéodory sense); then in (1.3), $g(w,\zeta) = 1/(\zeta - w) - 1/(\zeta - 1)$.

2. The proof of both theorems is geometric and relies on the properties of subharmonic functions of negative Gaussian curvature considered on the Teichmüller geodesic disks in the universal Teichmüller space.

Other holomorphic disks intrinsically connected with univalent functions are their homotopy disks, applied, for example, in [Kr4].

3. There have been many investigations addressed to distortion estimates for univalent functions with quasiconformal extensions (see, e.g., [GR], [Kr1], [KK], [Ku1], [Ku2], [Le], [Sc], [Sh]). All of those were concerned with much more specific functionals and involved completely different methods.

Rather complete distortion theory is now bilt up for quasiconformal maps with sufficiently small dilatations fow which many variational problems have been solved explicitly (see [Kr3], [Kr5]). Theorems 1.1 and 1.2 complete these results.

Creating a general distortion theory for univalent functions with k-quasiconformal extensions for generic $k < 1$ still remains a very complicate open problem.

The problem to get a general distortion theory for univalent functions with quasiconformal extensions for generic $k < 1$ still remains open.

4. In the case of the disk $\Delta^* := \{z \in \widehat{\mathbb{C}} : |z| > 1\}$, we get the well-known class $\Sigma = \Sigma(\Delta^*)$ of univalent $\widehat{\mathbb{C}}$-holomorphic functions with expansions (1.1) on Δ^*. For any $f \in \Sigma$, there is a sharp estimate $|b_1| \leq$ $\leq k(f)$, with equality only for $f(z) = z + b_0 + b_1 z^{-1}$ (see, e.g., [Ku1]).

Thus (1.8) implies the following corollary to Theorem 1.1 which is useful for investigation of functionals with $\|J'_{\mathrm{id}}\| = 1$.

Corollary 1.3. *For all holomorphic functionals J on Σ^* with $J(\mathrm{id}) = = 0$, $J'_{\mathrm{id}} \neq 0$,*

$$\max_{\|\mu\|_\infty \le \kappa} |J(f^\mu)| \ge \|J'_{\mathrm{id}}\| \, |b_1(f^\mu)| \, .$$

2. Proof of Theorem 1.1.

The proof is based on three lemmas. The first one relates to a generalization of the Gaussian curvature due to Royden (cf. [Ah], [He], [Ro]).

As is well known, the curvature of a C^2-smooth metric $\lambda > 0$ is defined by $\kappa_\lambda = -(\Delta \log \lambda)/\lambda^2$, where Δ means the Laplacian $4\partial\overline{\partial}$.

A subharmonic metric λ in a domain G on $\mathbb{C}$ (or on a Riemann surface) has curvature at most K **in the potential sense** at a point z_0 if there is a disk U about z_0 in which the function $\log \lambda + K \operatorname{Pot}_U(\lambda^2)$, where Pot_U denotes the logarithmic potential

$$\operatorname{Pot}_U h = \frac{1}{2\pi} \iint\limits_U h(\zeta) \log |\zeta - z| d\xi d\eta \quad (\zeta = \xi + i\eta), \tag{2.1}$$

is subharmonic. Since the Laplacian $\boldsymbol{\Delta} \operatorname{Pot}_U h = h$ (in the sense of distributions), one can replace U by any open subset $V \subset U$, because the function $\operatorname{Pot}_U(\lambda^2) - \operatorname{Pot}_V(\lambda^2)$ is harmonic on U. It is equivalent that the corresponding inequality

$$\boldsymbol{\Delta} \log \lambda \ge -K\lambda^2 \tag{2.2}$$

holds in the sense of distributions. This property is invariant under conformal maps.

We shall use Royden's lemma, estimating from below the circularly symmetric (radial) metrics satisfying (2.2).

Lemma 2.1 [Ro]. *If a circularly symmetric conformal metric $\lambda(|z|)|dz|$ in the unit disk has curvature at most -4 in the potential sense, then*

$$\lambda(r) \ge \frac{a}{1 - a^2 r^2} \, , \tag{2.3}$$

where $a = \lambda(0)$.

The right hand-side of (2.3) defines a supporting conformal metric for λ at the origin with constant Gaussian curvature -4 on the whole disk Δ.

Now consider the hyperbolic metric of the unit disk $ds = \lambda_\Delta(\zeta)|d\zeta|$ of curvature -4, with $\lambda_\Delta(\zeta) = 1/(1 - |\zeta|^2)$.

