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An extremal decomposition of a rectangle

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Dedicated to memory of Professor Promarz M. Tamrazov

For a given rectangular electric conductor we are looking for a cutting-up under some side conditions such that the enlargement of the resistance is minimal. There is a connection with conformal mapping theory.

1. The Problem. We take a metallic plate $\mathcal{R}$ in the form of a rectangle in the complex z-plane, with a constant thickness. We consider the electric resistance in the case of the flow from one side $\mathfrak{s}_1$ of $\mathcal{R}$ to the opposite side $\mathfrak{s}_2$, assuming at the whole $\mathfrak{s}_1$ and at the whole $\mathfrak{s}_2$, resp., a constant voltage. Apart from a physical constant and the thickness of $\mathcal{R}$, this resistance is the quotient of the lengths of $\mathcal{R}$ (distance of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ divided by the length of $\mathfrak{s}_1$ and $\mathfrak{s}_2$). If we make a cut in $\mathcal{R}$ from $\mathfrak{s}_1$ to $\mathfrak{s}_2$ then in general this resistance will increase. Only in the case of a cut along a segment orthogonal to $\mathfrak{s}_1$ and $\mathfrak{s}_2$, the resistance will be unchanged.

Now we mark in the interior of $\mathfrak{s}_1$ a point z_1 and in the interior of $\mathfrak{s}_2$ a point z_2. Then we have the

Physical Problem: For which cut from z_1 to z_2 the resistance is as small as possible?

Of course, with such cuts from z_1 to z_2, it is possible to get an arbitrary great resistance.

We can transform this physical problem into a problem of conformal mapping. Then it will become obvious that the problem in the original form in general has no solution. Namely, there is no cut for which the resistance assumes the infimum of the possible values.

To state our problem more precisely, we define $\mathcal{R}$ as the rectangle

$$\mathcal{R}\ :\quad 0 \le x \le a,\ 0 \le y \le b\ \ (a > 0, b > 0) \tag{1}$$

in the complex plane $z = x + iy$, further

$$\mathfrak{s}_1 :\quad 0 \le x \le a,\ y = 0; \qquad \mathfrak{s}_2 :\quad 0 \le x \le a,\ y = b\,. \tag{2}$$

For the marked points z_1 and z_2, we can assume

$$0 < \mathfrak{Re}\ z_1\ (= z_1) < \mathfrak{Re}\ z_2 < a \quad (\text{with}\ \ \mathfrak{Im}\ z_2 = b). \tag{3}$$

For the following, c.f. Fig. 1.

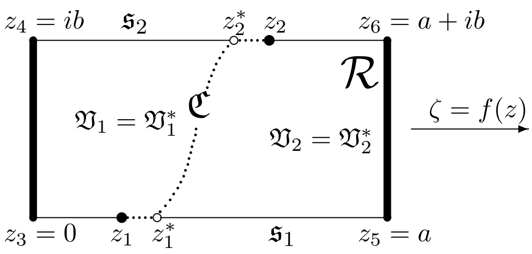

Fig.1: rectangle $\mathcal{R}$ in the z-plane with the extremal decomposition

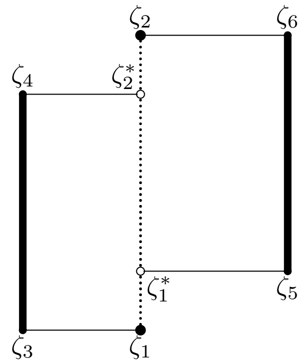

Fig.2: 8-gon in the ζ-plane

Now let $\mathfrak{V}_1$ and $\mathfrak{V}_2$ be two disjoint topological quadrilaterals (simply-connected domains with 4 marked boundary points and 2 marked opposite sides), contained in the interior of $\mathcal{R}$. Let one of the opposite sides of $\mathfrak{V}_1$ be situated on the segment $0 \le x \le \mathfrak{Re}\ z_1,\ y = 0$, the other side on $\mathfrak{s}_2$. Furthermore, let one of the opposite sides of $\mathfrak{V}_2$ be situated on the segment $\mathfrak{Re}\ z_2 \le x \le a,\ y = b$, the other side on $\mathfrak{s}_1$. The conformal module of, e.g., $\mathfrak{V}_1$ is defined by a schlicht conformal mapping of $\mathfrak{V}_1$ onto a rectangle, such that the opposite sides of $\mathfrak{V}_1$ transform onto opposite sides of the rectangle. Then the *module* $\mathfrak{V}_1$ is the length of these opposite sides divided by the length of the other sides (c.f., e.g., [Ku6]).

By Grötzsch's Principle (c.f. [Go], [Ku6]), we have

$$\textit{module}\ \mathfrak{V}_1 + \textit{module}\ \mathfrak{V}_2 \le \frac{a}{b} \tag{4}$$

in the class of all such admissible pairs $\mathfrak{V}_1, \mathfrak{V}_2$. Because of $\mathfrak{Re}\, z_1 < \mathfrak{Re}\, z_2$ (c.f. (3)), we never have equality in (4). Therefore, there arises the *problem* to determine

$$\sup \ (module\ \mathfrak{V}_1 + module\ \mathfrak{V}_2). \tag{5}$$

The connection with the physical problem of the beginning is obvious because, e.g., the *module* $\mathfrak{V}_1$ is (apart from a physical constant) the reciprocal resistance of $\mathfrak{V}_1$ (between the sides on $\mathfrak{s}_1$ and $\mathfrak{s}_2$). Of course, we can state problem (5) also as an extremal length problem for curve families in $\mathcal{R}$.

Problem (5) is slightly more general than the physical problem because we require only that, e.g., the second side of $\mathfrak{V}_1$ is situated on $\mathfrak{s}_2$ and not strongly on the segment $0 \leq x \leq \mathfrak{Re}\, z_2,\ y = b$. The proof covers also this more general configuration.

