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Functions on anisotropic spaces: points of local extremum

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Dedicated to memory of Professor Promarz M. Tamrazov

Below we introduce concepts of partial derivatives of functions on anisotropic spaces and prove necessary conditions of the local extremum extending criterions of the classical analysis.

1. Anisotropic space.

1.1. Let $\mathcal{X}$ be a nonempty set and let $r : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function with the following properties:

$$\begin{array}{ll} \alpha) & r(x,x) = 0 \text{ and } r(x,y) \geq 0 \text{ for all } x, y \in \mathcal{X}; \\ \beta) & r(x,y) \leq r(x,z) + r(z,y) \text{ for all } x, y, z \in \mathcal{X}. \end{array}$$

The pair $(\mathcal{X}, r)$ is called *anisotropic space,* and the function r is called *anisotropic metric*. Note that we do not assume here the symmetry of the anisotropic metric r, i.e. in general $r(x,y) \neq r(y,x)$.

Special cases of anisotropic spaces are pseudometric and metric spaces (see, for example, [1, §21]).

For other examples of anisotropic spaces arising in the theory of abstract surfaces see, for example, [2, Ch. 1].

Let $a \in \mathcal{X}$ and $\varepsilon > 0$ be a real number. Define ε-neighbourhood of a putting

$$O(a,\varepsilon) = \{x \in \mathcal{X} : \rho(a,x) < \varepsilon\} \quad (or \;\; O(a,\varepsilon) = \{x \in \mathcal{X} : \rho(x,a) < \varepsilon\})$$

and by standard way we define the basis topological concepts for anisotropic spaces.

1.2. Recall that the concept *anisotropy* (in Greek ánisos — unequal and trópos — direction) means the dependence of some properties of objects from directions. Thus, the space can be even metric, but to be *anisotropic*.

Simplest examples of anisotropic spaces are surfaces $\Omega = (D, ds_\Omega^2)$, where D is a domain in the Euclidean space $\mathbb{R}^n$ and ds_Ω is a length element, defined by the relation

$$ds_\Omega^2 = \sum_{k=1}^{n} g_k \, dx_k^2, \quad g_k \equiv \text{const} \quad (k = 1, 2, \ldots, n),$$

where coefficients g_k are not equal among themselves.

Anisotropic surfaces $\Omega = (D, ds_\Omega^2)$ with length elements

$$ds_\Omega^2 = \sum_{i,j=1}^{n} g_{ij}(x) \, dx_i \, dx_j,$$

where $g_{ij}(x)$, $i, j = 1, 2, \ldots, n$, are Lebesgue measurable functions, arise, for example, as graphs of locally Lipschitz functions $x_{n+1} = f(x_1, x_2, \ldots, x_n)$.

If the length element ds_Ω is defined on D by the relation

$$ds_\Omega^2 = \lambda^2(x) \sum_{k=1}^{n} dx_k^2,$$

then the variables $x_1, x_2, \ldots, x_n$ are called *isotermal coordinates* on the surface Ω.

With respect to existence isothermal coordinates on surfaces see, for example, [3].

1.3. Let $\mathcal{M}$ be an n-dimensional Riemannian C^2-manifold. As in [2, Ch. 1] for two-dimensional case, we define an *abstract surface* over a domain $D \subset \mathcal{M}$ by presetting the length element of curves lying on it, and the area element.

Let $\Gamma(D)$ be the set of all Jordan arcs or curves $\gamma \subset D$ locally rectifiable (with respect to the metric of $\mathcal{M}$). We will assume that along every $\gamma \in \Gamma(D)$ there is defined a direction. Every closed rectifiable arc γ can be given in the form $m = m(s) : [0, \text{length}\,(\gamma)] \to D$, where

$0 \le s \le \text{length}\,(\gamma)$ is the length of the arc, counting off the start point $m(0)$ up to the moving point $m(s)$ with respect to the direction along γ. Locally rectifiable curves γ can be parametrized evidently with length of the arc, counting off a fixed point in positive or negative directions along γ.

