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O. Martio

(University of Helsinki, Finland)

Solution classes for Riccati type equations

olli.martio@acadsci.fi

Dedicated to memory of Professor Promarz M. Tamrazov

Sufficient conditions in terms of $p > 1$ and $q \geq 0$ are found so that all p–superharmonic, or locally renormalized, solutions of the Riccati equation (*) $-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = |\nabla u|^q$ are ordinary weak solutions. Examples are constructed to show that (*) admits p–superharmonic solutions which are not weak solutions.

1. Introduction. The quasilinear Riccati equation

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) = |\nabla u|^q, \tag{1}$$

$p > 1$ and $q \geq 0$, admits several classes of solutions: very weak solutions, renormalized solutions, p-superharmonic solutions, weak solutions, smooth solutions etc. Usually the function space to which the solution belongs determines the class but there are other subtle differences, for example in which sense the gradient ∇u of a solution u is understood. In this paper we study p-superharmonic and weak solutions to a slightly more general equation

$$-\nabla \cdot A(x, \nabla u) = |\nabla u|^q \tag{2}$$

where $A(x,h) \cdot h \approx |h|^p$. Since weak solutions are also A-superharmonic solutions, our goal is to find such values for p and q that all A-superharmonic solutions are ordinary weak solutions. It seems that the structure of the polar set $\{u(x) = \infty\}$ of a solution u has an effect on this problem. By a recent result of Kilpeläinen–Kuusi–Tuhola–Kujanpää, see [1], locally renormalized solutions of (2) are A-superharmonic and hence these solutions are included in our study.

The Riccati equation (2) is a special case of the equation

$$-\nabla \cdot A(x, \nabla u) = \mu. \tag{3}$$

where μ is a non–negative Radon measure and the theory developed for (3) can be used to study the solutions of (2), see [2 — 4]. However, there are important differences since in (2) the Radon measure μ depends on the solution itself and the exponent q effects the function space to which the solutions belong.

After preliminaries in Section 2 we present the main results in Section 3. Explicit examples are constructed in Section 4.

2. Preliminaries. Throughout the paper we assume that $p > 1$ and that Ω is an open set in $\mathbf{R}^n$, $n \geq 2$. The space $W^{1,p}(\Omega)$ is the first order Sobolev space whose functions together with their distributional gradients are L^p–integrable in Ω; $W^{1,p}_{loc}(\Omega)$ stands for the corresponding local space.

We use the following standard assumptions. The mapping $A : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is a Caratheodory function such that for some $0 < \alpha \leq \beta < \infty$, all $h \in \mathbf{R}^n$ and a.e. $x \in \Omega$

$$A(x, h) \cdot h \geq \alpha |h|^p, \tag{4}$$

$$|A(x, h)| \leq \beta |h|^{p-1}, \tag{5}$$

$$(A(x, h_1) - A(x, h_2)) \cdot (h_1 - h_2) > 0 \tag{6}$$

whenever $h_1 \neq h_2$.

A solution $u \in W^{1,p}_{loc}(\Omega)$ of

$$\nabla \cdot A(x, \nabla u) = 0 \tag{7}$$

is a continuous version, called *A-harmonic*, of a weak solution u satisfying

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx = 0 \tag{8}$$

for all $\varphi \in C_0^\infty(\Omega)$. A function $u \in W^{1,p}_{loc}(\Omega)$ is an *A-supersolution* if

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx \geq 0 \tag{9}$$

for all nonnegative $\varphi \in C_0^\infty(\Omega)$. A lower semicontinuous function u in Ω is said to be A-*superharmonic* if u is not identically $+\infty$ in any component of Ω and for all open sets $D \subset\subset \Omega$ and for all $h \in C(\overline{D})$, A-harmonic in D, $h \le u$ on ∂D yields $h \le u$ in D. Instead of this definition we mainly employ an equivalent condition. A lower semicontinuous function u in Ω, not identically $+\infty$ in any component of Ω, is A-superharmonic if and only if for every $k \in \mathbf{R}$, $u_k = \min(u, k)$ is an A-supersolution. Note that u_k belongs to $W^{1,p}_{loc}(\Omega)$ and that every A-supersolution has a lower semicontinuous version and this is A-superharmonic. We shall always use the pointwise defined versions of A-superharmonic functions and A-supersolutions. For the theory of A-superharmonic functions see [2, Chapter 7].

