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On some analogues of the Hilbert formulas on the unit sphere for solenoidal and irrotational vector fields

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Dedicated to memory of Professor Promarz M. Tamrazov

In one complex variable, given a function holomorphic in the unit disc or in the upper half-plane the Hilbert formulas show the relation between the real components of its boundary value; in other words, the relation between the boundary values of a pair of conjugate harmonic functions. The main goal of this work is to establish some analogues of those formulas on the unit sphere for solenoidal and irrotational vector fields. Our formulas relate one of the real components of the boundary vector field of a solenoidal and irrotational vector field in the unit ball, with the rest of real components. Such formulas have been obtained using an intimate relation between solenoidal and irrotational vector fields and quaternionic analysis for the Moisil–Teodoresco operator; thus, the Hilbert formulas for the latter are obtained as well which relate a pair of real components of the boundary value of a quaternionic hyperholomorphic function in the unit ball with the other pair of real components.

1. Introduction. In one complex variable, given a function holomorphic in the unit disk or in the upper half-plane what is called usually the Hilbert formulas is the relation between the real components of the boundary value of the holomorphic function; in other words, it is a relation between the boundary values of a pair of conjugate harmonic functions. An encyclopedic source of information about them is the book [1].

The operator arising in these formulas bears the name of Hilbert operator, and it is a well-known transformation in mathematics and in signal processing; for example, in geophysics and astrophysics it deals with input signals. Examples of this type of signals are seismic, satellite and gravitational data; and the Hilbert operator proves to be useful for a local analysis of them, providing a set of rotation-invariant local properties: the local amplitude, local orientation and local phase, see, e.g., [2] and [3].

Since it turns out that in case of the half-plane the Hilbert operator coincides, up to a constant factor, with the singular Cauchy transform and in case of the unit disk it is tightly related with the latter, then the multidimensional generalizations of the classical Hilbert operator go mostly in the direction of generalizing the singular Cauchy transform, not the formulas themselves.

The main goal of our work is to establish some analogues of the Hilbert formulas on the unit sphere for solenoidal and irrotational vector fields in the unit ball. Our formulas relate one of the real components of the boundary vector field of a solenoidal and irrotational vector field in the unit ball, with the rest of real components; thus, their structure is deeply similar to that of their one-dimensional antecedents.

Such formulas have been obtained using an intimate relation between solenoidal and irrotational vector fields and quaternionic analysis for the Moisil–Theodoresco operator; thus, the Hilbert formulas for the latter have been obtained as well which relate a pair of real components of the boundary value of a quaternionic hyperholomorphic function in the unit ball with the other pair of real components.

Notice that certain analogue of the Hilbert formulas for the Moisil–Theodoresco system can be found in [4], pp. 186 — 191, where both the methods and the results are drastically different.

Although we do not introduce it explicitly but, of course, our formulas give rise to the generalizations of the Hilbert operator into the contexts of both considered theories. In this sense, our works fit into the research of other generalizations of the Hilbert operator. In the setting of differential forms one can find such generalizations in [5] and [6]. Many papers on the topic have been published in the framework of Clifford analysis which seems to be especially well-suited for a treatment of multidimensional phenomena encompassing all dimensions at once as an intrinsic feature, see, e.g., [7 — 15]. Directional Hilbert operators are considered in [16, 17]. For α-hyperholomorphic quaternionic function theory the Hilbert formulas and the Hilbert operator for a half-plane and a half space have been obtained in [18] and in the book [19] respectively.

The paper is organized as follows. Section 2 describes what occurs in the complex analysis situation and, thus, what is going to be generalized. In Section 3 the analogues of the Hilbert formulas on the unit sphere for solenoidal and irrotational vector fields are announced and their corollaries are formulated. Then, in Section 4 we present a brief review of the properties of hyperholomorphic functions in the sense of Moisil–Theodoresco which includes the notion of hyperconjugate harmonic functions. Section 5 presents the Hilbert formulas and their proof in the context of quaternionic analysis for the Moisil–Theodoresco operator which relate a pair of real components of the boundary value of a quaternionic hyperholomorphic function in the unit ball with the other pair of real components. Finally, in Section 6 we obtain the results announced in Section 3 using what is proved in Section 5.

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2. The Hilbert formulas for the unit disk in one complex variable.

2.1. Our aim is to establish some analogues of the Hilbert and Schwarz formulas of the holomorphic function theory in one complex variable. With this in mind, we shall describe here what occurs in the complex situation.

Let $\mathbb{S} = \mathbb{S}(0;1)$ be the unit circumference in the complex plane $\mathbb{C}$ which is the boundary of the unit disk $\mathbb{B} = \mathbb{B}(0;1)$. The following definitions are useful for us.

Definition 1. $\mathfrak{U}(\mathbb{B}(0;1); C^{0,\epsilon}(\mathbb{S}))$, $0 < \epsilon < 1$, *denotes the class of functions* $\tilde{f}$ *such that*

1) $\tilde{f} \in \mathrm{Hol}(\mathbb{B}(0;1))$*, the class of holomorphic in the unit disk functions;*
2) *there exists everywhere on* $\mathbb{S}$ *the limit* $\lim\limits_{\mathbb{B}(0;1)\ni x\to\tau\in\mathbb{S}} \tilde{f}(x) =: f(\tau)$ *generating the function* f *in* $C^{0,\epsilon}(\mathbb{S})$*.*

Definition 2. $\mathfrak{U}(\mathbb{B}(0;1); L_p(\mathbb{S}))$*,* $1 < p < \infty$*, denotes the class of functions* $\tilde{f}$ *such that*
1) $\tilde{f} \in \mathrm{Hol}(\mathbb{B}(0;1))$*;*
2) *there exists almost everywhere on* $\mathbb{S}$ *the limit* $\lim\limits_{\mathbb{B}(0;1)\ni x\to\tau\in\mathbb{S}} \tilde{f}(x) =:$ $=: f(\tau)$ *generating the function* f *in* $L_p(\mathbb{S})$*.*

Note that sometimes the class $\mathfrak{U}(\mathbb{B}(0;1); L_p(\mathbb{S}))$ is called the Hardy class $H_p(\mathbb{S})$.

In what follows, $\tilde{f}$ always denotes a function satisfying any of Definitions 1 or Definition 2, and f is its limit function. We want to express one of the real components of its limit function $f =: f_1 + if_2$ via the other. Or, which is equivalent, we wonder how to construct the limit function f knowing one of its real components. Of course, this has one more interpretation: since having the limit function f we have the function $\tilde{f}$ also, then an important consequence of the above is that we can reconstruct the function $\tilde{f}$ provided we know one of the real components of its limit function f.

On the linear spaces $C^{0,\epsilon}(\mathbb{S})$ and $L_p(\mathbb{S})$ define

$$H[g](\varphi) := \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\psi - \varphi}{2} g(\psi) d\psi, \quad \varphi \in [0, 2\pi], \tag{1}$$

$$M[g] := \frac{1}{2\pi} \int_0^{2\pi} g(\psi) d\psi, \tag{2}$$

where the integral $H[g]$ is understood in the sense of Cauchy's principal value, generating the so-called Hilbert operator with (real) kernel $\frac{1}{2\pi} \cot \frac{\psi-\varphi}{2}$ which is well defined on both spaces, and where M is a functional which can be seen as the average value of the function.