Obviously, our problem is a special case in the great field of conformal geometry of non-overlapping domains. That means extremal problems for functionals in which conformal moduli of non-overlapping domains, e.g., quadrilaterals or ring-domains are involved. The starting point for this great field was the work of H. Grötzsch, whose "strip-method" always works with non-overlapping domains. Then this direction was continued by O. Teichmüller (c.f., e.g., [Te], §4), and later in a systematic manner by U. Pirl [Pi1,2], J. A. Jenkins [Je] and K. Strebel (c.f. [St], Chapter VI). A new method was given by H. Renelt [Re]; c.f. there also additional references.

As always in this type of problems, the solution of problem (5) contains a quadratic differential, although in our case only hiddenly.

By the way, we remark also that related to this direction there is a lot of papers in which extremal problems with non-overlapping, mainly simply-connected domains were discussed; c.f., e.g., the surveys [Le], [BBZ] (preferable references in Russian and Ukrainian). In [Ku3] it was remarked that in principle extremal problems of this type can be considered (without clearing the question of equality in the obtained inequalities) also as a limit case for mappings of only one domain.

The aim of this paper is now only to show that in our special case a more concrete discussion is possible. Namely, we give an explicit formula for the solution by means of elliptic functions, and obtain a geometric property of $\mathfrak{V}_1$ and $\mathfrak{V}_2$ in the extremal case. It is useful and impressive to observe again the great strength and depth of the general results in [Pi1,2], [Je], [St], [Re] and now to discuss an example in detail and "to the end".

The following discussions are also possible in the more general case of a linear combination of *module* $\mathfrak{V}_1$ and *module* $\mathfrak{V}_2$ instead of (5), or in the more general case of more than one cuts to decompose $\mathcal{R}$. It is also possible to study the limit case in which the rectangle degenerates into a strip. But for simplicity, we will restrict ourselves to the case (5) which is simple but not too simple.

Perhaps, from the physical point of view, the Theorem 1 is in some sense surprising. But with the knowledge of Geometric Function Theory, it is clear that we will obtain a description of the solution with a quadratic differential which has a simple pole at z_1 and z_2 and further somewhere two simple zeros. Indeed, these facts are hiddenly included in Theorem 1.

2. The solution.

Theorem 1. *The problem* (5) *has exactly one solution, namely a pair of admissible quadrilaterals* $\mathfrak{V}_1^*, \mathfrak{V}_2^*$ *with*

$$\textit{module}\ \mathfrak{V}_1 + \textit{module}\ \mathfrak{V}_2 \leq \textit{module}\ \mathfrak{V}_1^* + \textit{module}\ \mathfrak{V}_2^* \tag{6}$$

for all admissible pairs $\mathfrak{V}_1$, $\mathfrak{V}_2$. *For this extremal pair we have the following properties.*

a) *There exists* (*under our general assumption* (3)) *a point* z_1^* *with*

$$\mathfrak{Im}\ z_1^* = 0,\ z_1 < \mathfrak{Re}\ z_1^* = z_1^* < a, \tag{7}$$

moreover a point z_2^* *with*

$$\mathfrak{Im}\ z_2^* = b,\ 0 < \mathfrak{Re}\ z_2^* < \mathfrak{Re}\ z_2, \tag{8}$$

and a closed analytic arc $\mathfrak{C}$ *with the endpoints* z_1^* *and* z_2^*, *contained* (*apart from these endpoints*) *in the interior of* $\mathcal{R}$, *such that one pair of the opposite sides of* $\mathfrak{V}_1^*$ *consists in*

$$\text{the segment } (0, z_1) \text{ and the segment } (ib, z_2^*)$$

while the other pair is

$$\text{the segment } (0, ib) \text{ and } \mathfrak{C} \cup \text{segment } (z_1, z_1^*).$$

The situation in the case of $\mathfrak{V}_2^*$ *is analogous; c.f. Fig. 1.*

b) *At the endpoint* z_1^* *of* $\mathfrak{C}$ *the angle between* $\mathfrak{C}$ *and the positive direction of the real axis equals* $\frac{\pi}{3}$ *while* $-\frac{2\pi}{3}$ *at the other endpoint* z_2^*.

c) If we go on $\mathfrak{C}$ *from* z_1^* *to* z_2^* *then both* $\mathfrak{Re}\, z$ *and* $\mathfrak{Im}\, z$ *are increasing, especially*

$$\mathfrak{Re}\, z_1^* < \mathfrak{Re}\, z_2^*. \tag{9}$$

d) With the Weierstraß σ*-function corresponding to the periods* $2\omega = = 2a$ *and* $2\omega' = 2ib$ *(c.f.* [TK]*), the function*

$$\zeta = f(z) = \int \sqrt{\frac{\sigma(z - z_1^*)\sigma(z + z_1^*)\sigma(z - z_2^*)\sigma(z + \overline{z_2^*})}{\sigma(z - z_1)\sigma(z + z_1)\sigma(z - z_2)\sigma(z + \overline{z_2})}}\, dz \tag{10}$$

yields a schlicht conformal mapping of $\mathcal{R}$ *onto the interior of a polygon ("8-gon", c.f. Fig.2) whose 8 sides are parallel to the real axis or to the imaginary axis, resp.. The 8 corners are the images* $\zeta_k = f(z_k)$ *of the 6 points* $z_1, z_2, z_3 = 0, z_4 = ib, z_5 = a, z_6 = a + ib$ *and the images* ζ_1^*, ζ_2^* *of the 2 points* z_1^*, z_2^**. At the points* $f(z_1^*)$ *and* $f(z_2^*)$ *the interior angle of this polygon is* $\frac{3}{2}\pi$*, while at the other corners the interior angle is* $\frac{1}{2}\pi$*. For example, the segment* $(f(z_3), f(z_1))$ *is parallel to the real axis. The corners* $f(z_1^*)$ *and* $f(z_2^*)$ *have the same real part, and the arc* $\mathfrak{C}$ *is the pre-image of the segment* $(f(z_1^*), f(z_2^*))$*. The segments* $(f(z_1), f(z_1^*))$ *and* $(f(z_2), f(z_2^*))$ *(lying on the same line as the image of* $\mathfrak{C}$*) have the same length.*

e) The last geometric description of the image of $\mathcal{R}$ *yields the following two equations for the analytic characterization of the unknown parameters* z_1^* *and* z_2^* *(always with the integral as in* (10)*)*

$$\int_{z_1}^{z_1^*} = \int_{z_2^*}^{z_2}, \tag{11}$$

$$\int_{0}^{z_1} = \int_{ib}^{z_2^*}, \tag{12}$$

while the essential arc $\mathfrak{C}$ *is characterized by*

$$\mathfrak{Re} \int_{z_1^*}^{z} = 0. \tag{13}$$

f) The maximal value (*right-hand side of* (6)) *of the extremal problem is given by*

$$\textit{module}\ \mathfrak{V}_1^* + \textit{module}\ \mathfrak{V}_2^* = \left(\int_0^{z_1} + \int_{z_1^*}^{a}\right) \Big/ \left(\frac{1}{i}\int_0^{ib}\right). \tag{14}$$