For a given sequence of holomorphic functionals $\{J_m(\varphi)\}$ on the space $\mathbf{T}$ with $|J_m(\varphi)| < 1$ and a holomorphic map $h: \ \Delta \to \mathbf{T}$, define on the holomorphic disk $h(\Delta)$ by pulling back λ_Δ the conformal metrics

$$\lambda_m(t) = (J_m \circ h)^* \lambda_\Delta = \frac{|(J_m \circ h)'(t)|}{1 - |(J_m \circ h)(t)|^2}, \tag{2.4}$$

whose Gaussian curvature equals -4 at all noncritical points. Consider the upper envelopes of these quantities

$$\mathcal{J}(\varphi) = \sup_m |J_m(\varphi)|$$

and

$$\lambda_{\mathcal{J}}(t) = \sup_m \lambda_m(t) \tag{2.5}$$

followed by their upper regularization $u^*(t) = \limsup\limits_{t' \to t} u(t')$.

The enveloping metric (2.5) is subharmonic and also has curvature at most -4 in the potential sense on Δ (cf. [Kr2], [Ro]). It can be regarded as the infinitesimal form of $\mathcal{J}$.

Note that the function $\mathcal{J}$ is contionuos. Indeed, for any fixed m, the function $J_m(\varphi) - J_m(\varphi_0)$ a holomorphic map of the ball

$$\{\varphi \in \mathbf{T}: \ \|\varphi - \varphi_0\|_{\mathbf{B}} < d\}, \quad d = \operatorname{dist}(\varphi_0, \partial\mathbf{T})$$

into the disk $\{|w| < 2\}$. Hence, by Schwarz's lemma,

$$|J_m(\varphi) - J_m(\varphi_0)| \le \frac{2}{d}\|\varphi - \varphi_0\|,$$

and

$$||h_{\mathbf{x}}(\varphi)| - |h_{\mathbf{x}}(\varphi_0)|| \le |h_{\mathbf{x}}(\varphi) - h_{\mathbf{x}}(\varphi_0)| \le \frac{2}{d}\|\varphi - \varphi_0\|,$$

which easily yieds that $\mathcal{J}(\varphi)$ satisfies locally the same estimate.

Since $\mathcal{J}(\varphi)$ admits the mean value inequality property, it is a plurisubharmonic function on $\mathbf{T}$. Its resriction to any holomorphic disk $h(\Delta)$ in $\mathbf{T}$ is subharmonic. In particular, the enveloping functional $\mathcal{J}$ and its metric $\lambda_{\mathcal{J}}$ are subharmonic on Teichmüller disks

$$\Delta(\psi_0) = \{t|\psi_0|/\psi_0: \ |t| < 1\} \subset \mathbf{Belt}(\Delta)_1.$$

In addition, we have

Lemma 2.2. *For any sequence* J_m *with* $\limsup_{m\to\infty} |J_m(\varphi)| < 1$*, the limit functional*

$$\mathcal{J}(\varphi) = \limsup_{m\to\infty} |J_m(\varphi)|$$

is reconstructed on Teichmüller disks $\Delta(\psi_0)$ *from its metric* $\lambda_{\mathcal{J}}$*, defined similar to (2.5), by*

$$\tanh^{-1}[\mathcal{J}(F^{r|\psi_0|/\psi_0})] = \int\limits_0^r \lambda_{\mathcal{J}}(t)dt, \quad 0 < r < 1. \tag{2.6}$$

More special lemmas on the reconstruction of the enveloping functional by its infinitesimal metric were applied in [Kr2], [Kr4].

Proof. Fix $r \in (0, 1)$. Then for any appropriate J_m, we have the equalities

$$\tanh^{-1}[J_m(r)] = \int\limits_0^{J_m(r)} \frac{|dt|}{1 - |t|^2} = \int\limits_0^r \lambda_{J_m}(t)|dt|,$$

and, taking a monotone increasing subsequence

$$\lambda_1 = \lambda_{J_{m_1}}, \ \lambda_2 = \max(\lambda_{J_{m_1}}, \lambda_{J_{m_2}}), \ \lambda_3 = \max(\lambda_{J_{m_1}}, \lambda_{J_{m_2}}, \lambda_{J_{m_3}}), \ \dots$$

so that $\lim\limits_{p\to\infty} \lambda_p(t) = \sup_m \lambda_{J_m}(t) = \limsup\limits_{m\to\infty} \lambda_{J_m}(t)$,

$$\tanh^{-1}[\mathcal{J}(f^{r|\psi_0|/\psi_0})] = \sup_m \int\limits_0^r \lambda_{J_m}(t)|dt| = \int\limits_0^r \sup_m \lambda_{J_m}(t)|dt|. \tag{2.7}$$