Given a limit function f, denote $g(\varphi) := f(e^{i\varphi})$, $g = g_1 + ig_2$. Then the real components of g, g_1 and g_2, are related by the following formulas known as the Hilbert formulas for the unit disk, or for the unit circumference:

$$M[g_1] + H[g_2] = g_1, \qquad M[g_2] - H[g_1] = g_2. \tag{3}$$

2.2. There are known many proofs of them, see, for instance, [1]. We give here one of them which admits its exact analogues for some classes of hyperholomorphic functions.

Let S denote the singular integration operator along an appropriate curve γ:

$$S[f](t) := \frac{1}{\pi i}\int_\gamma \frac{f(\tau)}{\tau - t}d\tau,\ t \in \gamma. \tag{4}$$

It is known that S is an involution on $C^{0,\epsilon}(\gamma)$, $0 < \epsilon < 1$, and on $L_p(\gamma)$, $1 < p$: $S^2 = I$, the identity operator. Besides, a necessary and sufficient condition for f to be the limit function of a function $\tilde{f}$ is

$$f = S[f]. \tag{5}$$

Let now $\gamma = \mathbb{S}$, then one has:

$$\begin{aligned} S[f](t) &= \frac{1}{\pi i}\int_{\mathbb{S}} \frac{f(\tau)}{\tau - t}d\tau = \frac{1}{\pi i}\int_{\mathbb{S}} \frac{\bar{\tau} - \bar{t}}{|\tau - t|^2} f(\tau)d\tau = \\ &= \frac{1}{\pi i}\int_{\mathbb{S}} \frac{\bar{\tau} - \bar{t}}{|\tau - t|^2} in(s) f(\tau) ds_\tau = \frac{1}{\pi}\int_{\mathbb{S}} \frac{\bar{\tau} - \bar{t}}{|\tau - t|^2}\tau f(\tau) ds_\tau = \\ &= \frac{1}{\pi}\int_{\mathbb{S}} \frac{|\tau|^2 - \bar{t}\tau}{|\tau - t|^2} f(\tau) ds_\tau = \frac{1}{\pi}\int_{\mathbb{S}} \frac{1 - \bar{t}\tau}{|\tau - t|^2} f(\tau) ds_\tau. \end{aligned}$$

Let Ξ be the angle between the vectors t and τ then

$$\bar{t}\tau = \cos\Xi + i\sin\Xi, \qquad |\tau - t|^2 = 2 - 2\cos\Xi,$$

and the formula for $S[f]$ becomes:

$$\begin{aligned} S[f](t) &= \frac{1}{\pi}\int_{\mathbb{S}} \frac{1 - \cos\Xi - i\sin\Xi}{2 - 2\cos\Xi} f(\tau) ds_\tau = \\ &= \frac{1}{\pi}\int_{\mathbb{S}} \frac{1 - \cos\Xi}{2 - 2\cos\Xi} f(\tau) ds_\tau - i\frac{1}{\pi}\int_{\mathbb{S}} \frac{\sin\Xi}{2 - 2\cos\Xi} f(\tau) ds_\tau = \\ &= \frac{1}{2\pi}\int_{\mathbb{S}} f(\tau) ds_\tau - i\frac{1}{2\pi}\int_{\mathbb{S}} \cot\left(\frac{\Xi}{2}\right) f(\tau) ds_\tau. \end{aligned} \tag{6}$$

Take now a function $\tilde{f}$ then for its limit function f formulas (5) and (6) give:

$$f(t) = \frac{1}{2\pi}\int_{\mathbb{S}} f(\tau) ds_\tau - i\frac{1}{2\pi}\int_{\mathbb{S}} \cot\left(\frac{\Xi}{2}\right) f(\tau) ds_\tau. \tag{7}$$

Making the change of variables $\tau = e^{i\varphi}$, $0 \leq \varphi \leq 2\pi$, and recalling that $g(\varphi) = f(e^{i\varphi})$, $g = g_1 + ig_2$, we get:

$$g_1 + ig_2 = (M - iH)[g_1 + ig_2],$$

and separating the real and the imaginary parts we obtain the Hilbert formulas (3).

We have done the change of variables in order to obtain the Hilbert formulas in their classical form, but we can separate the real and the imaginary parts directly from (7) and then to make the change of variables.

2.3. Because of the relation between g_1 and g_2, the function g can be written as

$$g = g_1 - iH[g_1] + iM[g_2] = M[g_1] + H[g_2] + ig_2,$$

which means that up to an additive constant the function g is determined by any one of its real components. Applying the Cauchy Integral Formula we obtain for $\tilde{f}$ the following formula:

$$\begin{aligned} \tilde{f}(z) &= \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t - z} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi}}{e^{i\varphi} - z} g(\varphi) d\varphi = \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\varphi}}{e^{i\varphi} - z} \left(g_1(\varphi) - iH[g_1](\varphi) + iM[g_2]\right) d\varphi, \end{aligned} \tag{8}$$

which after some computations becomes

$$\tilde{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} f_1(e^{i\varphi}) \frac{e^{i\varphi} + z}{e^{i\varphi} - z} d\varphi + if_{2,0}, \tag{9}$$

where $f_{2,0} := \frac{1}{2\pi} \int_0^{2\pi} f_2(e^{i\varphi}) d\varphi$.

This formula is known as the Schwarz formula for the unitary disk and shows how the function itself can be reconstructed from its real part on the boundary up to a purely imaginary constant; a similar formula holds for the imaginary part.

2.4. Assume now that $g \in \ker M$, that is, its average is zero which is not an onerous restriction, then the Hilbert formulas take the form

$$H[g_2] = g_1, \qquad -H[g_1] = g_2. \tag{10}$$

These formulas are often called the mutually inverse Hilbert formulas.

In this case the function g, and thus the function f, is determined uniquely by any one of its real components:

$$g = g_1 - iH[g_1] = H[g_2] + ig_2.$$

The function $\tilde{f}$ in (9) does not contain any additive constant.

3. Analogues of the Hilbert formulas on the unit sphere for solenoidal and irrotational vector fields.

3.1. Let Ω be a domain in $\mathbb{R}^3$ and let Γ be its boundary; we assume that Γ is smooth enough. Consider a vector field $\vec{F} \in C^1(\Omega;\mathbb{R}^3)$, if $\vec{F} = F_1 i_1 + F_2 i_2 + F_3 i_3$ satisfies the following system:

$$\begin{cases} \operatorname{div}\vec{F} = 0, \\ \operatorname{rot}\vec{F} = 0, \end{cases} \tag{11}$$

then it is called a solenoidal and irrotational vector field (SI-vector field). The set of SI-vector fields in Ω will be denoted by $\vec{\mathfrak{M}}(\Omega)$.