Because the integrals in (11), (12), (13), (14) represent some length of sides of the image of $\mathcal{R}$, we obviously can replace these integrals by other integrals (side length).

There is also another possibility to describe analytically the solution of our problem. Namely, after a transformation of $\mathcal{R}$ by a Weierstraß $\wp$-function onto a half-plane, we have a usual Schwarz–Christoffel transformation, again with a parameter problem; c.f. also [Pi2], § 9. The disadvantage of this procedure is that this needs two mapping steps.

3. Proof of Theorem 1. We start with the inverse situation. Namely, we set in the ζ-plane an 8-gon as in Fig. 2 and prescribed after (10) and use the conformal mapping onto a rectangle $\mathcal{R}$ in the z-plane, with the corners $0, a, a+ib, ib$ and with some $a>0$ and $b>0$ such that ζ_3 transforms onto $z_3 = 0$, and so on. First we ignore the restriction $\mathfrak{Im}\,\zeta_2 > \mathfrak{Im}\,\zeta_4$, using only $\mathfrak{Im}\,\zeta_4 - \mathfrak{Im}\,\zeta_3 = \mathfrak{Im}\,\zeta_6 - \mathfrak{Im}\,\zeta_5$.

We obtain $\mathfrak{C}$ as the image of the segment $[\zeta_1^*, \zeta_2^*]$.

With the so obtained points z_1 and z_2 (first not necessarily with $\mathfrak{Re}\,z_1 < < \mathfrak{Re}\,z_2$), the inverse mapping $f(z)$ has the extremal property (6). To confirm this, in our case we only have to apply Grötzsch's strip method in the usual manner. That means we only have to use Grötzsch's fundamental inequality (c.f. [Ku6], p. 104) for both the obtained quadrilaterals $\mathfrak{V}_1^*$ and $\mathfrak{V}_2^*$ and then to add both inequalities.

Now the problem is: Is it possible to choose the 8-gon such that we obtain the rectangle $\mathcal{R}$ with points z_1 and z_2 (as images of ζ_1 and ζ_2) at the "correct places"? Observe that we have at the 8-gon and also at $\mathcal{R}, z_1, z_2$ essentially three real parameters. The solution of this existence problem follows as in [Pi1,2] by Koebe's classical method of continuity (the necessary uniqueness follows from the extremal property), in [Je] by a variational method, or in [Re] by means of Dirichlet integrals.

It is also possible to transform our problem for quadrilaterals into a problem for ring-domains. For this reason, without loss of generality we can choose $b = \pi$, use reflection at the real axis and then transform by the complex logarithm.

After this existence proof we obtain

$$\mathfrak{Re}\,z_1 < \mathfrak{Re}\,z_1^*,\ \mathfrak{Re}\,z_2^* < \mathfrak{Re}\,z_2 \quad \text{if} \quad \mathfrak{Im}\,\zeta_2 > \mathfrak{Im}\,\zeta_4.$$

Of course, in the other case $\mathfrak{Im}\,\zeta_2 < \mathfrak{Im}\,\zeta_4$ the situation is analogous.

Now we will derive in the case $\mathfrak{Im}\,\zeta_2 > \mathfrak{Im}\,\zeta_4$ the property c).

We use the fact that at the boundary of $\mathcal{R}$ the function $f'^2(z)$ has only real values, with the exception of the points z_1^*, z_2^*, z_1, z_2 where we have the expansions (15), (16), (17), (18) below. Namely, the boundary of $\mathcal{R}$ consists of segments between the critical points $z_3 = 0, z_1, z_1^*$, $z_5 = a, z_6 = a + ib, z_2, z_2^*, z_4 = ib$. At all open segments, the function $f'(z)$ is non-vanishing and real, with the exception of the segments $(z_1,\ z_1^*)$ and (z_2, z_2^*) where $f'(z)$ is non-vanishing and imaginary. This means we have $f'^2 > 0$, resp. $f'^2 < 0$.

Now we have to study the behavior at the mentioned critical points.

At the corners z_3, z_5, z_6, z_4 of $\mathcal{R}$, the function $f'(z)$ is analytic and $f'(z) > 0$.

At the point z_1^*, the function $(f(z) - \zeta_1^*)^{2/3}$ is analytic with a positive derivative:

$$(f(z) - \zeta_1^*)^{2/3} = A_1(z - z_1^*) + \dots,\quad A_1 > 0.$$

From here it follows that also $f'^2(z)$ is analytic with a zero of first order:

$$f'^2(z) = \frac{9}{4} A_1^3 (z - z_1^*) + \dots. \tag{15}$$

The same holds at z_2^*:

$$f'^2(z) = -\frac{9}{4} A_2^3 (z - z_2^*) + \dots,\ A_2 > 0, \tag{16}$$

with some constants $A_1,\ A_2$.

In the same manner, we obtain for the function $f'^2(z)$ a simple pole at z_1 and z_2:

$$f'^2(z) = -\frac{B_1}{4} \frac{1}{z - z_1} + \text{analytic function},\ B_1 > 0, \tag{17}$$

$$f'^2(z) = \frac{B_2}{4} \frac{1}{z - z_2} + \text{analytic function},\ B_2 > 0, \tag{18}$$

with some constants $B_1,\ B_2$.