Suppose that along every $\gamma \in \Gamma(D)$ it is given some nonnegative, Lebesgue measurable function $h_\gamma(m)$. The set of all such functions for the arcs $\gamma \in \Gamma(D)$, we will designate by the symbol $\mathcal{H} = \{h_\gamma\}$.

We will say that the set of functions $\mathcal{H}$ is *coordinated* at the point $a \in D$, if for all curves $\gamma \in \Gamma(D)$, passing through the point a in the same direction ξ (i.e. having the same tangent vector $\xi \in T_a(\mathcal{M})$, $|\xi| = 1$) at a, values $h_\gamma(a)$ coincide.

Suppose that the set of functions $\mathcal{H}$ is coordinated almost every on the domain D. Thus, for almost everywhere $m \in D$ and all directions $\xi \in T_m(\mathcal{M})$, $|\xi| = 1$, there is defined an nonegative function $H(m, \xi)$. Extend H with respect to the second variable onto the all space $T_m(\mathcal{M})$, using the following rule $H(m, \lambda\,\xi) = \lambda\, H(m, \xi)\,, \quad \lambda \ge 0\,$. As the result of such extension, along every $\gamma \in \Gamma(D)$ we have everywhere

$$H(m, \overrightarrow{ds}_{\mathcal{M}}) = h_\gamma(m)\, |\overrightarrow{ds}_{\mathcal{M}}|\,. \tag{1.1}$$

Here $\overrightarrow{ds}_{\mathcal{M}}$ is a vector of length $ds_{\mathcal{M}}$ on $T_m(\mathcal{M})$ with its beginning at the point m.

Fix arbitrarily an nonnegative function $\sigma(m)$ defined almost everywhere and Lebesgue measurable on D.

An arbitrary triple $\Omega = (D, H, \sigma)$ of the described form is called the *abstract surface*.

The quantity

$$ds_\Omega = h_\gamma(m(s))\, |ds_{\mathcal{M}}|\,, \tag{1.2}$$

is called the *length element* of $\gamma \in \Gamma(D)$ at the point $m \in D$, and the quantity

$$d\Omega = \sigma(m)\, *\mathbf{1}_{\mathcal{M}} = \sigma(m)\, d\mathcal{H}^n_{\mathcal{M}} \tag{1.3}$$

is called the *area element* of Ω.

Here by $*\mathbf{1}_{\mathcal{M}}$ we denote the volume form on the manifold $\mathcal{M}$.

The metric (1.2) is a Finsler metric (see [4, 5]).

Let $\Omega = (D, H, \sigma)$ be an abstract surface. For an arbitrary oriented, locally rectifiable arc (or curve) $\gamma \subset D$, the length of γ with respect to the metric (1.2) is given by the following formula

$$\text{length}_\Omega \gamma = \int\limits_\gamma H(m, \overrightarrow{ds}_{\mathcal{M}})\,. \tag{1.4}$$

Observe, that in the general case, the $\text{length}_\Omega \gamma$ depends on the direction, choose on γ, and the metric ds_Ω, defined by the relation (1.2), is *anisotropic.*

Let $\Omega = (D, H, \sigma)$ be an abstract surface. Below we will need a function dual to the function $H(x, \xi)$:

$$G(x, \eta) = \sup_{\xi \in \Xi(x)} \langle \xi, \eta \rangle, \tag{1.5}$$

where $\Xi(x) = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$ and $\langle \xi, \eta \rangle$ is the standard scalar product of vectors ξ and η on $T_x(\mathcal{M})$.

We put $G^+(x) = \sup_{|\eta|=1} \sup_{G(x,\xi)=1} \langle \xi, \eta \rangle$.

It is not difficult to prove that the function $G(x, \eta)$ satisfies the conditions: $G(x, \eta) \geq 0$ and for an arbitrary $x \in D$ the set $\{\eta \in T_x(\mathcal{M}) : H(x, \eta) < 1\}$ is convex. Moreover, everywhere on D the following property holds

$$G(x, \xi) = \sup_{\eta : H(x,\eta) \neq 0} \frac{\langle \xi, \eta \rangle}{H(x, \eta)} \tag{1.6}$$

(see [6]).