It may happen that the distributional derivative of an A-superharmonic function u is not a function. However, this drawback can be partly repaired using the limit process $u = \lim_{k\to\infty} u_k$ as follows. Let $\nabla u = \lim_{k\to\infty} \nabla u_k$. Then ∇u defines a weak gradient of u as a function and this coincides to the usual distributional gradient if the latter exists as a function. Only for $p \le 2 - 1/n$ there are A-superharmonic functions who do not belong to $W^{1,1}_{loc}(\Omega)$ and then the weak gradient need not be the distributional gradient of u. We shall always use the weak gradient of u unless otherwise stated. Note that for an A-superharmonic function u, $|\nabla u|^{p-1}$ and $A(x, \nabla u)$ belong to $L^s_{loc}(\Omega)$ for $1 \le s < n/(n-1)$, see [2, 7.42].

If u is A-superharmonic in Ω, then there is a unique non–negative Radon measure $\mu = \mu(u)$, called the *Riesz measure* or the *Riesz mass* of u, such that

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\mu \tag{10}$$

for every $\varphi \in C_0^\infty(\Omega)$, see [2, p. 281 and Chapter 21]. For $s > 0$ we write $L^s(\mu, U)$ for the set of q–integrable functions in U with respect to the measure μ and abbreviate $L^s(U) = L^s(m, U)$ for the Lebesgue measure m.

Let $q \ge 0$. A function $u \in W^{1,s}_{loc}(\Omega)$, $s = \max(p, q)$, is a *weak solution* of the Riccati equation (2) if

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \varphi |\nabla u|^q \, dx \tag{11}$$

for every $\varphi \in C_0^\infty(\Omega)$, i.e. u is an A-supersolution of (10) with $d\mu = |\nabla u|^q \, dx$.

Let $t = \max(p-1, q)$. A function u is an *A-superharmonic solution* of (2) if u is A-superharmonic, $\nabla u \in L^t_{loc}(\Omega)$ and equation (11) holds. In general, the weak derivative is now used on both sides of (11) if $t \le 2-1/n$.

A weak solution of (2) is an A-superharmonic solution and our purpose is to find those values p and q that the converse is true. Note that for $q \ge p$ or $p > n$ every A-superharmonic solution is a weak solution. For $q \ge p$ this is clear and for $p > n$ A-superharmonic functions are continuous A-supersolutions, see [2, Theorem 7.25]. Note that an A-superharmonic solution of (2) which is not a weak solution is always locally unbounded since locally bounded A-superharmonic functions are A-supersolutions.

In [4, Theorems 3.1 and 3.4] the A-supersolution property of an A-superharmonic function u was investigated in terms of the Riesz mass μ of u.

Lemma 1. *Suppose that $1 < p \le n$. Then an A-superharmonic function u in Ω is an A-supersolution if and only if $u \in L^1_{loc}(\mu, \Omega)$ and $u|\nabla u|^{p-1} \in$ $\in L^1_{loc}(\Omega)$. If $p > n-1$ or*

$$\limsup_{x \to y} u(x) < \infty \tag{12}$$

for every $y \in \partial\Omega$, then the condition $u \in L^1_{loc}(\mu, \Omega)$ alone suffices.

In the boundary condition (12) the point ∞ is considered as a boundary point of Ω if Ω is unbounded.

3. A-superharmonic versus weak solutions to the Riccati equation. We first derive from Lemma 1 necessary and sufficient conditions that an A-superharmonic solution of the Riccati equation (2) is an ordinary weak solution. As remarked above we need to consider the values $1 < p \le n$ and $0 \le q < p$ only.

Lemma 2. *Suppose that $1 < p \le n$. Then an A-superharmonic solution u of (2) in Ω is a weak solution of (2) if and only if one of the conditions below is satisfied.*

$$0 \le q \le p-1,\ p \le n-1 \text{ and } u|\nabla u|^{p-1} \in L^1_{loc}(\Omega), \tag{13}$$

$$p-1 \le q < p,\ p \le n-1 \text{ and } u|\nabla u|^{q} \in L^1_{loc}(\Omega), \tag{14}$$

$$0 \le q < p,\ p > n-1 \text{ and } u|\nabla u|^{q} \in L^1_{loc}(\Omega). \tag{15}$$

Moreover, if an A-superharmonic solution u of (2) *in* Ω *satisfies the boundary condition* (12), *then for* $0 \le q < p$, *u is a weak solution if and only if* $u|\nabla u|^q \in L^1_{loc}(\Omega)$.