The expansions (15), (16), (17), (18) guarantee that the values of the function $f'^2(z)$ in a neighborhood of the critical points are lying in the upper half-plane. Therefore, *all* values of $f'^2(z)$ in the (open) rectangle $\mathcal{R}$ must lie in the upper half-plane. This follows by the maximum principle, applied for the harmonic function $\mathfrak{Im}\, f'^2(z)$ in $\mathcal{R}$ after deleting small half-discs with centers at z_1 resp. z_2.

From $\mathfrak{Im}\, f'^2(z) > 0$ for $z \in \mathcal{R}$, now it follows

$$\mathfrak{Re}\, f'(z) > 0, \ \ \mathfrak{Im}\, f'(z) > 0 \text{ for all } \ z \in \mathcal{R}. \tag{19}$$

The other case $\mathfrak{Re}\, f'(z) < 0$, $\mathfrak{Im}\, f'(z) < 0$ is impossible what we can see, e.g., for the points in a neighborhood of z_1^* (c.f. the development (15)). Now (19) yields at $\mathfrak{C}$ the orientation of dz as asserts c). From c) it follows also $\mathfrak{Re}\, z_1 < \mathfrak{Re}\, z_2$. (Of course, in the other case $\mathfrak{Im}\, \zeta_2 < \mathfrak{Im}\, \zeta_4$ the situation is analogous, especially then $\mathfrak{Re}\, z_1 > \mathfrak{Re}\, z_2$.)

In the opposite direction, it follows now $\mathfrak{Im}\, \zeta_2 > \mathfrak{Im}\, \zeta_4$, that means $\mathfrak{Re}\, z_1 < \mathfrak{Re}\, z_1^* < \mathfrak{Re}\, z_2^* < \mathfrak{Re}\, z_2$ if $\mathfrak{Re}_1 < \mathfrak{Re}\, z_2$.

The assertion b) of the Theorem is now obvious by conformal mapping theory.

For the proof of the analytic representation (10) in the assertion d) we use again the fact that $f'^2(z)$ is real at the boundary of $\mathcal{R}$. Using analytic continuation by reflection, we see that $f'^2(z)$ is an elliptic function with the periods $2a$ and $2b$, with simple poles at $z_1, z_2, -\overline{z_1}, -\overline{z_2}$ (mod periods), and simple zeroes at $z_1^*, z_2^*, -\overline{z_1^*}, -\overline{z_2^*}$ (mod periods). Form here it follows (c.f. [TK], p.292), apart from a positive multiplicative constant, the representation for $f'^2(z)$, which is equivalent to (10). This multiplicative constant is positive because, e.g., at $z = 0$ the function $f'^2(z)$ is positive. Without loss of generality, we can choose this constant as 1.

This procedure to obtain the analytic expression for $f(z)$ is the same as in [Ku2] (p. 101); c.f. also [DT] (p. 49).

4. A surprising phenomenon. Here we will show that in Theorem 1 the arc $\mathfrak{C}$ always has an orthogonal projection at the basic side $[0, a]$ of the rectangle $\mathcal{R}$ which is in some sense always strongly smaller than this basic side. We demonstrate this in the simplest

$$\text{symmetric case} \quad z_1 - z_3 = z_6 - z_2 \tag{20}$$

(c.f. the notations in Fig. 1). In Fig. 2 then we have also a centrally symmetric situation.

In what follows, without loss of generality we can assume $a = 1$.

Theorem 2. *In this symmetric case with $a = 1$, there is a universal positive constant c (independent of the hight b of the rectangle $\mathcal{R}$ and independent of the prescribed points z_1 and z_2) such that $\mathfrak{C}$ always lies in the strip $c < \mathfrak{Re}\ z < a - c$ (or equivalently: it holds $z_1^* > c$, $z_2^* - ib = a - z_1^* < a - c$).*

The determination of the greatest possible value c (= a "Weltkonstante" in the sense of Edmund Landau) is a great desideratum. This needs a subtle discussion of the corresponding Schwarz-Christoffel formula for the 8-gon of Fig.2. However by the first step of the following proof it is enough to study the much simpler limit case (a 6-gon) of Fig.4 (c.f. similar discussions in [KS], p. 234).

In the *Proof of Theorem* 2 we make a discussion by the free parameters in the ζ-plane. The proof is based on the comparison of the conformal module of quadrilaterals, in several steps using Grötzsch's principle (c.f. [Go]) which goes back to Paul Koebe (c.f. an essential footnote in [Gr], p. 62).

(i) *First step* of the Proof. Here we show that it is enough to prove the Theorem 2 for the *limit case* with $z_1 \to z_3,\ z_2 \to z_6$. Without loss of generality, in this case we can assume

$$z_3 = z_1 = 0,\ z_2 = z_6 = 1 + ib \quad \text{and} \quad \zeta_1^* = 0,\ \zeta_5 = 1,\ \zeta_4 = -1 + ih, \tag{21}$$

$$\zeta_2^* = ih \ \text{ with some } h > 0,\ \ \zeta_1 = \zeta_3 = -i\infty,\ \zeta_2 = \zeta_6 = +i\infty.$$

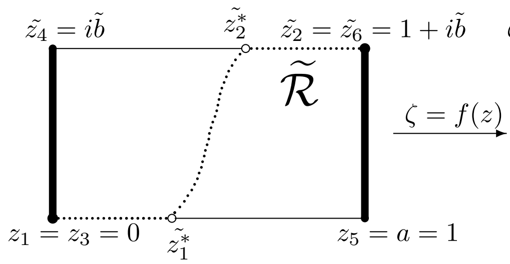

Fig.3: rectangle $\widetilde{\mathcal{R}}$ in the z-plane with the extremal decomposition, in the limit case $z_1 = z_3,\ \tilde{z}_2 = \tilde{z}_6$ $(z_1 = \tilde{z}_1, z_3 = \tilde{z}_3, z_5 = \tilde{z}_5\ )$

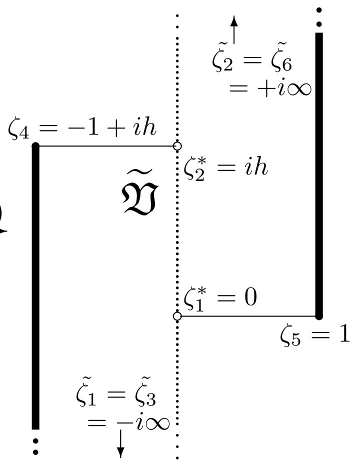

Fig.4: limit case in the ζ-plane (an unbounded 6-gon)

Now we have only the essential parameter b resp. h (= a function of b, and vice versa).