In general, the function $G(x, \eta)$ assumes on $D \times T_x(\mathcal{M})$ values from $\overline{\mathbb{R}}^1$. Infinite values $G(x, \eta)$ arise, for example, in cases, when the convex set $\Xi(x)$ is unbounded. On the other hand, it is not difficult to prove, that *the set* $\Xi(x)$ *is bounded if and only if* $G^+(x) < +\infty$.

Example 1.1. Let $(e_1, e_2, \ldots, e_n)$ be an orthonormal basis in $\mathbb{R}^n$ and let $H(x, \xi) = |\langle e_1, \xi \rangle|$. Then the set

$$\Xi(x) = \{\xi : |\langle e_1, \xi \rangle| < 1\} = \{\xi \in \mathbb{R}^n : |\xi_1| < 1\}.$$

Here the dual function has the form

$$G(x, \eta) = \begin{cases} |\eta_1|, & \text{if } \eta_i = 0,\ i = 2, 3, \ldots, n \\ +\infty, & \text{if there exists } i \geq 2 : \eta_i \neq 0 \end{cases}$$

and has infinite values. The set $\Xi(x)$ is the open interval $(-1, +1)$, lying on the axis $O\eta_1$. The function $G^+(x) \equiv +\infty$.

Example 1.2. Let k_1, $k_2 \geq 0$ are constants. Consider the function $k(\xi) : \mathbb{R}^1 \to \mathbb{R}^1$, where

$$k(\xi) = \begin{cases} k_1 \xi & \text{if} \quad \xi \geq 0\,, \\ k_2 \xi & \text{if} \quad \xi < 0\,. \end{cases}$$

Let $(a, b) \subset \mathbb{R}^1$ be an arbitrary interval and let $h(x) : (a, b) \to \mathbb{R}^1$ be an nonnegative measurable function. The function

$$H(x, \xi) = h(x)\, k(\xi) : (a, b) \times \mathbb{R}^1 \to \mathbb{R}^1$$

is homogeneous with respect to the variable ξ. The triple $\Omega = ((a, b), H, \sigma)$, where the function $\sigma(x) : (a, b) \to \mathbb{R}^1$ is nonegative and Lebesgue measurable, gives the simplest example of the abstract surface.

Example 1.3. Observe, that an arbitrary p-dimensional surface Σ, given by a locally Lipschitz vector function $f : D \subset \mathbb{R}^p \to \mathbb{R}^n\,, \quad p < n\,,$ is an abstract surface. In this case the vector function f is absolutely continuous along every locally rectifiable arc $\gamma \subset D$, described by the relations $x = x(s) : [0, \mathrm{length}\, \gamma] \to D$. Here we have

$$h_\gamma(x(s)) = \left| \frac{df(x(s))}{ds} \right| = \left(\sum_{i=1}^{n} \left| \frac{df_i(x(s))}{ds} \right|^2 \right)^{1/2}.$$

The vector function $f(x)$ has a total differential almost everywhere on the domain D. The family $\mathcal{H}$ is coordinated at every point, where $f(x)$ is differentiable, moreover

$$H(x, \xi) = \left(\sum_{i,j=1}^{p} g_{ij}(x)\, \xi_i\, \xi_j \right)^{1/2}$$

with real Lebesgue measurable coefficients

$$g_{ij}(x) = \left\langle \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_j} \right\rangle, \quad i, j = 1, 2, \ldots, p,$$

defined almost everywhere on D.