One would like to have an analogue of the Cauchy-type integral for the class $\vec{\mathfrak{M}}(\Omega)$ in the same sense as its analogue for holomorphic functions in one variable. It turns out that for having it one should restrict the class of SI-vector fields. Our methods impose the following condition: if $\vec{f} \in C(\Gamma)$ then it should satisfy for $\vec{u} \in \mathbb{R}^3 \setminus \Gamma$, the identity

$$0 = \int_\Gamma \frac{1}{|\vec{u}-\vec{v}|^3} \left\{ \left\langle [(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})]\, , \vec{f}(\vec{v}) \right\rangle \right\} d\Gamma_{\vec{v}}\, . \tag{12}$$

For such vector fields the Cauchy-type integral is defined by

$$\begin{aligned} K_{SI}[\vec{f}](\vec{u}) :=& \frac{1}{4\pi} \int_\Gamma \frac{1}{|\vec{u}-\vec{v}|^3} \Big\{ - \langle (\vec{u}-\vec{v}), \vec{n}(\vec{v}) \rangle\, \vec{f}(\vec{v}) + \\ &+ \Big[[(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})] \times \vec{f}(\vec{v}) \Big] \Big\}\, d\Gamma_{\vec{v}}\, , \end{aligned} \tag{13}$$

where $\vec{n}(\vec{v})$ is an outward pointing normal to the surface Γ at a point $\vec{v}$.

The fine point here is that if $\vec{F}$ is an SI-vector field in Ω which is continuous up the boundary where it satisfies the condition (12) then the Cauchy integral representation with the integral (13) holds.

The origin of this restriction will be made more precise below. There are also physical reasons for considering vector fields with condition (12) as explained in [3], see pages 120 — 128 and Appendix A. A similar reasoning can be found in [20, 21].

Assuming additionally that $\vec{f}$ is a Hölder vector field on Γ then the limits of the right-hand side of (12) when $\vec{u}$ tends to $\vec{x} \in \Gamma$ from the inside or outside exist and thus

$$0 = \int_{\Gamma} \frac{1}{|\vec{x}-\vec{v}|^3} \left\{ \left\langle [(\vec{x}-\vec{v}) \times \vec{n}(\vec{v})] \, , \vec{f}(\vec{v}) \right\rangle \right\} d\Gamma_{\vec{v}} \, , \tag{14}$$

where the integral is understood in the sense of Cauchy's principal value. Also, we can consider $\vec{f} \in L_p(\Gamma)$ with $1 < p < \infty$ and the previous conclusions are valid almost everywhere on Γ.

3.2. Let $\mathbb{S}^2 = \mathbb{S}^2(0;1)$ be the unit sphere in $\mathbb{R}^3$ which is the boundary of the unit ball $\mathbb{B}^2 = \mathbb{B}^2(0;1)$, the following formulas define linear bounded operators on both spaces of our interest which are $C^{0,\epsilon}(\mathbb{S}^2)$, $0 < \epsilon < 1$, and $L_p(\mathbb{S}^2)$, $1 < p < \infty$:

$$M^0_{\mathbb{S}^2}[f](\vec{x}) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{1}{2|\vec{x}-\vec{y}|} f(\vec{y}) d\Gamma_{\vec{y}} \, , \tag{15}$$

$$\mathcal{H}^1_{\mathbb{S}^2}[f](\vec{x}) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{(x_2y_3 - x_3y_2)}{|\vec{x}-\vec{y}|^3} f(\vec{y}) d\Gamma_{\vec{y}} \, , \tag{16}$$

$$\mathcal{H}^2_{\mathbb{S}^2}[f](\vec{x}) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{(x_3y_1 - x_1y_3)}{|\vec{x}-\vec{y}|^3} f(\vec{y}) d\Gamma_{\vec{y}} \, , \tag{17}$$

$$\mathcal{H}^3_{\mathbb{S}^2}[f](\vec{x}) := \frac{1}{2\pi} \int_{\mathbb{S}^2} \frac{(x_1y_2 - x_2y_1)}{|\vec{x}-\vec{y}|^3} f(\vec{y}) d\Gamma_{\vec{y}} \, . \tag{18}$$

These operators are well-defined for functions taking values in $\mathbb{R}$ and can be extended component-wise to vector fields. Observe that the operator $M^0_{\mathbb{S}^2}[f]$ has a singularity of order one, thus it is understood as improper; meanwhile the integrals $\mathcal{H}^k_{\mathbb{S}^2}[f]$ with $k = 1, 2, 3$ have singularities of order two (for reasons that will we be clear later) and the integrals have to be understood in the sense of Cauchy's principal value.

The integral $M^0_{\mathbb{S}^2}[f]$ sometimes is called a boundary value simple-layer Newton potential (or Riesz potential) with density f; it is a harmonic function on $\mathbb{R}^3 \backslash \mathbb{S}^2$ and continuos on $\mathbb{R}^3$. Moreover, this operator is compact on both spaces (see [22]).

3.3. Our techniques require to consider the following real linear spaces:

$$\hat{C}^{0,\epsilon}(\Gamma) := \{\vec{f} \in C^{0,\epsilon}(\Gamma), \; 0 < \epsilon < 1; \; (14) \text{ is valid}\}.$$

$$\hat{L}_p(\Gamma) := \{\vec{f} \in L_p(\Gamma), \; 1 < p < \infty; \; (14) \text{ is valid}\}.$$

Definition 3. $\vec{\mathfrak{U}}(\mathbb{B}^2(0;1);\hat{C}^{0,\epsilon}(\mathbb{S}^2))$, $0<\epsilon<1$, *denotes the class of vector fields* $\vec{F}$ *such that*

1) $\vec{F}\in\vec{\mathfrak{M}}(\mathbb{B}^2(0;1))$;
2) *there exists everywhere on* $\mathbb{S}^2$ *the limit* $\lim\limits_{\mathbb{B}^2(0;1)\ni\vec{u}\to\vec{x}\in\mathbb{S}^2}\vec{F}(\vec{u})=:\vec{f}(\vec{x})$ *generating the vector field* $\vec{f}$ *in* $\hat{C}^{0,\epsilon}(\mathbb{S}^2)$.

Definition 4. $\vec{\mathfrak{U}}(\mathbb{B}^2(0;1);\hat{L}_p(\mathbb{S}^2))$, $1<p<\infty$, *denotes the class of vector fields* $\vec{F}$ *such that*

1) $\vec{F}\in\vec{\mathfrak{M}}(\mathbb{B}^2(0;1))$;
2) *there exists almost everywhere on* $\mathbb{S}^2$ *the limit* $\lim\limits_{\mathbb{B}^2(0;1)\ni\vec{u}\to\vec{x}\in\mathbb{S}^2}\vec{F}(\vec{u})=:$ $=:\vec{f}(\vec{x})$ *generating the vector field* $\vec{f}$ *in* $\hat{L}_p(\mathbb{S}^2)$.

Note that $\vec{\mathfrak{U}}(\mathbb{B}^2(0;1);\hat{L}_p(\mathbb{S}^2))$ may be reasonably called the Hardy space for SI-vector fields.