The reduction of the general case (Fig. 1 and 2) to this limit case runs as follows. We have to prove that Theorem 2 is true if it is always true for the limit case.

We start with an 8-gon as in the ζ-plane of Fig. 2 (but now more special with central symmetry analogous as in (20)), with the corresponding $\mathcal{R}$ as in Fig.1 (also with central symmetry). Beside $a = 1$ we can assume $\zeta_1^* = 0$, $\zeta_5 = 1$. That means we have with fixed values $h > 0$, $H > 0$ $(h < H)$ the corners (in the following order) $0\ (= \zeta_1^*)$, $1\ (= \zeta_5)$, $1 + iH\ (= \zeta_6$, with some $H > 0)$, $iH\ (= \zeta_2)$, $ih\ (= \zeta_2^*)$, $-1 + ih\ (= \zeta_4)$, $-1 - i(H - h)\ (= \zeta_3)$, $-i(H - h)\ (= \zeta_1)$. We consider this 8-gon as a quadrilateral $\mathfrak{V}$ with the opposite sides (both consisting in 3 segments)

$$[\zeta_4, \zeta_2^*] \cup [\zeta_2^*, \zeta_2] \cup [\zeta_2, \zeta_6] \quad \text{and} \quad [\zeta_3, \zeta_1] \cup [\zeta_1, \zeta_1^*] \cup [\zeta_1^*, \zeta_5].$$

Further, we create from $\mathfrak{V}$ an unbounded (also centrally symmetric) 6-gon $\tilde{\mathfrak{V}}$ (of limit case type, c.f. Fig. 4) by shifting the segment $[\zeta_2, \zeta_6]$ to the point $i\infty$ (in Fig. 4 denoted by $\tilde{\zeta_2} = \tilde{\zeta_6}$) in the direction of the positively imaginary axis and by shifting the segment $[\zeta_3, \zeta_1]$ to the point $-i\infty$ (in Fig. 4 denoted by $\tilde{\zeta_1} = \tilde{\zeta_3}$) in the direction of the negatively imaginary axis (the points $\zeta_4, \zeta_5, \zeta_1^*, \zeta_2^*$ unchanged). We consider this $\tilde{\mathfrak{V}}(\supset \mathfrak{V})$ as a quadrilateral with the opposite sides $[\zeta_4, \zeta_2^*] \cup [\zeta_2^*, +i\infty]$ (the latter ray containing the original ζ_2) and $[-i\infty, \zeta_1^*] \cup [\zeta_1^*, \zeta_5]$.

Evidently, we have *module* $\mathfrak{V} >$ *module* $\tilde{\mathfrak{V}}$. This means $\tilde{b} > b$ for the corresponding rectangles $\mathcal{R}$ resp. $\tilde{\mathcal{R}}$ (c.f. Fig. 3) with corners $0, 1,\ ib,\ 1+ib$ resp. $0,\ 1,\ i\tilde{b},\ 1 + i\tilde{b}$ (the usual boundary correspondence at the corner, as above).

Furthermore, for the images z_2^* resp. $\tilde{z_2^*}$ of ζ_2^* (with $\mathfrak{Re}\ z_2^* > \frac{1}{2}$, $\mathfrak{Re}\ \tilde{z_2^*} > \frac{1}{2}$ by (9)) we have at the boundary of $\mathcal{R}$ resp. $\tilde{\mathcal{R}}$

$$\mathfrak{Re}\ z_2^* < \mathfrak{Re}\ \tilde{z_2^*} \tag{22}$$

(an analogous inequality for the centrally symmetric points). For otherwise, we additionally consider the image of the 8-gon $\mathfrak{V}$ (as a part of the 6-gon $\tilde{\mathfrak{V}}$) by the mapping of the *whole* 6-gon onto a part of $\tilde{\mathcal{R}}$ and observe the module of this part as a quadrilateral with the opposite sides $[i\tilde{b}, \tilde{z_2^*}]$ and $[\tilde{z_1^*}, 1]$. If (22) not were true then we obtain a contradiction by comparing with the (equal) module of the by the similarity

$$\zeta \quad \longrightarrow \quad \frac{\tilde{b}}{b}\left(\zeta - \frac{1}{2} - i\frac{b}{2}\right) + \frac{1}{2} + i\frac{\tilde{b}}{2}$$

enlarged $\mathcal{R}$ (with analogously opposite sides).

The inequality (22) was the aim of this first part (i) of the proof.

(ii) In the *second step* of the proof we demonstrate the assertion of Theorem 2 in the limit case (Fig. 3 and 4) for all $\tilde{\mathcal{R}}$ *under the restriction*

$$\tilde{b} \geq c_1 > 0, \tag{23}$$

where c_1 is an arbitrary given positive constant. In the first place, it then always follows $h \geq c_2$ (= a positive constant, depending only on c_1), because $\tilde{b}$ is also the module of the 6-gon in the ζ-plane (Fig. 4) with the opposite sides $[1, i\infty]$ and $[-1+ih, -i\infty]$ (two rays parallel to the imaginary axis), and this module obviously (by a comparison of modules) is a monotonic function of h. (It is possible to calculate a concrete c_2, because the mentioned module of the 6-gon is smaller than the module of the strip $-1 < \mathfrak{Re}\,\zeta < 1$ slitted along the rays $[ih, +i\infty]$ and $[0, -i\infty]$, with the lines $\mathfrak{Re}\,\zeta = -1$ and $\mathfrak{Re}\,\zeta = 1$ as opposite sides. And the calculation of the module of this slitted strip is possible by considering the fourth part, namely the half-strip defined by $0 < \mathfrak{Re}\,\zeta < 1$, $\mathfrak{Im}\,\zeta > h/2$ and conformal mapping onto a half-plane).