We put $g(x) = \det\,(g_{ij}(x))\,,\ \sigma(x) = \sqrt{g(x)}\,,\ d\mathcal{H}^p_\Sigma = \sqrt{g(x)}\, dx_1 \wedge \ldots \wedge$ $\wedge\, dx_p\,, \quad g^{ij}(x) = (g_{ij}(x))^{-1}\,, \quad i, j = 1, 2, \ldots, p\,,$ and

$$G(x, \xi) = \left(\sum_{i,j=1}^{p} g^{ij}(x)\, \xi_i\, \xi_j \right)^{1/2}.$$

Thus, we obtain the abstract surface $(D, H, \sqrt{g})$.

See details, for example, in [4, Ch. 1, § 8], [2, Sections 1.1 – 1.7].

2. Anisotropic metric in special coordinates.

2.1. Let D be a subdomain of $\mathbb{R}^n$ and let r be an anisotropic metric on D. We put

$$\Lambda_r(a) = \limsup_{a'\to a} \frac{r(a,a')}{|a'-a|}, \tag{2.7}$$

where $|a'-a|$ is the Euclidean distance between the points $a, a' \in D$.

For an arbitrary pair of points $a', a'' \in D$ let $\gamma(a', a'')$ denote an oriented, locally rectifiable arc in D, leading from a' to a''.

Fix arbitrarily a set of points

$$a_1, a_2, \ldots, a_k \in \gamma(a', a''),$$

following one to another in the positive direction from a' to a''. We have

$$\begin{aligned} r(a', a'') &\le r(a', a_1) + r(a_1, a_2) + \ldots + r(a_k, a'') = \\ &= \frac{r(a', a_1)}{|a'-a_1|}|a'-a_1| + \frac{r(a_1, a_2)}{|a_1-a_2|}|a_1-a_2| + \ldots + \frac{r(a_k, a'')}{|a_k-a''|}|a_k-a''| \le \\ &\le \sup_{a\in\gamma(a',a_1)} \Lambda_r(a)|a'-a_1| + \sup_{a\in\gamma(a_1,a_2)} \Lambda_r(a)|a_1-a_2| + \ldots + \\ &\quad + \sup_{a\in\gamma(a_k,a'')} \Lambda_r(a)|a_k-a''|. \end{aligned}$$

If the function $\Lambda_r(x)$ is continuous on D, then the quantity on the right side of this relation is the upper integral Darboux sum for the integral

$$\int\limits_{\gamma(a',a'')} \Lambda_r(x)|dx| = \int\limits_0^{s(\gamma)} \Lambda_r(x(s))ds,$$

where $x(s) : [0, s(\gamma)] \to \gamma(a', a'')$ is the natural parametrization of the arc $\gamma(a', a'')$. Thus, making the partition of $\gamma(a', a'')$ with points $a_1, a_2, \ldots, a_k$ sufficiently small, we obtain

Theorem 2.1. *If the quantity $\Lambda_r(x)$ is continuous on the domain D, then for an arbitrary pair of points $a', a'' \in D$ the following property holds*

$$r(a', a'') \le \inf_{\gamma(a',a'')} \int\limits_{\gamma(a',a'')} \Lambda_r(x)\,|dx|. \tag{2.8}$$

2.2. If an anisotropic metric r belongs to the class $C^1(D \times D)$, then the differential

$$dr(x,a) = \sum_{i=1}^{n} r'_{x_i}(x,a)\big|_{x=a}\, dx_i$$

exists at every point $a \in D$ and continuous. The function

$$H_r(a,\xi) = \left| \sum_{i=1}^{n} r'_{x_i}(x,a)\big|_{x=a}\, \xi_i \right| : D \times \mathbb{R}^n \to \mathbb{R}^1$$

satisfies the conditions, imposed on the function H in (1.1), and the Finsler metric (1.2) is defined.

Theorem 2.2. *Let $D \subset \mathbb{R}^n$ be a domain and let r be an anisotropic distance on D. If $r \in C^1(D \times D)$, then*

$$r(a',a'') \le \inf_{\gamma(a',a'')} \int\limits_{\gamma(a',a'')} H_r(x,dx). \tag{2.9}$$

Proof. For the proof it is sufficient to consider the case, where the arc $\gamma = \gamma(a',a'')$ in the right side of (2.9) is smooth. Namely, for an arbitrarily pair of points $a', a'' \in D$ let γ means an oriented arc of the class C^1 on D, leading from a' to a''.