3.4. We announce now several statements whose profs will be given later, in Section 6.

Theorem 5 (Analogues of the Hilbert formulas for SI-vector fields). *Let* $\vec{F}\in\vec{\mathfrak{U}}(\mathbb{B}^2(0;1);\hat{C}^{0,\epsilon}(\mathbb{S}^2))$ *or* $\vec{F}\in\vec{\mathfrak{U}}(\mathbb{B}^2(0;1);\hat{L}_p(\mathbb{S}^2))$. *A vector field* $\vec{f}=i_1f_1+i_2f_2+i_3f_3$ *is the limit function of* $\vec{F}$ *if and only if the following relations between its components hold:*

$$\begin{aligned} f_1&=M^0_{\mathbb{S}^2}[f_1]+\mathcal{H}^2_{\mathbb{S}^2}[f_3]-\mathcal{H}^3_{\mathbb{S}^2}[f_2],\\ f_2&=M^0_{\mathbb{S}^2}[f_2]+\mathcal{H}^3_{\mathbb{S}^2}[f_1]-\mathcal{H}^1_{\mathbb{S}^2}[f_3],\\ f_3&=M^0_{\mathbb{S}^2}[f_3]+\mathcal{H}^1_{\mathbb{S}^2}[f_2]-\mathcal{H}^2_{\mathbb{S}^2}[f_1].\end{aligned}\tag{19}$$

The reader may compare the formulas (19) with (3), they have the same structure relating now the real components of the limit function $\vec{f}$. We can conclude that on $\ker M^0_{\mathbb{S}^2}$ formulas (19) become

$$\begin{aligned} f_1&=\mathcal{H}^2_{\mathbb{S}^2}[f_3]-\mathcal{H}^3_{\mathbb{S}^2}[f_2],\\ f_2&=\mathcal{H}^3_{\mathbb{S}^2}[f_1]-\mathcal{H}^1_{\mathbb{S}^2}[f_3],\\ f_3&=\mathcal{H}^1_{\mathbb{S}^2}[f_2]-\mathcal{H}^2_{\mathbb{S}^2}[f_1],\end{aligned}\tag{20}$$

compare with (10), although now we do not have an explicit description of $\ker M^0_{\mathbb{S}^2}$.

Corollary 6. *Let $\vec{f}$ be the limit function of $\vec{F}$, then $\vec{f}$ is determined by*

$$\begin{aligned}\vec{f} &= (M^0_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2])i_1 + f_2 i_2 + f_3 i_3 = \\ &= f_1 i_1 + (M^0_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3])i_2 + f_3 i_3 = \\ &= f_1 i_1 + f_2 i_2 + (M^0_{\mathbb{S}^2}[f_3] + \mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1])i_3.\end{aligned}$$

In particular, if $\vec{f} \in \ker M^0_{\mathbb{S}^2}$ then

$$\begin{aligned}\vec{f} &= (\mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2])i_1 + f_2 i_2 + f_3 i_3 = \\ &= f_1 i_1 + (\mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3])i_2 + f_3 i_3 = \\ &= f_1 i_1 + f_2 i_2 + (\mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1])i_3.\end{aligned}$$

That is, if $\vec{f} \in \ker M^0_{\mathbb{S}^2}$ then $\vec{f}$ is determined by any two of its real components.

Corollary 7 (Analogue of the Schwarz formula for SI-vector fields). *Let $\vec{f}$ be the limit function of $\vec{F}$, then $\vec{F}$ is determined by*

$$\begin{aligned}\vec{F} &= K_{SI}\left[(M^0_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2])i_1 + f_2 i_2 + f_3 i_3\right] = \\ &= K_{SI}\left[f_1 i_1 + (M^0_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3])i_2 + f_3 i_3\right] = \\ &= K_{SI}\left[f_1 i_1 + f_2 i_2 + (M^0_{\mathbb{S}^2}[f_3] + \mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1])i_3\right],\end{aligned}$$

where K_{SI} is the Cauchy-type integral for SI-vector fields defined by (13). *In particular, if $\vec{f} \in \ker M^0_{\mathbb{S}^2}$ then*

$$\begin{aligned}\vec{F} &= K_{SI}\left[(\mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2])i_1 + f_2 i_2 + f_3 i_3\right] = \\ &= K_{SI}\left[f_1 i_1 + (\mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3])i_2 + f_3 i_3\right] = \\ &= K_{SI}\left[f_1 i_1 + f_2 i_2 + (\mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1])i_3\right].\end{aligned}$$

That is, if $\vec{f} \in \ker M^0_{\mathbb{S}^2}$ then $\vec{F}$ is determined by any two of the limit functions of its real components.

4. Rudiments of quaternionic analysis for the Moisil–Theodoresco operator.

4.1. In the previous sections, we announced the results related to the Hilbert formulas for SI-vector fields. On the other hand, it turns out that there exists a quaternionic function theory that have shown to be a generalization of the theory of functions of one complex variable and which includes the SI-vector theory as a particular case (see [19]).

We shall denote by $\mathbb{H}$ the set of real quaternions, each quaternion w is represented in the form

$$w = w_0 + w_1 i_1 + w_2 i_2 + w_3 i_3.$$

The coefficients $\{w_k\}$ are real; $\{i_1, i_2, i_3\}$ are the quaternionic imaginary units. $\mathbb{H}$ has the structure of a real non-commutative, associative division algebra.

The representation $w = w_0 + \vec{w}$, where $\vec{w} := \mathrm{Vec}(w) := w_1 i_1 + w_2 i_2 + $ $+ w_3 i_3$, will rather essential for our purposes. $w_0 := \mathrm{Sc}(w)$ will be called the scalar part and $\vec{w}$ the vector part of the quaternion w.

For a quaternion w we will consider its quaternionic conjugate $\bar{w}$ defined byThe coefficients

$$\bar{w} := w_0 - \vec{w}.$$

4.2. We will consider $\mathbb{H}$–valued functions given in a domain $\Omega \subset \mathbb{R}^3$ with Γ its boundary which is smooth enough. On the set $C^1(\Omega; \mathbb{H})$ the well-known Moisil–Theodoresco operator is defined by the formula:

$$D_{MT} := \sum_{k=1}^{3} i_k \partial_k, \tag{21}$$

where $\partial_k := \frac{\partial}{\partial x_k}$. In vectorial language, the Moisil–Theodoresco system

$$D_{MT}[f] = 0,$$

can be rewritten as

$$\mathrm{div} \vec{f} = 0, \tag{22}$$

$$\mathrm{grad} f_0 + \mathrm{rot} \vec{f} = 0. \tag{23}$$

The system (22), (23) is considered by many as the most natural and simple spatial generalization of the Cauchy–Riemann equations. Note that SI-vector fields satisfy the previous system.

A function $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$ satisfying (21) in Ω, i.e., $f \in$ $\in \ker D_{MT}(\Omega) =: \mathfrak{M}(\Omega)$, will be called hyperholomorphic in Ω in the sense of Moisil–Theodoresco, or MT-hyperholomorphic function.

From the definition of the operator D_{MT} its important property follows:

$$-D_{MT}^2 = \Delta_{\mathbb{R}^3}, \tag{24}$$

where $\Delta_{\mathbb{R}^3} := \sum_{k=1}^{3} \partial_k^2$ is the Laplace operator.

The Cauchy-type integral operator is defined by

$$K_{\mathbb{H}}[f](\vec{u}) := \frac{1}{4\pi}\int_{\Gamma}\frac{(\vec{u}-\vec{v})}{|\vec{u}-\vec{v}|^3}\vec{n}(\vec{v})f(\vec{v})d\Gamma_{\vec{v}}\,, \quad u \in \mathbb{R}^3 \setminus \Gamma, \tag{25}$$

where $(\vec{u}-\vec{v}) := (u_1 - v_1)i_1 + (u_2 - v_2)i_2 + (u_3 - v_3)i_3$ and $\vec{n}(\vec{v})$ is an outward pointing normal to the surface Γ at the point $\vec{v} \in \Gamma$, which can be written in the quaternionic form: $\vec{n} := n_1 i_1 + n_2 i_2 + n_3 i_3$.