By $h \geq c_2$ now it follows that there is also an analogous estimation of the module of the 6-gon of Fig. 4, but now with the opposite sides $[ih, +i\infty]$ and $[-1+ih, -i\infty]$ (two rays parallel to the imaginary axis). Namely, this module is also a monotonic function of h. Therefore we get again an estimation of this module, of the form $\geq c_3$ with a positive constant c_3 (again only depending on c_1). This means the same inequality for the module of $\tilde{\mathcal{R}}$, but now with the opposite sides $[0, i\tilde{b}]$ and $[\tilde{z}_2^*, 1+i\tilde{b}]$. All the more, this inequality $\geq c_3$ follows for the (greater) module of the half-strip $\mathfrak{Im}\,z < \tilde{b}$, $0 < \mathfrak{Re}\,z < 1$ as a quadrilateral with the opposite sides $[i\tilde{b}, -i\infty]$ and $[\tilde{z}_2^*, 1+i\tilde{b}]$. Because this latter module is a monotonic function of the distance between $\tilde{z}_2^*$ and $1+i\tilde{b}$, this distance is also always greater then a positive constant (only depending on c_1). This second step of our proof is completed.

(iii) In the last *third step* of the proof it is enough to show the correctness of the assertion of Theorem 2 in the limit case, for all $\tilde{\mathcal{R}}$ (c.f. again Fig. 3 and 4) and for sufficiently small $\tilde{b}$ (corresponding to sufficiently small h), that means under the restriction

$$\tilde{b} \leq c^* \tag{24}$$

with some constant $c^* > 0$ which is fixed and sufficiently small.

We start with a situation which in some sense is (after a similarity) the limit case $\tilde{b} \to 0$ of Fig. 3 and 4. Namely, we consider the schlicht conformal mapping $\check{\zeta} = \check{\zeta}(\check{z})$ of

the strip $0 < \mathfrak{Im}\ \check{z} < 1$ onto the unbounded 4-gon

$$\left\{-\pi < \arg(\check{\zeta} - i) < \frac{\pi}{2}\right\} \cap \left\{0 < \arg \check{\zeta} < \frac{3\pi}{2}\right\}, \tag{25}$$

under the side conditions

$$\check{z} = \frac{i}{2} \text{ onto } \check{\zeta} = \frac{i}{2}, \quad \check{z} = +\infty \text{ onto } \check{\zeta} = +\infty, \quad \check{z} = -\infty \text{ onto } \check{\zeta} = -\infty \tag{26}$$

(c.f. Fig. 5 and 6). Then an (centrally symmetric) arc $\check{\mathfrak{C}}$ in the strip corresponds to the segment $[0, i]$ in the $\check{\zeta}$-plane. Both domains in (25) are centrally symmetric with respect to the point $\frac{i}{2}$. Also this mapping $\check{\zeta} = \check{\zeta}(\check{z})$ is centrally symmetric with respect to the point $\frac{i}{2}$.

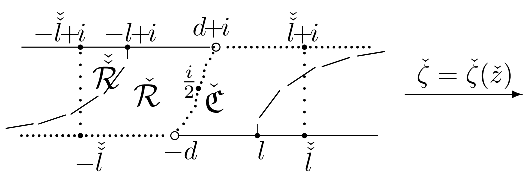

Fig.5: $\check{z}$-plane and $\check{\check{z}}$ -plane

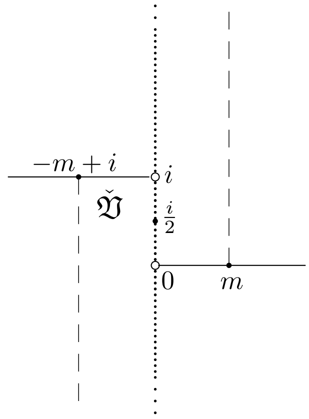

Fig.6: 4-gon in the $\check{\zeta}$-plane

We have for this limit case, as in Theorem 1.c (c.f. the proof by (19)): If we run on the arc $\check{\mathfrak{C}}$ from the endpoint at $\mathfrak{Im}\ \check{z} = 0$ to the endpoint at $\mathfrak{Im}\ \check{z} = 1$ then both $\mathfrak{Re}\ \check{z}$ and $\mathfrak{Im}\ \check{z}$ are monotonically increasing. The same also holds for the pre-images of the rays with $\mathfrak{Re}\ \check{\zeta} = \text{const}$.

Now we take into account that part $\check{\mathfrak{V}}$ of our 4-gon which is lying in the strip $-m < \mathfrak{Re}\ \check{\zeta} < m$ with a $m > 0$. Let $\check{\mathcal{R}}$ be the pre-image of $\check{\mathfrak{V}}$ in the $\check{z}$ -plane. Its boundary consists in two segments on $\mathfrak{Im}\ \check{z} = 0$ and $\mathfrak{Im}\ \check{z} = 1$, and further on an unbounded curve starting at a point l with $\check{\zeta}(l) = m$ and running to $\check{z} = \infty$, and the centrally symmetric curve starting at $-l + i$ (c.f. the two broken curves in Fig. 5).

Of course, our goal is now to pas over from $\check{\mathcal{R}}$ to a $\tilde{\mathcal{R}}$ of the form of Fig. 3. For this reason, we first take the conformal mapping $\check{\check{z}} = \check{\check{z}}(\check{z})$ of $\check{\mathcal{R}}$ (as a quadrilateral) onto a rectangle $\check{\check{\mathcal{R}}}$ with corners $\check{\check{l}} = \check{\check{z}}(l)$, $-\check{\check{l}} = \check{\check{z}}(-\infty)$, $\check{\check{l}} + i = \check{\check{z}}(i + \infty)$, $-\check{\check{l}} + i = \check{\check{z}}(-l + i)$, with some constant $\check{\check{l}} > 0$ depending on l.