Fix arbitrarily a collection of points $a_0 = a', a_1, a_2, \ldots, a_{m+1} = a'' \in \gamma$, following one to another in the positive direction from a' to a''. Denote by γ_k the part of γ lying between the points a_k and a_{k+1}.

For an arbitrary k, $0 \le k \le m$, let

$$x_k(s) : (0, s(\gamma_k)) \to D, \quad x_k(0) = a_k, \quad x_k(s(\gamma_k)) = a_{k+1},$$

be the natural parametrization of γ_k and let s_k be the Euclidean length of γ_k. We have

$$r(a',a'') \le \sum_{k=0}^{m} r(a_k,a_{k+1}) = \sum_{k=0}^{m} r(x_k(0), x_k(s_k)) =$$

$$= \sum_{k=0}^{m} \int\limits_0^{s_k} \frac{dr}{ds}(x_k(s))\, ds = \sum_{k=0}^{m} \int\limits_0^{s_k} \sum_{i=1}^{n} r'_{x_i}(x_k,a_k) \frac{dx_i}{ds}(x_k)\bigg|_{x_k = x_k(s)} ds =$$

$$= \sum_{k=0}^{m} \int\limits_0^{s_k} \sum_{i=1}^{n} r'_{x_i}(x_k,a_k)\bigg|_{x_k=x_k(s)} dx_i(s). \tag{2.10}$$

On the other hand,

$$\int\limits_{\gamma(a',a'')} H_r(x,dx) = \sum_{k=0}^{m} \int\limits_{0}^{s_k} H_r(x_k(s),dx) =$$

$$= \sum_{k=0}^{m} \int\limits_{0}^{s_k} \left| \sum_{i=1}^{n} r'_{x_i}(x,a_k)|_{x=a_k} dx_i(s) \right| . \tag{2.11}$$

Next we observe, that because the anisotropic metric r belongs to the class $C^1(D\times D)$, then for sufficiently small partitions of the arc $\gamma(a',a'')$ with points $a_1,\dots,a_m$, the quantities

$$\left| \sum_{i=1}^{n} r'_{x_i}(x_k,a_k)|_{x_k=x_k(s)} - \sum_{i=1}^{n} r'_{x_i}(x,a_k) \right|$$

are uniformly small. Comparing (2.10) and (2.11), we conclude the validity of (2.9).

3. Derivative and Differential. Below we recall some concepts of [7]. These concepts are generalizations of classical notions.

3.1. Because an anisotropic space is a regular topological space, then by the standard way we can define in it oriented continuous arcs and curves. The anisotropic metric r permits to define the (oriented) length element ds_γ of an oriented arc (or curve) γ, and also the (oriented) linear measure on it.

An oriented arc (curve) is called *rectifiable*, if its length is finite.

For an arbitrary set $D\subset\mathcal{X}$ by the symbol $\Gamma(D)$ we will denote the family of the simple arcs or simple curves (open or closed), lying on D. We will assume also that on every $\gamma\in\Gamma(D)$ there is showed an direction (in particular, from one end point to another). Every closed, locally rectifiable arc $\gamma\in\Gamma(D)$ can be given in the following form

$$a = a(s) : [0, \text{length}\,(\gamma)] \to D\,,$$

where $0\le s\le \text{length}\,(\gamma)$ is the length of the arc between the start point $a(0)$ and the moving point $a(s)$ with the given along γ direction. The locally rectifiable arcs $\gamma\in\Gamma(D)$ can be evidently parametrized with the length of arc between a fixed point in the positive and negative directions along γ.

Let $D \subset \mathcal{X}$ be an nonempty set. By a *foliation*[1] x_D we will call a family $\{\gamma\}$ of arcs (or curves) $\gamma \in \Gamma(D)$ with the property: through every point $a \in D$ one and only one arc (or curve) $\gamma \in \Gamma(D)$ passes.