4.3. One more structure of quaternions proved to be useful for our purposes. Let $f \in \mathbb{H}$, then

$$f = \sum_{k=0}^{3} f_k i_k = (f_0 + f_1 i_1) + (f_2 + f_1 i_1)i_2 =: F_1 + F_2 i_2.$$

F_1, F_2 are of the form $a + bi_1$ with a, b usual real numbers, and thus they are complex numbers with the imaginary unit i_1: F_1 and F_2 belongs to $\mathbb{C}(i_1)$. F_1 and F_2 will be called the complex components of f.

If f is a hyperholomorphic function in the sense of Moisil–Theodoresco then

$$D_{MT}[f] = i_1\frac{\partial f}{\partial x_1} + i_2\frac{\partial f}{\partial x_2} + i_3\frac{\partial f}{\partial x_3} = 0,$$

now witting f in terms of its complex components F_1 and F_2 and separating the complex components in the above equality we obtain:

$$\begin{aligned} i_1\frac{\partial F_1}{\partial x_1} - \left(\frac{\partial \bar{F}_2}{\partial x_2} + i_1\frac{\partial \bar{F}_2}{\partial x_3}\right) &= 0,\\ i_1\frac{\partial F_2}{\partial x_1} + \left(\frac{\partial \bar{F}_1}{\partial x_2} + i_1\frac{\partial \bar{F}_1}{\partial x_3}\right) &= 0. \end{aligned} \tag{26}$$

Define

$$2\frac{\partial}{\partial \bar{z}_{2,3}} := \left(\frac{\partial}{\partial x_2} + i_1\frac{\partial}{\partial x_3}\right), \tag{27}$$

then we will say that the complex-valued functions of three real variables F_1 and F_2 are hyperconjugate harmonic functions in the sense of Moisil–Theodoresco if they satisfy the system

$$\begin{aligned} i_1\frac{\partial F_1}{\partial x_1} - 2\frac{\partial \bar{F}_2}{\partial \bar{z}_{2,3}} &= 0,\\ i_1\frac{\partial F_2}{\partial x_1} + 2\frac{\partial \bar{F}_1}{\partial \bar{z}_{2,3}} &= 0, \end{aligned} \tag{28}$$

which can be considered as an analogue of the complex Cauchy–Riemann equations for this case. In the paper [23] the pair $(f_0, \vec{f})$ is called a hyperconjugate harmonic pair if they satisfy the system (22), (23).

5. The Hilbert formulas on the unit sphere for MT-hyperholomorphic functions.

5.1. We will use the combinations of the operators introduced in (15) — (18)

$$M_{\mathbb{S}^2}[f](\vec{x}) : = M^0_{\mathbb{S}^2}[f](\vec{x}) + i_1\mathcal{H}^1_{\mathbb{S}^2}[f](\vec{x}) = \\ = \frac{1}{2\pi}\int_{\mathbb{S}^2}\left(\frac{1}{2|\vec{x}-\vec{y}|} + \frac{(x_2y_3 - x_3y_2)i_1}{|\vec{x}-\vec{y}|^3}\right) f(\vec{y})d\Gamma_{\vec{y}}\,, \qquad (29)$$

$$\mathcal{H}_{\mathbb{S}^2}[f](\vec{x}) : = -\mathcal{H}^2_{\mathbb{S}^2}[f](\vec{x}) - i_1\mathcal{H}^3_{\mathbb{S}^2}[f](\vec{x}) = \\ = -\frac{1}{2\pi}\int_{\mathbb{S}^2}\left(\frac{(x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)i_1}{|\vec{x}-\vec{y}|^3}\right) f(\vec{y})d\Gamma_{\vec{y}}\,, \quad (30)$$

$$\bar{M}_{\mathbb{S}^2}[f](\vec{x}) := \frac{1}{2\pi}\int_{\mathbb{S}^2}\left(\frac{1}{2|\vec{x}-\vec{y}|} - \frac{(x_2y_3 - x_3y_2)i_1}{|\vec{x}-\vec{y}|^3}\right) f(\vec{y})d\Gamma_{\vec{y}}\,, \qquad (31)$$

$$\bar{\mathcal{H}}_{\mathbb{S}^2}[f](\vec{x}) := -\frac{1}{2\pi}\int_{\mathbb{S}^2}\left(\frac{(x_3y_1 - x_1y_3) - (x_1y_2 - x_2y_1)i_1}{|\vec{x}-\vec{y}|^3}\right) f(\vec{y})d\Gamma_{\vec{y}} \quad (32)$$

which are well-defined on $C^{0,\epsilon}(\mathbb{S}^2)$, $0 < \epsilon < 1$, and on $L_p(\mathbb{S}^2)$, $1 < p < \infty$; the integrals are understood in the sense of Cauchy's principal value.

Definition 8. $\mathfrak{U}(\mathbb{B}^2(0;1); C^{0,\epsilon}(\mathbb{S}^2))$, $0 < \epsilon < 1$, *denotes the class of functions* $\tilde{f}$ *such that*

1) $\tilde{f} \in \mathfrak{M}(\mathbb{B}^2(0;1))$;
2) *there exists everywhere on* $\mathbb{S}^2$ *the limit* $\lim\limits_{\mathbb{B}^2(0;1)\ni\vec{u}\to\vec{x}\in\mathbb{S}^2} \tilde{f}(\vec{u}) =: f(\vec{x})$ *generating the function* f *in* $C^{0,\epsilon}(\mathbb{S}^2)$.

Definition 9. $\mathfrak{U}(\mathbb{B}^2(0;1); L_p(\mathbb{S}^2))$, $1 < p < \infty$, *denotes the class of functions* $\tilde{f}$ *such that*

1) $\tilde{f} \in \mathfrak{M}(\mathbb{B}^2(0;1))$;
2) *there exists almost everywhere on* $\mathbb{S}^2$ *the limit* $\lim\limits_{\mathbb{B}^2(0;1)\ni\vec{u}\to\vec{x}\in\mathbb{S}^2} \tilde{f}(\vec{u}) =:$ $=: f(\vec{x})$ *generating the function* f *in* $L_p(\mathbb{S}^2)$.

Note that, again, the class $\mathfrak{U}(\mathbb{B}^2(0;1); L_p(\mathbb{S}^2))$ may be called the (quaternionic) Hardy space for MT-hyperholomorphic functions.

5.2.

Theorem 10 (Analogue of the Hilbert formulas for MT-hyperholomorphic functions). *Let* $\tilde{f} \in \mathfrak{U}(\mathbb{B}^2(0;1); C^{0,\epsilon}(\mathbb{S}^2))$ *or* $\tilde{f} \in \mathfrak{U}(\mathbb{B}^2(0;1); L_p(\mathbb{S}^2))$. *A function* $f = F_1 + F_2 i_2$ *is the limit function of* $\tilde{f}$ *if and only if* F_1 *and* F_2 *are related by the formulas:*

$$F_1 = M_{\mathbb{S}^2}[F_1] + \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2], \qquad F_2 = M_{\mathbb{S}^2}[F_2] - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1]. \tag{33}$$

Proof. Let $S_{\mathbb{H}}$ denote de singular integral operator along $\mathbb{S}^2$:

$$S_{\mathbb{H}}[f](\vec{x}) := \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\vec{n}(\vec{y}) f(\vec{y}) d\Gamma_{\vec{y}}\,, \quad \vec{x} \in \Gamma, \tag{34}$$

where $(\vec{x}-\vec{y}) := (x_1 - y_1)i_1 + (x_2 - y_2)i_2 + (x_3 - y_3)i_3$.