We have the following essential inequality

$$l < \check{\check{l}} < l + f \tag{27}$$

with some universal constant $f > 0$ (that means f is independent of l resp. m). The left-hand side of this inequality (27) is obvious using a module comparison. The right-hand side of (27) means an inequality for the module $2\check{\check{l}}$ of the quadrilateral $\check{\check{\mathcal{R}}}$ with the boundary parts (two rays) on $\mathfrak{Im}\ \check{z} = 0$ and $\mathfrak{Im}\ \check{z} = 1$ as the opposite sides. To get this estimation, we use a formula for the extremal length of an one-parameter family of curves, given in [Ku1] (c.f. also [Ku6], p. 109). Denoting by $2l(t)$ the length of the greatest segment in $\check{\mathcal{R}}$ parallel to the real axis and lying on $\mathfrak{Im}\ \check{z} = t$, $0 < t < 1$, we obtain

$$\frac{1}{2\check{\check{l}}} \geq \int_0^1 \frac{dt}{2l(t)} = \int_{1/2}^1 \frac{dt}{l + \frac{1}{2}L_l(t)}\ . \tag{28}$$

Here we write $2l(t) = 2l + L_l(t)$ with a new function $L_l(t) \geq 0$, also depending on l.

For the following we have to take into account that $L_l(t) \to +\infty$ if $t \to +1$ and if $t \to -1$ whereby the following integrals are always convergent. To obtain this convergence we see after analytic continuation by reflection that the function $\exp\left\{\frac{\pi}{2}\check{z}(\check{\zeta})\right\}$ has at $\check{\zeta} = \infty$ a simple pole. This yields the development

$$\check{z} = \frac{2}{\pi}\log\check{\zeta} + \frac{2}{\pi}\log\lambda + \frac{A}{\check{\zeta}} + \ldots \quad \text{with some } \lambda > 0,\quad A > 0,$$

and further coefficients. If we now put $\check{\zeta} = m + i\tau$ with a constant $m > 0$ and a parameter $\tau > 0$ then we get

$$\check{z} = \frac{2}{\pi}\log\tau + \frac{2}{\pi}\log\lambda + i + \frac{2p}{\pi i\tau} + \frac{A}{m + i\tau} + \ldots,$$

from here by eliminating of τ the development of $\mathfrak{Re}\ \check{z}$ as a function of $\mathfrak{Im}\ \check{z}$. Because here always the essential term is $\log\tau$, we see the convergence of the following integrals and the possibility to get universal estimates.

It follows by (28)

$$\frac{1}{2\check{\check{l}}} > \int_{1/2}^{1} \frac{1}{l}\left(1 - \frac{1}{2}\frac{L_l(t)}{l}\right) dt =$$

$$= \frac{1}{2}\int_{0}^{1} \frac{1}{l}\left(1 - \frac{1}{2}\frac{L_l(t)}{l}\right) dt = \frac{1}{2l}\left(1 - \frac{1}{2l}\int_{0}^{1} L_l(t)dt\right) dt,$$

$$\check{\check{l}} < l\left(1 + \frac{1}{l}\int_{0}^{1} L_l(t)dt\right) = l + 2\int_{0}^{1} L_l^*(t)dt \quad \text{(for large } l\text{)}, \tag{29}$$

where now $L_l^*(t)$ is the length of that part of $l(t)$ which is lying in the half-plane $\mathfrak{Re}\ \check{z} \geq l$. Because of the foregoing, universal estimates (independent of l) for the integrals in (29) are possible and we can obtain a constant f for the inequality (27).

For the point $d + i$ with $\check{\zeta}(d + i) = i$ (c.f. Fig. 5 and 6), we need in addition the point $\check{\check{d}} = \check{\check{z}}(d + i)$ because our aim is an estimate of the distance $\check{\check{l}} - \check{\check{d}}$, more precisely an inequality of the form

$$\frac{\check{\check{l}} - \check{\check{d}}}{\check{\check{l}}} > \quad \text{const} \quad > 0 \tag{30}$$

for sufficiently great $\check{\check{l}}$ (c.f. a similarity of $\mathcal{R}$ such that the basic side of length $2\check{\check{l}}$ transforms onto the length 1, as with $\tilde{R}$ in Fig. 3). To this end we denote by M the conformal module of $\check{\check{R}}$ as a quadrilateral with the opposite sides $[-\check{\check{l}}, \check{\check{l}}]$ and $[\check{\check{d}}+i, \check{\check{l}}+i]$. Observing the pre-image $\check{\mathcal{R}}$ (containing the rectangle $d < \mathfrak{Re}\ \check{z} < l,\ 0 < \mathfrak{Im}\ \check{z} < 1$) we obtain

$$M > l - d\,. \tag{31}$$

(It is true that $\check{\check{d}}$ depends on p but not on d). On the other side, we have $M < M'$ where M' is the module of the half-strip $\mathfrak{Re}\ \check{z} < \check{\check{l}},\ \ 0 < \mathfrak{Im}\ \check{z} < 1$, considered as a quadrilateral with the opposite sides $[-\infty, \check{\check{l}}]$ (ray on the real axis) and $[\check{\check{d}} + i, \check{\check{l}} + i]$.

To avoid the calculation of the exact value of M' with elliptic integrals, we estimate

$$M' < \frac{\pi/2}{\mathfrak{ArSin}\left(1/(\check{\check{l}} - \check{\check{d}})\right)}\,. \tag{32}$$

To prove (32) we observe after a similarity the module of the half-strip $0 < \mathfrak{Re}\,\eta, 0 < \mathfrak{Im}\,\eta < H\ (H > 0)$ in a η-plane as a quadrilateral with the opposite sides $[0,1]$ and $[iH,+\infty]$ (a ray) with $H = 1/(\check{\check{l}} - \check{\check{d}})$. Transforming this half-strip by $\arcsin\eta$ the image contains the rectangle $0 < \mathfrak{Re}\,\arcsin\eta < \frac{\pi}{2}$, $0 < \mathfrak{Im}\,\arcsin\eta < \mathfrak{ArSin}\,H$. This yields (32).