For any fixed p,

$$\int\limits_0^r \lambda_p(t)|dt| < \tanh^{-1}[\mathcal{J}(f^{r|\psi_0|/\psi_0})],$$

thus (2.7) yields

$$\int\limits_0^r \lambda_{\mathcal{J}}(t)dt \le \tanh^{-1}[\mathcal{J}(f^{r|\psi_0|/\psi_0})].$$

The inverse inequality follows in a similar way. This implies the desired equality (2.6), completing the proof of the lemma.

We proceed to the proof of the theorem and select on the boundary quasicircle L a dense subset

$$e = \{z_1, z_2, \dots, \ z_m, \dots\},$$

getting a sequence of holomorphic maps

$$J_m(t) = \widetilde{J}(S_{f^{t\mu_0}}, f(z_m)) : \ \Delta \to \Delta, \quad m = 1, 2, \dots$$

with $\mu_0 = |\psi_0|/\psi_0$. Using these functions, we define $\mathcal{J}(t) = \sup_m |J_m(t)|$ and the corresponding subharmonic conformal metrics (2.5) of the curvature -4 at noncritical points.

Similarly to above, the envelope metric $\lambda_{\mathcal{J}}(t) = \sup_m \lambda_m(t)$ is subharmonic on Δ and, due to [Kr3], its curvature is at most -4 in the potential sence. Lemma 2.2 yields that $\mathcal{J}$ is the integrated form of this metric along the radii in a Teichmüller geodesic disk:

$$\tanh^{-1}[\mathcal{J}(f^{t|\psi_0|/\psi_0})] = \int\limits_0^{|t|} \lambda_{\mathcal{J}}(re^{i\theta}) dr, \quad \theta = \arg t. \tag{2.8}$$

Now we make use of circular averaging

$$\mathcal{M}u(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} u(re^{i\theta}) d\theta.$$

This mean is well defined for any measurable real-valued function u on the disk, locally bounded from above, and inherits certain important properties of its original function. For example, if u is subharmonic on Δ, then so is $\mathcal{M}u$. Moreover, in this case $\mathcal{M}u(r)$ is convex with respect to $\log r$. By Jensen's inequality, for any convex function ω on a real interval containing the values of both u and $\mathcal{M}u$, we have the inequality $\omega(\mathcal{M}u) \le \mathcal{M}\omega(u)$. We also have

Lemma 2.3 [Ro]. *Let $\lambda|dz|$ be a conformal metric on the unit disk which has curvature at most -4 in the potential sense. Then the metric $\widetilde{\lambda} = e^{\mathcal{M}u}$, where $u = \log \lambda$, also has curvature at most -4 in the potential sense.*

Letting in (2.8) $|t| = \kappa$ and averaging both sides, we get

$$\int_0^{\kappa} \mathcal{M}\lambda_{\mathcal{J}}(r)dr = \frac{1}{2\pi}\int_0^{\kappa}\int_0^{2\pi} \lambda_{\mathcal{J}}(re^{i\theta})d\theta dr = \mathcal{M}\tanh^{-1}[\mathcal{J}(f^{t|\psi_0|/\psi_0})] \geq$$

$$\geq \tanh^{-1}[\mathcal{M}\mathcal{J}(f^{t|\psi_0|/\psi_0})]\,.$$

On the other hand, by Lemmas 2.1 and 2.3,

$$\int_0^{\kappa} \mathcal{M}\lambda_{\mathcal{J}}(r)dr \geq \int_0^{\kappa} \frac{adr}{1-a^2r^2} = \tanh^{-1}(a\kappa),$$

while from (1.2) and definition of $\lambda_{\mathcal{J}}$,

$$a = \mathcal{M}\lambda_{\mathcal{J}}(0) = \lambda_{\mathcal{J}}(0) = \|J'_{\mathrm{id}}\|.$$