It is known that $S_{\mathbb{H}}$ is an involution on $C^{0,\epsilon}$, $0 < \epsilon < 1$, and on L_p, $1 < p < \infty$: $S_{\mathbb{H}}^2 = I$, the identity operator. Also, a necessary and sufficient condition for f to be the limit value of a function $\tilde{f}$ is that

$$f = S_{\mathbb{H}}[f], \tag{35}$$

for proofs of these statements see for instance [19].

The definition of the operator $S_{\mathbb{H}}$ implies that

$$S_{\mathbb{H}} = M_{\mathbb{S}^2} - \mathcal{H}_{\mathbb{S}^2} i_2,$$

where the last operator acts by the formula

$$\mathcal{H}_{\mathbb{S}^2} i_2[f](\vec{x}) = -\frac{1}{2\pi}\int_{\mathbb{S}^2}\left(\frac{(x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)i_1}{|\vec{x}-\vec{y}|^3}\right) i_2 f(\vec{y}) d\Gamma_{\vec{y}}\,.$$

Then

$$\begin{aligned} S_{\mathbb{H}}[f] &= (M_{\mathbb{S}^2} - \mathcal{H}_{\mathbb{S}^2} i_2)[f] = (M_{\mathbb{S}^2} - \mathcal{H}_{\mathbb{S}^2} i_2)[F_1 + F_2 i_2] = \\ &= \left(M_{\mathbb{S}^2}[F_1] + \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2]\right) + \left(M_{\mathbb{S}^2}[F_2] - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1]\right) i_2. \end{aligned}$$

Making use of condition (35) leads to (33). Theorem is proved.

The reader may compare the formulas (33) with (3), they have the same structure relating now the complex components of the limit function f. Again, we can conclude that on $\ker M_{\mathbb{S}^2}$ formulas (33) become

$$F_1 = \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2], \qquad F_2 = -\mathcal{H}_{\mathbb{S}^2}[\bar{F}_1], \tag{36}$$

compare with (10), although now we do not have an explicit description of $\ker M_{\mathbb{S}^2}$.

Remark 11. *Using now the fact that the operator $S_{\mathbb{H}}$ is an involution we have that*

$$Id = S_{\mathbb{H}}^2 = M_{\mathbb{S}^2}^2 - \mathcal{H}_{\mathbb{S}^2}\bar{\mathcal{H}}_{\mathbb{S}^2} - (\mathcal{H}_{\mathbb{S}^2}\bar{M}_{\mathbb{S}^2} + M_{\mathbb{S}^2}\mathcal{H}_{\mathbb{S}^2})i_2,$$

from which one concludes that

$$Id = M_{\mathbb{S}^2}^2 - \mathcal{H}_{\mathbb{S}^2}\bar{\mathcal{H}}_{\mathbb{S}^2} \quad and \quad \mathcal{H}_{\mathbb{S}^2}\bar{M}_{\mathbb{S}^2} + M_{\mathbb{S}^2}\mathcal{H}_{\mathbb{S}^2} = 0.$$

We believe that it would not be so easy to prove these properties directly.

5.3. In analogy with the complex case Theorem 10 implies several corollaries.

Corollary 12. *Given two complex-valued functions of three real variables F_1 and F_2 of the classes $C^{0,\epsilon}(\mathbb{S}^2)$ or $L_p(\mathbb{S}^2)$, they are the limit functions of a pair $\tilde{F}_1$ and $\tilde{F}_2$ of hyperconjugate harmonic functions in the sense of Moisil-Theodoresco if and only if they satisfy the following formulas:*

$$F_1 = M_{\mathbb{S}^2}[F_1] + \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2], \qquad F_2 = M_{\mathbb{S}^2}[F_2] - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1].$$

In particular, if $F_1, F_2 \in \ker M_{\mathbb{S}^2}$ then

$$F_1 = \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2], \qquad F_2 = -\mathcal{H}_{\mathbb{S}^2}[\bar{F}_1].$$

Of course this is just a reformulation of Theorem 10.

Corollary 13. *Let $f = F_1 + F_2 i_2$ be the limit function of $\tilde{f}$, then f is determined by*

$$f = F_1 + \left(M_{\mathbb{S}^2}[F_2] - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1]\right) i_2 = M_{\mathbb{S}^2}[F_1] + \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2] + F_2 i_2.$$

In particular, if $f \in \ker M_{\mathbb{S}^2}$ then

$$f = F_1 - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1] i_2 = \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2] + F_2 i_2.$$

That is, if $f \in \ker M_{\mathbb{S}^2}$ then f is determined by any one of its complex components.

Corollary 14 (Analogue of the Schwarz formula for MT-hyperholomorphic functions). *Let $f = F_1 + F_2 i_2$ be the limit function of $\tilde{f}$, then $\tilde{f}$ is determined by*

$$\tilde{f} = K_{\mathbb{H}}\left[F_1 + \left(M_{\mathbb{S}^2}[F_2] - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1]\right) i_2\right] = K_{\mathbb{H}}\left[M_{\mathbb{S}^2}[F_1] + \mathcal{H}_{\mathbb{S}^2}[\bar{F}_2] + F_2 i_2\right].$$

In particular, if $f \in \ker M_{\mathbb{S}^2}$ then

$$\tilde{f} = K_{\mathbb{H}}\left[F_1 - \mathcal{H}_{\mathbb{S}^2}[\bar{F}_1] i_2\right] = K_{\mathbb{H}}\left[\mathcal{H}_{\mathbb{S}^2}[\bar{F}_2] + F_2 i_2\right].$$

That is, if $f \in \ker M_{\mathbb{S}^2}$ then $\tilde{f}$ is determined by any one of the limit functions of its complex components.