Bringing now all together we get

$$\frac{\check{\check{l}} - \check{\check{d}}}{\check{\check{l}}} > \frac{1}{\check{\check{l}}\;\mathfrak{Sin}\,\frac{\pi/2}{\check{\check{l}}-f-d}}\,. \tag{33}$$

Because here on the right-hand side the limit for $\check{\check{l}} \to \infty$ is $2/\pi \neq 0$, (30) is proven. This was the aim of the third and last step (iii) of the proof.

5. Remarks.

1. Because of the conformal invariance of the conformal module we can restrict ourselves in Theorem 1 to the case of a rectangle $\mathcal{R}$. That means we obtain mutatis mutandis an analogous result for a general quadrilateral instead of a rectangle $\mathcal{R}$.

2. Our problem (5) can be interpreted as a special case or limit case of the general problem of the reduction of the module of a rectangle by an inscribed obstacle. A qualitative solution was already given in [Gr] (Hilfssatz 3a). This is important in the application of Grötzsch's strip method.

3. Of course, after Theorem 1 and 2 there arises the question for further discussions of the geometric properties of the for our solution essential curve $\mathfrak{C}$. For example, we have the

Problem: Is it true that z_1^* is monotonically increasing if z_1 is monotonically increasing?

4. Discussions as in Theorem 1 and 2, of the extremal cutting $\mathfrak{C}$ are also possible in related and in more general situations. We remark here only the following configuration which is closely related to the Problem of Section 1.

Let a concentric circular ring $\mathcal{R}$ be given and at the two boundary circles two points z_1 and z_2. We consider all cuttings from z_1 to z_2 and ask for extremal cuttings. Of course, here we have additionally to fix a homotopy class for the cuttings, because here a "winding" in $\mathcal{R}$ is possible.

Instead of the circular ring $\mathcal{R}$, it is also possible to consider the conformally equivalent situation on a circular cylinder, between two circles.

5. We can generalize the problem of Section 1 in the following manner. Now let $\mathcal{R}$ be not necessarily homogeneous. That means, the conductivity is not necessarily a constant but depends on the point z in $\mathcal{R}$ (we can think,

e.g., of a metallic plate with a non-constant thickness). Then by [Ku5] we have to take into account more generally a so-called p-module where p is a real function of z. That means, we have to replace the conformal mappings of $\mathcal{R}, \mathfrak{V}_1, \mathfrak{V}_2$ by mappings $\zeta = u(x,y) + iv(x,y)$ which are solutions of the elliptic system

$$u_x = \frac{1}{p}\, v_y,\ u_y = -\frac{1}{p}\, v_x. \tag{34}$$

Without loss of generality, we can assume $p \geq 1$ for the fixed function $p = p(z)$. We obtain the solution of the physical problem and of the corresponding mathematical problem (5) (now with the p-module) similarly as in Section 2 and 3 if we can solve the corresponding existence problem. The latter remains as a great desideratum. (The inverse problem corresponding to the beginning of Section 3 is simple — this means only the solution of a Beltrami system.) Only the proof of the extremal property is exactly the same as in the conformal case, now using [Ku4,5], because Grötzsch's fundamental inequality remains true for solutions of (34), after a suitable new formulation — [Ku4] (c.f. there (3)).

The situation becomes much simpler if we weaken the side condition that the cut $\mathfrak{C}$ has two fixed endpoints z_1 and z_2. If we fix, e.g., only z_1 as one endpoint while z_2 is free on $\mathfrak{s}_2$, then we obtain the cut $\mathfrak{C}$ with the smallest enlargement of the resistance if we transform $\mathcal{R}$ onto a rectangle (beside the usual side conditions) by a solution of the system (34). Then the extremal cut is the pre-image of the vertical segment starting at $\zeta(z_1)$. Of course, for a function $p(z)$ which is a constant, this extremal cut $\mathfrak{C}$ is a vertical segment (c.f. the beginning of Section 1).

6. It is also possible to consider an analogue of problem (5) in space. Here we can state again a problem like (5), now with partitioning a resistor in space by disjoint resistors. Again we can attack the extremal property of a "suitable" partitioning, because in space the resistance can characterized by the space form extremal length introduced by J. Hersch [He1,2]. (This extremal length definition is different from the extremal length definition used in quasiconformal mapping theory in space; c.f. [Ku6], p. 114.) But the existence of the corresponding extremal partitioning is now a great desideratum. Namely, the method, e.g., in [Pi1,2] fails because of the use of conformal mapping theory. Perhaps here the method in [Re] could be successful.

It is also difficult to give a non-trivial spatial example. We can present here only the following. Let be given in the x, y, t- space the resistor $0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq t \leq c$ with the flow from the boundary rectangle

$0 \le x \le a,\ 0 \le t \le c,\ y = 0$ to the rectangle $0 \le x \le a,\ 0 \le t \le c,\ y = b$. We divide this resistor by a surface whose boundary consists in the segments $x = x_1,\ y = 0,\ 0 \le t \le c$ and $x = x_2,\ y = b,\ 0 \le t \le c$ with fixed x_1 and x_2, and two arcs $\mathfrak{C}_1$ and $\mathfrak{C}_2$ in the planes $t = 0$ and $t = c$ joining the points $x = x_1,\ y = 0,\ \ t = 0$ and $x = x_2,\ y = b,\ t = 0$ resp. $x = x_1,\ y = 0,\ t = c$ and $x = x_2,\ y = b,\ t = c$. Let be $\mathfrak{C}_1$ the orthogonal projection of $\mathfrak{C}_2$ at the plane $t = 0$. We have again the question: For which cutting-up of the resistor, the enlargement of the resistance is minimal? If we take into account only such cutting-up whose complete projection at the plane $t = 0$ is also $\mathfrak{C}_1$ then the solution is obvious by Theorem 1. Otherwise, for a more general cutting-up we have to use [He1,2].

Furthermore, for this space problem it should be mentioned that an additional complication can appear because we have to take into account the more difficult topological situation. That means there is the possibility to prescribe in the partitioning a knotting of the resistors.

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