These relations result in (1.8). The sharpness of this bound follows from

Proposition 2.4 [Kr5]. *For any holomorphic functional J on $\Sigma^0(D^*)$ whose range domain has more than two boundary points, there exists a number $k_0(J) > 0$ such that the values of J on the ball*

$$\mathbf{Belt}(D)_k = \{\mu \in \mathbf{Belt}(D)_1 : \ \|\mu\|_\infty \leq k\}$$

for all $k \leq k_0(J)$ are located in the closed disk $\Delta(J(\mathrm{id}), M_k)$ centered at the point $J(\mathrm{id})$ and with radius

$$M_k(F) = \max_{|t|=k} |J(f^{t|\psi_0|/\psi_0}) - J(\mathrm{id})|.$$

The boundary points of this disk correspond to $\mu = t|\psi_0|/\psi_0$ with $|t| = k$.

This completes the proof of the theorem.

3. Proof of Theorem 1.2.

Let V be a domain in a complex Banach space endowed with a pseudo-distance ρ. A holomorphic map $h : \ \Delta \to V$ is called a **complex ρ-geodesic** if there exist $t_1 \neq t_2$ in Δ such that

$$d_\Delta(t_1, t_2) = \rho(h(t_1), h(t_2));$$

one says also that the points $h(t_1)$ and $h(t_2)$ can be joined by a complex ρ-geodesic (see [Ve])).

If h is a complex c_V-geodesic then it is also d_V-geodesic and the above equality holds for all points $t_1,\ t_2 \in \Delta$, so $h(\Delta)$ is a holomorphic disk in V hyperbolically isometric to Δ.

Certain sufficient conditions ensuring the existence of complex geodesics have been given in [Di], [DTV] for convex Banach domains. The main underlying properties are the equality of invariant metrics and weak* compactness. Theorem 1.1 allows us to apply the same arguments.

Recall that a Banach space X is called the **dual** of a Banach space Y if $X = Y'$, that is, X is the space of bounded linear functionals $x(y) = \langle x, y \rangle$ on Y; then Y is called the **predual** of X. The **weak* topology** $\sigma(X, Y)$ on X determined by Y is the topology of pointwise convergence on points of Y, i.e., $x_n \in X \to x \in X$ in $\sigma(X, Y)$ as $n \to \infty$ if and only if $x_n(y) \to x(y)$ for all $y \in Y$.

If X has a predual Y then, by the Alaoglu–Bourbaki theorem, the closure $\overline{X}_1$ of its open unit ball is weakly* compact.

We model $\mathbf{T}(D)$ as a bounded domain $\mathbf{D}$ in the corresponding Banach space $\mathbf{B}(D^*)$ and note that this space is dual to the space $A_1(D^*)$ of integrable holomorphic functions on D^* (satisfying $f(z) = O(z^{-4})$ as $z \to \infty$ (see, e.g. [Be1]). As was mentioned above, by the Alaoglu-Boubaki theorem *the weak*-closure of $\mathbf{D}$ in $\sigma(\mathbf{B}(\Gamma), A_1(\Gamma))$ is compact.*

Now let φ_1 and φ_2 be distinct points in $\mathbf{D}$. By Theorem 1.1,

$$d_{\mathbf{D}}(\varphi_1, \varphi_2) = c_{\mathbf{D}}(\varphi_1, \varphi_2) = \inf\{d_\Delta(h^{-1}(\varphi_1), h^{-1}(\varphi_2)) : \ h \in \mathrm{Hol}(\Delta, \mathbf{D})\};$$

hence there exist sequences $\{h_n\} \subset \mathrm{Hol}(\Delta, \mathbf{D})$ and $\{r_n\},\ 0 < r_n < 1$, such that $h_n(0) = \varphi_1$ and $h_n(r_n) = \varphi_2$ for all n, $\lim\limits_{n\to\infty} r_n = r < 1$ and $c_{\mathbf{T}}(\varphi_1, \varphi_2) = d_\Delta(0, r)$. Let $h_n(t) = \sum\limits_{m=0}^{\infty} a_{n,m} t^m$ for all $t \in \Delta$ and n.