Proof. In the case of (13) the condition $u|\nabla u|^q \in L^1_{loc}(\Omega)$ follows from $u|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$. Since $d\mu = |\nabla u|^q dx$, Lemma 1 implies the result. A similar reasoning applies to (14) and (15). The rest of the proof follows from the boundary condition part of Lemma 1.

In Lemma 2 the conditions for $u \in W^{1,p}_{loc}(\Omega)$ depend on u itself. In the following we present conditions depending on p, q and n only. The results are based on the integrability properties of A-superharmonic functions.

Theorem 1. *Let* $1 < p \le n$ *and* $0 \le q < p$. *Suppose that u is an A-superharmonic solution of the Riccati equation* (2) *in* Ω. *If either* (16) *or* (17) *below is satisfied, then u is a weak solution of* (2).

$$n \ge 4,\ (n+1)/2 < p \le n-1,\ 0 \le q \le \frac{p(n+1)-2n}{n-1}\,. \tag{16}$$

$$n \ge 2,\ \max(\frac{4}{3}, n-1) < p \le n,\ 0 \le q < \frac{p(n+1)-2n}{n-1}\,. \tag{17}$$

Proof. Let $n \ge 2$, $0 \le q \le p-1$ and $p \le n-1$. By (13) in Lemma 2 we need to show that $u|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$. Since an A-superharmonic function belongs to $L^t_{loc}(\Omega)$ for $0 < t < n(p-1)/(n-p)$ and its (weak) gradient to $L^\eta_{loc}(\Omega)$ for $0 < \eta < n(p-1)/(n-1)$, see [2, p. 154], we can use the Hölder inequality for $1 < t < n(p-1)/(n-p)$ to obtain

$$\int_C |u||\nabla u|^{p-1}\,dx \le (\int_C |u|^t\,dx)^{1/t}(\int_C |\nabla u|^{t(p-1)/(t-1)}\,dx)^{(t-1)/t}$$

where $C \subset \Omega$ is compact. If now

$$\frac{t(p-1)}{t-1} < \frac{n(p-1)}{n-1}\,, \tag{18}$$

then $u|\nabla u|^{p-1} \in L^1(C)$. Inequality (18) gives $t > n$ and since $t < n(p-1)/(n-p)$ we end at the condition $p > (n+1)/2$. Note that $p > (n+1)/2$ also gives $n(p-1)/(n-p) > 1$ for $n \ge 2$ and we can choose $t > 1$.

In (16) we have $(n+1)/2 < n-1$ and hence $n \ge 4$. We have proved that if

$$n \ge 4,\ (n+1)/2 < p \le n-1,\ 0 \le q \le p-1 \tag{19}$$

holds, then u is a weak solution.

Next let $p-1 < q$. By Lemma 2 (14) we need to show $u|\nabla u|^q \in L^1_{loc}(\Omega)$. We use the same method and notation as in the previous proof. If $1 < t < n(p-1)/(n-p)$, then

$$\int_C |u||\nabla u|^q \, dx \le (\int_C |u|^t \, dx)^{1/t} (\int_C |\nabla u|^{tq/(t-1)} \, dx)^{(t-1)/t}$$

and hence $u|\nabla u|^q \in L^1(C)$ provided that $tq/(t-1) < n(p-1)/(n-1)$. This yields

$$t > \frac{n(p-1)}{n(p-1) - q(n-1)} \tag{20}$$

where $q < n(p-1)/(n-1)$ has also been used. Now $t < n(p-1)/(n-p)$ gives

$$q < \frac{p(n+1) - 2n}{n-1} \, .$$

Note that this also implies $q < n(p-1)/(n-1)$ and that

$$p \le n-1, \; p-1 < \frac{p(n+1) - 2n}{n-1}$$

yield $n \ge 4$. Thus (16) follows from this and (19).