6. Proof of the Hilbert formulas for SI-vector fields.

6.1. The aim of this section is to prove the statements of Section 3.

Let's analyze the structure of the singular integral operator $S_{\mathbb{H}}$; since on the unit sphere $\vec{n}(\vec{y}) = \vec{y}$ then we have that

$$S_{\mathbb{H}}[f](\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\vec{n}(\vec{y}) f(\vec{y}) d\Gamma_{\vec{y}} = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\vec{y} f(\vec{y}) d\Gamma_{\vec{y}}\,; \quad (37)$$

if Ξ is the angle between $\vec{x}$ and $\vec{y}$ then $|\vec{x}-\vec{y}|^2 = 2 - 2\cos\Xi$, therefore

$$\frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3}\vec{y} = \frac{1 - \langle \vec{x}, \vec{y}\rangle}{|\vec{x}-\vec{y}|^2} \cdot \frac{1}{|\vec{x}-\vec{y}|} + \frac{[\vec{x}\times\vec{y}]}{|\vec{x}-\vec{y}|^3} =$$

$$= \frac{1-\cos\Xi}{2(1-\cos\Xi)} \cdot \frac{1}{|\vec{x}-\vec{y}|} + \frac{[\vec{x}\times\vec{y}]}{|\vec{x}-\vec{y}|^3} = \frac{1}{2|\vec{x}-\vec{y}|} + \frac{[\vec{x}\times\vec{y}]}{|\vec{x}-\vec{y}|^3}.$$

Thus

$$S_{\mathbb{H}}[f](\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{1}{2|\vec{x}-\vec{y}|} f(\vec{y}) d\Gamma_{\vec{y}} + \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{[\vec{x}\times\vec{y}]}{|\vec{x}-\vec{y}|^3} f(\vec{y}) d\Gamma_{\vec{y}},$$

If we write $[\vec{x}\times\vec{y}] = |\vec{x}||\vec{y}|\sin\Xi \cdot \vec{r} = \sin\Xi \cdot \vec{r}$ with $\vec{r}$ a unitary vector perpendicular to the plane that contains $\vec{x}$ and $\vec{y}$ in the direction given by the right-hand rule, then $|[\vec{x}\times\vec{y}]| = \sin\Xi$ while $|\vec{x}-\vec{y}| = 2\sin\frac{\Xi}{2}$; so when $\vec{x}$ tends to $\vec{y}$ we have that $\frac{|[\vec{x}\times\vec{y}]|}{|\vec{x}-\vec{y}|^3} \approx \frac{\Xi}{\Xi^3} = \frac{1}{\Xi^2}$ which shows that the second integral has a strong singularity (of order two).

Thus

$$S_{\mathbb{H}}[f](\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^2}\left(\frac{1}{2|\vec{x}-\vec{y}|}+\right.$$

$$\left.+\frac{(x_2y_3-x_3y_2)i_1+(x_3y_1-x_1y_3)i_2+(x_1y_2-x_2y_1)i_3}{|\vec{x}-\vec{y}|^3}\right)f(\vec{y})d\Gamma_{\vec{y}} =$$

$$= M^0_{\mathbb{S}^2}[f](\vec{x}) + i_1\mathcal{H}^1_{\mathbb{S}^2}[f](\vec{x}) + i_2\mathcal{H}^2_{\mathbb{S}^2}[f](\vec{x}) + i_3\mathcal{H}^3_{\mathbb{S}^2}[f](\vec{x}), \tag{38}$$

where the operators $M^0_{\mathbb{S}^2}$, $\mathcal{H}^1_{\mathbb{S}^2}$, $\mathcal{H}^2_{\mathbb{S}^2}$ and $\mathcal{H}^3_{\mathbb{S}^2}$ are defined in formulas (15)-(18). Recall that the operator $M^0_{\mathbb{S}^2}$ is understood as improper. If now we write $[\vec{x}\times\vec{y}] = \sin\Xi\cdot\vec{r} = \sin\Xi\cdot r_1i_1 + \sin\Xi\cdot r_2i_2 + \sin\Xi\cdot r_3i_3$ we see that the operators $\mathcal{H}^k_{\mathbb{S}^2}$ with $k=1,2,3$ have singularities of order 2.

6.2. Since we have that: $S^2_{\mathbb{H}} = Id$, we come to the following equality:

$$\begin{aligned} Id = S^2_{\mathbb{H}} = {} & \left((M^0_{\mathbb{S}^2})^2 - (\mathcal{H}^1_{\mathbb{S}^2})^2 - (\mathcal{H}^2_{\mathbb{S}^2})^2 - (\mathcal{H}^3_{\mathbb{S}^2})^2\right) + \\ & + i_1\left(M^0_{\mathbb{S}^2}\mathcal{H}^1_{\mathbb{S}^2} + \mathcal{H}^1_{\mathbb{S}^2}M^0_{\mathbb{S}^2} + \mathcal{H}^2_{\mathbb{S}^2}\mathcal{H}^3_{\mathbb{S}^2} - \mathcal{H}^3_{\mathbb{S}^2}\mathcal{H}^2_{\mathbb{S}^2}\right) + \\ & + i_2\left(M^0_{\mathbb{S}^2}\mathcal{H}^2_{\mathbb{S}^2} + \mathcal{H}^2_{\mathbb{S}^2}M^0_{\mathbb{S}^2} + \mathcal{H}^3_{\mathbb{S}^2}\mathcal{H}^1_{\mathbb{S}^2} - \mathcal{H}^1_{\mathbb{S}^2}\mathcal{H}^3_{\mathbb{S}^2}\right) + \\ & + i_3\left(M^0_{\mathbb{S}^2}\mathcal{H}^3_{\mathbb{S}^2} + \mathcal{H}^3_{\mathbb{S}^2}M^0_{\mathbb{S}^2} + \mathcal{H}^1_{\mathbb{S}^2}\mathcal{H}^2_{\mathbb{S}^2} - \mathcal{H}^2_{\mathbb{S}^2}\mathcal{H}^1_{\mathbb{S}^2}\right), \end{aligned}$$

therefore

$$\begin{aligned} Id &= (M^0_{\mathbb{S}^2})^2 - (\mathcal{H}^1_{\mathbb{S}^2})^2 - (\mathcal{H}^2_{\mathbb{S}^2})^2 - (\mathcal{H}^3_{\mathbb{S}^2})^2, \\ 0 &= M^0_{\mathbb{S}^2}\mathcal{H}^1_{\mathbb{S}^2} + \mathcal{H}^1_{\mathbb{S}^2}M^0_{\mathbb{S}^2} + \mathcal{H}^2_{\mathbb{S}^2}\mathcal{H}^3_{\mathbb{S}^2} - \mathcal{H}^3_{\mathbb{S}^2}\mathcal{H}^2_{\mathbb{S}^2}, \\ 0 &= M^0_{\mathbb{S}^2}\mathcal{H}^2_{\mathbb{S}^2} + \mathcal{H}^2_{\mathbb{S}^2}M^0_{\mathbb{S}^2} + \mathcal{H}^3_{\mathbb{S}^2}\mathcal{H}^1_{\mathbb{S}^2} - \mathcal{H}^1_{\mathbb{S}^2}\mathcal{H}^3_{\mathbb{S}^2}, \\ 0 &= M^0_{\mathbb{S}^2}\mathcal{H}^3_{\mathbb{S}^2} + \mathcal{H}^3_{\mathbb{S}^2}M^0_{\mathbb{S}^2} + \mathcal{H}^1_{\mathbb{S}^2}\mathcal{H}^2_{\mathbb{S}^2} - \mathcal{H}^2_{\mathbb{S}^2}\mathcal{H}^1_{\mathbb{S}^2}. \end{aligned}$$

As we noticed in Remark 11, we believe that it would not be so easy to prove these properties directly.

6.3. We are ready to get obtaining the results of Section 3 as a particular case of the results of the MT-hyperholomorphic function theory.