Take a ball $B(0, R) = \{\varphi \in \mathbf{B}(\Gamma) : \ \|\varphi\| < R\}$ containing $\mathbf{D}$. For any $\varphi \in B(0, R)$, the Cauchy inequalities imply $\|a_{n,m}\|_{\mathbf{B}} \le R$ for all n and m. Passing, if needed, to a subsequence of $\{h_n\}$, one can suppose that for a fixed m, the sequence $a_{n,m}$ is weakly* convergent to $a_m \in \mathbf{B}(\Gamma)$ as $n \to \infty$, that is

$$\lim_{n\to\infty} \langle a_{n,m}, \psi \rangle_\Delta = \langle a_m, \psi \rangle_\Delta \quad \text{for any} \quad \psi \in A_1.$$

Hence $h(t) = \sum\limits_{m=0}^{\infty} a_m t^m$ defines a holomorphic function from Δ into $\mathbf{B}(\Gamma)$. Since $a_{n,0} = \varphi_1$ for all n, we have $h(0) = \varphi_1$.

Now, let $\alpha,\ 0<\alpha<1$, and $\varepsilon>0$ be given. Choose m_0 so that

$$r\sum_{m=m_0}^{\infty}\alpha^m<\varepsilon.$$

If $\psi\in A_1(\Gamma),\ \|\psi\|=1$, then

$$\sup_{|t|\le\alpha}|\langle h_n(t)-h(t),\psi\rangle_\Delta|\le\sum_{m=1}^{m_0-1}|\langle a_{n,m}-a_m,\psi\rangle_\Delta|+2r\sum_{m=m_0}^{\infty}\alpha^m$$

for all n, which implies that h_n is convergent to h in $\sigma(\mathbf{B}(\Gamma),A_1(\Gamma))$ uniformly on compact subsets of Δ^* as $n\to\infty$. Since the closure $\overline{\mathbf{D}}$ is $\sigma(\mathbf{B}(\Gamma),A_1(\Gamma))$ compact, $h(\Delta)\subset\overline{\mathbf{D}}$. Using that h is holomorphic and $h(0)\in\mathbf{D}$, one concludes that $h(\Delta)\subset\mathbf{D}$. For $r<r'<1$,

$$\omega_2=h_n(r_n)=\frac{1}{2\pi i}\int\limits_{|t|=r'}\frac{h_n(t)dt}{t-r_n}\to\frac{1}{2\pi i}\int\limits_{|t|=r'}\frac{h(t)dt}{t-r}=h(r)$$

as $n\to\infty$. Hence,

$$d_\Delta(0,r)=d_{\mathbf{D}}(\varphi_1,\varphi_2)=c_{\mathbf{D}}(h(0),h(r)),$$

and h is simultaneously complex $d_{\mathbf{D}}$ and $c_{\mathbf{D}}$ geodesics.

In a similar way, one obtains that there exists a holomorphic map $h:\Delta\to\mathbf{D}$ such that for any two points $t_1,t_2\in\Delta$,

$$d_\Delta(t_1,t_2)=d_{\mathbf{D}}(h(t_1),h(t_2))=c_{\mathbf{D}}(h(t_1),h(t_2)),$$

and that for any point $\psi\in\mathbf{T}(D)$ and any nonzero tangent vector v at this point, there exists at least one complex geodesic $h^*:\ \Delta\to\mathbf{T}(D)$ such that $h(0)=\psi$ and $h'(0)$ is colinear to v.

Since both metrics $d_{\mathbf{T}}$ and $c_{\mathbf{T}}$ of the space $\mathbf{T}(D)$ are equal to its Teichmüller metric, the above complex geodesic maps h and g define the corresponding extremal disks $h(\Delta)$ and $g(\Delta)$ in this space.

4. Additional remarks.

4.1. Univalent functions in bounded domains. In a similar way, one can consider holomorphic functionals on the class $S^0(D)$ of univalent functions in a bounded quasidisk D with expansions $F(z)=z+\sum\limits_2^\infty a_nz^n$ near the origin $z=0\in D$ with quasiconformal extensions to D^*.

Their inversions $f_F(z) = 1/F(1/z)$ are univalent and do not vanish in $D' = \{z : \ 1/z \in D\}$, and one derives from Theorem 1.1 a similar result for $S^0(D)$.

4.2. Generalization. Both theorems can be extended to more general functionals $J(f)$ whose derivative (1.4) has a finite number of simple poles in the domain D, where the maps are quasiconformal. Accordingly, instead of (1.6), one can estimate the functionals of the form

$$J(f) = J(f(a_1), f(a_2), \dots , f(a_m); \ f(z_1), f'(z_1), \dots , f^{(\alpha_1)}(z_1); \dots ;$$

$$f(z_p), f'(z_p), \dots , f^{(\alpha_p)}(z_p)) ,$$

where $a_1, a_2, \dots, \ a_m$ are distinct fixed points in D, and $z_1, z_2, \dots , z_p$ are distinct fixed points in D^* with assigned orders $\alpha_1, \alpha_2, \ \dots \ , \alpha_p$, respectively.