Next let $n-1 < p \le n$. By Lemma 2 (15) we need to show $u|\nabla u|^q \in L^1_{loc}(\Omega)$ and the proof is similar to the previous case. The method leads to

$$0 \le q < \min\left(\frac{n}{n-1}(p-1), \frac{p(n+1) - 2n}{n-1}\right)$$

and since $n(p-1) \ge p(n+1) - 2n$ for $p \le n$ we end at (17). Note that the condition $p > 2n/(n+1)$ is needed for $t > 1$ but this holds for $p > n-1$ if $n \ge 3$ and for $p > 4/3$ if $n = 2$.

Theorem 1 has been proved.

If an A-superharmonic solution u satisfies the boundary condition (12), then we can employ the proof for the case (17) above to obtain

Theorem 2. *Suppose that u is A-superharmonic solution of* (2) *in* Ω *and u satisfies* (12), *then for*

$$\frac{2n}{n+1} < p \le n, \; 0 \le q < \frac{p(n+1) - 2n}{n-1} \tag{21}$$

u is a weak solution of (2).

Theorems 1 and 2 do not work for p close to 1. However, there is another method, based on the Wolff potential, which produces a better result for $p \le \sqrt{n}$.

Theorem 3. *Let $1 < p \le n$ and $0 \le q < p$. Suppose that u is A-superharmonic solution of* (2) *in Ω and either $p > n-1$ or u satisfies the boundary condition* (12), *then for*

$$0 \le q < \frac{n(p-1)^2}{(n-1)p} \tag{22}$$

u is a weak solution of (2).

Remark 1. Note that

$$\frac{n(p-1)^2}{(n-1)p} \le p-1$$

for $1 < p \le n$. Hence Theorem 3 does not apply to the case (14) in Lemma 2.

Proof for Theorem 3. In [4, Lemma 2.1] it is shown that the condition $u \in L^1_{loc}(\mu, \Omega)$ for non-negative A-superharmonic solutions of (3) is equivalent to the condition that each $x_o \in \Omega$ has $r > 0$ such that

$$\int_{B(x_o,r)} W_\mu(x,r)\, d\mu(x) < \infty. \tag{23}$$

Here $W_\mu(x,r)$ is the Wolff potential of μ defined for $x \in \Omega$ and $0 < r < d(x, \partial\Omega)$ as

$$W_\mu(x,r) = \int_0^r \left(\frac{\mu(B(x,t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t}\,. \tag{24}$$

Since we may assume that the A-superharmonic solution of the Riccati equation is non-negative, we can use this result to conclude that $u|\nabla u|^q \in L^1_{loc}(\Omega)$. Thus we can use this criterion in the cases (14) and (15) of Lemma 2 as well as when the boundary condition applies.

To this end let $x_o \in \Omega$ and choose $r > 0$ such that $B(x_o, 3r) \subset \Omega$. Set $\nabla u = 0$ in $\mathbf{R}^n \setminus B(x_o, 2r)$ and let

$$M_f(x) = \sup_{t>0} \frac{1}{m(B(x,t))} \int_{B(x,t)} |f(y)|\, dy$$

denote the Hardy–Littlewood maximal function of $f \in L^1(\mathbf{R}^n)$. For $d\mu = |\nabla u|^q dx$ and $x \in B(x_o, r)$ we obtain

$$W_\mu(x,r) = \int_0^r \left(\frac{\mu(B(x,t))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t} =$$

$$= \int_0^r \left(\frac{1}{t^n}\int_{B(x,t)} |\nabla u|^q \, dy\right)^{1/(p-1)} t^{1/(p-1)}\, dt \le$$

$$\le c\int_0^r M_{|\nabla u|^q}(x)^{1/(p-1)} t^{1/(p-1)}\, dt \le cM_{|\nabla u|^q}(x)^{1/(p-1)}$$

where c depends only on n, p and r. Since $|\nabla u|^q \le M_{|\nabla u|^q}$ a.e., we get

$$\int_{B(x_o,r)} W_\mu(x,r)\, d\mu(x) = \int_{B(x_o,r)} W_\mu(x,r)|\nabla u|^q\, dx \le$$

$$\le c\int_{B(x_o,r)} M_{|\nabla u|^q}(x)^{p/(p-1)}\, dx < \infty$$

provided that $|\nabla u|^q \in L^{p/(p-1)}(\mathbf{R}^n)$ because then by the Hardy–Littlewood maximal function function theorem, see e.g. [5, p. 5], $M_{|\nabla u|^q} \in L^{p/(p-1)}(\mathbf{R}^n)$.