First, we write (for an arbitrary surface Γ, not necessarily the unit sphere) the Cauchy-type integral (25) in vectorial form:

$$K_{\mathbb{H}}[f](\vec{u}) = \frac{1}{4\pi}\int_{\Gamma} \frac{(\vec{u}-\vec{v})}{|\vec{u}-\vec{v}|^3} n(\vec{v}) f(\vec{v}) d\Gamma_{\vec{v}} =$$

$$= \frac{1}{4\pi}\int_{\Gamma} \frac{1}{|\vec{u}-\vec{v}|^3} \{- \langle (\vec{u}-\vec{v}), \vec{n}(\vec{v})\rangle f_0(\vec{v}) -$$

$$- \left\langle [(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})], \vec{f}(\vec{v}) \right\rangle - \langle (\vec{u}-\vec{v}), \vec{n}(\vec{v})\rangle \vec{f}(\vec{v}) +$$

$$+ [(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})] f_0(\vec{v}) + \left[[(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})] \times \vec{f}(\vec{v}) \right] \Big\} d\Gamma_{\vec{v}} . \quad (39)$$

Separating the scalar and vector parts we get:

$$\mathrm{Sc}(K_{\mathbb{H}}[f](\vec{u})) = \frac{1}{4\pi}\int_{\Gamma} \frac{1}{|\vec{u}-\vec{v}|^3} \{- \langle (\vec{u}-\vec{v}), \vec{n}(\vec{v})\rangle f_0(\vec{v}) -$$

$$- \left\langle [(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})], \vec{f}(\vec{v}) \right\rangle \Big\} d\Gamma_{\vec{v}} ,$$

$$\mathrm{Vec}(K_{\mathbb{H}}[f](\vec{u})) = \frac{1}{4\pi}\int_{\Gamma} \frac{1}{|\vec{u}-\vec{v}|^3} \Big\{ - \langle (\vec{u}-\vec{v}), \vec{n}(\vec{v})\rangle \vec{f}(\vec{v}) +$$

$$+ [(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})] f_0(\vec{v}) + \left[[(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})] \times \vec{f}(\vec{v}) \right] \Big\} d\Gamma_{\vec{v}} ,$$

which for a vector field $f = \vec{f}$, i.e., $f_0 = 0$, become

$$\mathrm{Sc}(K_{\mathbb{H}}[\vec{f}](\vec{u})) = \frac{1}{4\pi}\int_{\Gamma} \frac{1}{|\vec{u}-\vec{v}|^3} \Big\{ - \left\langle [(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})], \vec{f}(\vec{v}) \right\rangle \Big\} d\Gamma_{\vec{v}} ,$$

$$\mathrm{Vec}(K_{\mathbb{H}}[\vec{f}](\vec{u})) = \frac{1}{4\pi}\int_{\Gamma} \frac{1}{|\vec{u}-\vec{v}|^3} \Big\{ - \langle (\vec{u}-\vec{v}), \vec{n}(\vec{v})\rangle \vec{f}(\vec{v}) +$$

$$+ \left[[(\vec{u}-\vec{v}) \times \vec{n}(\vec{v})] \times \vec{f}(\vec{v}) \right] \Big\} d\Gamma_{\vec{v}} .$$

We see that $K_{SI}[\vec{f}]$ in (13) coincides with $\mathrm{Vec}(K_{\mathbb{H}}[\vec{f}])$ and besides we want $\mathrm{Sc}(K_{\mathbb{H}}[\vec{f}])$ to be identically zero. This explains the condition (12) and the fact that $K_{SI}[\vec{f}]$ is an SI-vector field: this is because the quaternionic Cauchy-type integral of a vector field $\vec{f}$ is an MT-hyperholomporphic function and since its scalar part is identically zero then it is an SI-vector field.

Restricting our consideration to the unit sphere again, the action of the singular integral operator in vectorial terms takes the form:

$$S_{\mathbb{H}}[f](\vec{x}) = \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} n(\vec{y}) f(\vec{y}) d\Gamma_{\vec{y}} =$$

$$= \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{1}{|\vec{x}-\vec{y}|^3} \Big\{ (1-\langle \vec{x},\vec{y}\rangle)\, f_0(\vec{y}) - \Big\langle [\vec{x}\times\vec{y}]\,, \vec{f}(\vec{y}) \Big\rangle +$$

$$+ (1-\langle \vec{x},\vec{y}\rangle)\, \vec{f}(\vec{y}) + [\vec{x}\times\vec{y}]\, f_0(\vec{y}) + \Big[[\vec{x}\times\vec{y}] \times \vec{f}(\vec{y}) \Big] \Big\}\, d\Gamma_{\vec{y}}\,.$$

Condition (12) implies the following: for $\vec{x}\in\mathbb{S}^2$

$$0 = \int_{\mathbb{S}^2} \frac{1}{|\vec{x}-\vec{y}|^3} \Big\langle [\vec{x}\times\vec{y}]\,, \vec{f}(\vec{y}) \Big\rangle\, d\Gamma_{\vec{y}}\,, \tag{40}$$

giving also the condition $\mathrm{Sc}(S_{\mathbb{H}}[f]) = 0$, thus the following integral defines in fact the singular integral operator for SI-vector fields:

$$S_{SI}[\vec{f}](\vec{x}) := \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{1}{|\vec{x}-\vec{y}|^3} \Big\{ (1-\langle \vec{x},\vec{y}\rangle)\, \vec{f}(\vec{y}) + \Big[[\vec{x}\times\vec{y}] \times \vec{f}(\vec{y}) \Big] \Big\}\, d\Gamma_{\vec{y}}. \tag{41}$$

6.4. Now we are in a position to prove Theorem 5. By (38), for SI-vector fields $f = \vec{f}$ we get the following equality:

$$\begin{aligned} S_{\mathbb{H}}[\vec{f}] &= M^0_{\mathbb{S}^2}[\vec{f}] + i_1\mathcal{H}^1_{\mathbb{S}^2}[\vec{f}] + i_2\mathcal{H}^2_{\mathbb{S}^2}[\vec{f}] + i_3\mathcal{H}^3_{\mathbb{S}^2}[\vec{f}] = \\ &= \big(\mathcal{H}^1_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_3]\big) + \\ &\quad + \big(M^0_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2]\big)\, i_1 + \\ &\quad + \big(M^0_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3]\big)\, i_2 + \\ &\quad + \big(M^0_{\mathbb{S}^2}[f_3] + \mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1]\big)\, i_3. \end{aligned} \tag{42}$$

We have that (40) is satisfied and can be be rewritten as

$$0 = \mathcal{H}^1_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_3].$$

In addition, recalling the condition (35) we obtain

$$\begin{aligned} \vec{f} &= \big(M^0_{\mathbb{S}^2}[f_1] + \mathcal{H}^2_{\mathbb{S}^2}[f_3] - \mathcal{H}^3_{\mathbb{S}^2}[f_2]\big)\, i_1 + \\ &\quad + \big(M^0_{\mathbb{S}^2}[f_2] + \mathcal{H}^3_{\mathbb{S}^2}[f_1] - \mathcal{H}^1_{\mathbb{S}^2}[f_3]\big)\, i_2 + \\ &\quad + \big(M^0_{\mathbb{S}^2}[f_3] + \mathcal{H}^1_{\mathbb{S}^2}[f_2] - \mathcal{H}^2_{\mathbb{S}^2}[f_1]\big)\, i_3. \end{aligned} \tag{43}$$

Therefore, separating the coefficients of the imaginary units i_1, i_2, i_3 in (43) we get the Hilbert formulas (19). Theorem is proved.

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