This involve the invariant metrics on Teichmüller spaces of the punctured quasidisks $D \setminus \{a_1, a_2, \dots, a_m\}$. The details will be presented elsewhere.

References

[Ah] Ahlfors L. *An extension of Schwarz's lemma* // Trans. Amer. Math. Soc. — 1938. — **43**. — P. 359—364.

[Be1] Bers L. *A non-standard integral equation with applications to quasiconformal mappings* // Acta Math. — 1966. — **116**. — P. 113—134.

[Be2] Bers L. *Fiber spaces over Teichmüller spaces* // Acta Math. — 1973. — **130**. — P. 89—126.

[Di] Dineen S. *The Schwarz Lemma*. — Oxford: Clarendon Press, 1989.

[DTV] Dineen S., Timoney R. M., Vigué J. P. *Pseudodistances invariantes sur les domains d'une espace localement convexe* // Ann. Scuola. Norm. Sup. Pisa Cl. Sci. — 1985. — **12**, № 4. — P. 515—529.

[GL] Gardiner F. P., Lakic N. *Quasiconformal Teichmüller Theory*. — Amer. Math. Soc., 2000.

[GR] Gutlyanskii V. Ya., Ryazanov V. I. *Geometric and Topological Theory of Functions and Mappings (Geometricheskaya i topologicheskaya teoria funkcii i otobrazhenii)*. — Kiev: Naukova Dumka, 2011 (Russian).

[He] Heins M. *A class of conformal metrics* // Nagoya Math. J. — 1962. — **21**. — P. 1—60.

[Kr1] KRUSHKAL S.L. *Quasiconformal Mappings and Riemann Surfaces.* — New York: Wiley, 1979.

[Kr2] KRUSHKAL S.L. *Strengthened Moser's conjecture, geometry of Grunsky inequalities and Fredholm eigenvalues* // Central European J. Math. — 2007. — **5**, № 3. — P. 551—580.

[Kr3] KRUSHKAL S.L. *Extremal problems for Fredholm eigenvalues* // Israel J. Math. — 2009. — **172**. — P. 279—307.

[Kr4] KRUSHKAL S.L. *Proof of the Zalcman conjecture for initial coefficients* // Georgian Math. J. — 2010. — **17**. — P. 663—681.

[Kr5] KRUSHKAL S.L. *Hyperbolic metrics on universal Teichmüller space and extremal problems* // Ukrains'kii Matematychnii Visnyk. — 2011. — **8**. — P. 557—579; J. of Math. Sciences. — 2012. — **182**, № 1. — P. 70—86.

[KK] KRUSHKAL S.L., KÜHNAU R. *Quasikonforme Abbildungen - neue Methoden und Anwendungen.* — Teubner-Texte zur Math. — **54**. — Leipzig, 1983.

[Ku1] KÜHNAU R. *Verzerrungssätze und Koeffizientenbedingungen vom Grunskyschen Typ für quasikonforme Abbildungen* // Math. Nachr. — 1971. — **48**. — P. 77—105.

[Ku2] KÜHNAU R. *Funktionalabschätzungen bei quasikonformen Abbildungen mit Fredholmschen Eigenwerten* // Comment. Math. Helv. — 1981. — **56**. — P. 297—306.

[Le] LEHTO O. *Quasiconformal mappings and singular intergrals* // Istituto Nazionale di Alta Matematica. Simposia Mathematica. — **18**. — Academic Press, 1976. — P. 429—453.

[Ro] ROYDEN H. L. *The Ahlfors-Schwarz lemma: the case of equality* // J. Analyse Math. — 1986. — **46**. — P. 261—270.

[Sc] SCHIFFER M. *A variational method for univalent quasiconformal mappings* // Duke Math. J. — 1966. — **33**. — P. 395—411.

[Sch] SCHOBER G. *Univalent Functions — Selected Topics.* — Lecture Notes in Math. — **478**. — Berlin: Springer, 1975.

[Sh] SHERETOV V. G. *Analytic Functions with Quasiconformal Extensions.* — Tver State Univ., 1991 (Russian).

[Ve] VESENTINI E. *Complex geodesics and holomorphic mappings* // Sympos. Math. — 1982. — **26**. — P. 211—230.