Since an A-superharmonic solution u to (2) has the property that $|\nabla u| \in L^s_{loc}(\Omega)$ for $s < n(p-1)/(n-1)$, we see that for $qp/(p-1) < n(p-1)/(n-1)$, or in other words

$$q < \frac{n(p-1)^2}{(n-1)p}\,,$$

$u|\nabla u|^q \in L^1_{loc}(\Omega)$ as required. The theorem follows.

Since

$$\frac{n(p-1)^2}{(n-1)p} \ge \frac{p(n+1)-2n}{n-1}$$

for $p \le \sqrt{n}$, we obtain from Theorems 2 and 3 the following corollaries.

Corollary 1. *An A-superharmonic solution u of (2) in $\Omega \subset \mathbf{R}^2$ is a weak solution of (2) if*

$$0 \le q < \begin{cases} \frac{2(p-1)^2}{p}, & 1 < p \le \sqrt{2}, \\ 3p-4, & \sqrt{2} < p \le 2. \end{cases}$$

Remark 2. Theorem 3 does not improve the case (17) in Theorem 1 for $n \geq 3$.

Since $2n/(n+1) < \sqrt{n}$ for $n \geq 2$, Theorems 2 and 3 yield

Corollary 2. *If an A-superharmonic solution u of* (2) *in* $\Omega \subset \mathbf{R}^n$ *satisfies the boundary condition* (12), *then u is a weak solution of* (2) *provided that*

$$0 \leq q < \begin{cases} \frac{n(p-1)^2}{(n-1)p}, & 1 < p \leq \sqrt{n}, \\ \frac{p(n+1)-2n}{n-1}, & \sqrt{n} < p \leq n. \end{cases}$$

4. Examples. In order to show that a function u is an A-superharmonic solution of (2) in Ω but not a weak solution of the same equation one has to show that $u_k = \min(u, k)$ is A-superharmonic for each $k \in \mathbf{R}$, u satisfies equation (11) and $u \notin W^{1,p}_{loc}(\Omega)$. In the extreme case, i.e. when the polar set $\{u(x) = \infty\}$ consists of a single point or, more generally, of a compact set of zero n–capacity, the following lemma is handy for constructing examples.

Lemma 3. *Let $1 < p \leq n$ and $0 \leq q < p$. Suppose that $u : \Omega \to \mathbf{R} \cup \{+\infty\}$ is a continuous function such that u satisfies the boundary condition* (12) *and u is a weak solution of* (2) *in $\Omega \setminus C$ where $C = \{u = \infty\}$ is a compact set of zero n–capacity in Ω. If $\nabla u \in L^q_{loc}(\Omega) \setminus L^p_{loc}(\Omega)$, and*

$$q \geq \frac{n}{n-1}(p-1), \tag{25}$$

then u is an A-superharmonic solution of (2) *in Ω but not a weak solution.*

Proof. Since $u \notin W^{1,p}_{loc}(\Omega)$, u is not a weak solution of (2).

Next we show that u is A-superharmonic in Ω. Let $u_k = \min(u, k)$, $k \in \mathbf{R}$. Since $u_k \in C(\Omega)$, the set $\Omega_k = \{u < k\}$ is open and (11) holds for all $\varphi \in C^\infty_0(\Omega_k)$. Note that the gradient ∇u in Ω_k is defined as a usual distributive gradient of $u_k \in W^{1,p}_{loc}(\Omega_k)$. Thus $u = u_k$ is an A-supersolution in Ω_k. Since u_k is continuous in Ω, the Pasting Lemma [2, Pasting lemma p. 134] implies that u_k is A-superharmonic in Ω and thus an A-supersolution there because $u_k \in W^{1,p}_{loc}(\Omega)$. Since $u = \lim_{k\to\infty} u_k$ and not ∞ in any component of Ω, u is A-superharmonic in Ω, see [2, Chapter 7].

It remains to show that u satisfies (11) in Ω. Fix $\varphi \in C^\infty_0(\Omega)$ and let $\varphi_i \in C^\infty_0(\mathbf{R}^n)$ be a sequence of functions such that $0 \leq \varphi_1 \leq \varphi_2 \leq \ldots \leq 1$,

$spt\,\varphi_i \cap C = \emptyset$, $\varphi_i(x) \to 1$ for each $x \in \mathbf{R}^n \setminus C$ and $\nabla\varphi_i \to 0$ in $L^n(\mathbf{R}^n)$. This is possible because C is of zero n-capacity.

Now $\varphi_i\varphi \in C_0^\infty(\Omega \setminus C)$ and since u is a weak solution of (2) in $\Omega \setminus C$ we obtain

$$\int_{\Omega\setminus C} A(x,\nabla u)\cdot\nabla(\varphi_i\varphi)\,dx = \int_{\Omega\setminus C} \varphi_i\varphi|\nabla u|^q\,dx.$$

Since $m(C) = 0$, see [2, Lemma 2.10], we have

$$\int_{\Omega\setminus C} \varphi_i\varphi|\nabla u|^q\,dx \to \int_\Omega \varphi|\nabla u|^q\,dx$$

by the Lebesgue dominated convergence theorem. On the other hand

$$\int_{\Omega\setminus C} A(x,\nabla u)\cdot\nabla(\varphi_i\varphi)\,dx = \int_\Omega A(x,\nabla u)\cdot\nabla\varphi_i\,\varphi\,dx + \int_\Omega A(x,\nabla u)\cdot\nabla\varphi\,\varphi_i\,dx$$

and for $i \to \infty$ we get

$$\int_\Omega A(x,\nabla u)\cdot\nabla\varphi\,\varphi_i\,dx \to \int_\Omega A(x,\nabla u)\cdot\nabla\varphi\,dx$$

and

$$|\int_\Omega A(x,\nabla u)\cdot\nabla\varphi_i\,\varphi)\,dx| \le c\int_{spt\varphi} |\nabla u|^{p-1}|\nabla\varphi_i|\,dx$$

$$\le c\left(\int_{spt\varphi} |\nabla u|^{n(p-1)/(n-1)}\,dx\right)^{(n-1)/n}\left(\int_\Omega |\nabla\varphi_i|^n\,dx\right)^{1/n} \to 0 \qquad (26)$$

where $c < \infty$ depends only on the structure constant β and φ. By (25) the first integral in (26) is finite and the second integral converges to 0. Thus (11) holds and the lemma follows.

The p-harmonic Riccati equation (1) takes the form

$$|u'|^{p-2}((p-1)u'' + \frac{n-1}{r}u') = -|u'|^q \qquad (27)$$

in the spherical coordinates of $\mathbf{R}^n$; here $u = u(r)$, $r > 0$. We are looking for p-superharmonic solutions of (1) depending only on r in $B(0,1)$ with a singularity at 0.

Assume $p-1<q<p<n$ and let $v=u'$. For v equation (27) reads as

$$(p-1)v' + \frac{n-1}{r}v = -|v|^{q-p+2} \tag{28}$$

and we seek a solution $v=v(r)=cr^{\alpha}$, i.e. the solution u has the form

$$u(r) = \frac{c}{\alpha+1}(r^{\alpha+1}-1)$$

where $c<0$ and $\alpha+1<0$. A computation gives

$$\alpha = -\frac{1}{q-p+1} < 0$$

and

$$c = -[(p-1)\alpha + n - 1]^{-\alpha}$$

provided that $q > n(p-1)/(n-1)$. The last condition also yields $\nabla u \in L^q(B(0,1))$ and the condition (25) of Lemma 3. On the other hand $\nabla u \notin L^p(B(0,1))$ if $q \le \frac{n+1}{n}p - 1$. Hence by Lemma 3 for

$$\frac{n}{n-1}(p-1) < q \le \frac{n+1}{n}p - 1 \tag{29}$$

u is a p-superharmonic solution of (1) in $B(0,1)$ that is not a weak solution. Note that there is a gap on the values between the conditions in (29) and in Corollary 2.

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