Curves $\gamma \in x_D$ are called *layers* of the foliation x_D.

Two foliations $x_{1D} = \{\gamma_1\}$ and $x_{2D} = \{\gamma_2\}$ *coincide*, if families of layers $\{\gamma_1\}$ and $\{\gamma_2\}$ coincide and the given on them orientations coincide also.

3.2. Let $x_D = \{\gamma\}$ be a foliation of a domain $D \subset \mathcal{X}$. We will name x_D *coordinate foliation* if there exists a function $x : x_D \to \mathbb{R}^1$, which is constant on every layer $\gamma \in x_D$.

Thus, if a coordinate foliation x_D is given, then a function

$$x : M \in D \to x_D(M) \in \mathbb{R}^1 \tag{3.12}$$

is defined. This function we will call by the *coordinate function* of the foliation x_D. A foliation x_D is called *continuous* (at a point, or a set), if the corresponding coordinate function is continuous.

Example 3.1. Let $D = \mathbb{R}^2$ be the plane with coordinates (x_1, x_2) and Euclidean distance $r = \sqrt{x_1^2 + x_2^2}$. We define the foliation x_D by assignment in the capacity of layers $\gamma \in x_D$ the straight lines parallel to the axis $0x_1$ and lying on the half-plane $\Pi_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \le 0\}$, and also rays, formed by intersections of the straight lines, perpendicular to the axis $0x_1$, with the half-plane $\Pi_2 = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_2\}$.

In the capacity of the coordinate function we put

$$x(x_1, x_2) = \begin{cases} x_2 & \text{if} \quad (x_1, x_2) \in \Pi_1, \\ 1/x_2 & \text{if} \quad (x_1, x_2) \in \Pi_2. \end{cases}$$

It is clear, that this coordinate foliation is discontinuous on the set $\partial\Pi_1 \cap \partial\Pi_2$.

If $x : D \to \mathbb{R}^1$ is a coordinate function of a foliation x_D and $\varphi : x(D) \to \mathbb{R}^1$ is a strongly monotone function, then $\varphi(x) : D \to \mathbb{R}^1$ is also a coordinate function of the foliation x_D.

Suppose that on the set $D \subset \mathcal{X}$ it is given a system of coordinate foliations $x_{\overline{1,n}D} = (x_{1D}, x_{2D}, \dots, x_{nD})$, $1 \le n \le \infty$, and, therefore, it is defined the mapping

$$x_{\overline{1,n}D} : M \in D \to (x_{1D}, x_{2D}, \dots, x_{nD}) \in V, \tag{3.13}$$

where V is some vector space.

[1]) 1-dimension foliation

If the system of foliations such that the mapping (3.13) is one-to-one, then we will call the system $x_{\overline{1,n}D}$ by the *coordinate system* on D, and the quantity n by the *dimension* of the set $D \subset \mathcal{X}$, and write $\dim D = n$.

In the case of two-dimensional surfaces M, prescribed by locally bi-Lipschitz immersions to $\mathbb{R}^n$, $n \geq 2$, examples of mappings (3.13) are conformal mappings $M \to \mathbb{R}^2$, which introduce *isothermal coordinates* on M.

If an anisotropic metric space $\mathcal{X}$ such that every point $a \in \mathcal{X}$ has a neighbourhood D, in which there exists a coordinate system, then we will call

$$x_{1D}, x_{2D}, \ldots, x_{nD}$$

local coordinates in $\mathcal{X}$.

In particular, if $V = \mathbb{R}^n$ and the mapping (3.13) is one-to-one, then investigation of the geometric structure of the anisotropic metric (sub)space

$$(D, r), \quad D \subset \mathcal{X}, \quad r = r(a', a''),$$

is equivalent to investigation of the anisotropic metric space

$$(\Delta, \overline{r}), \quad \Delta = x_{\overline{1,n}D}(D) \subset \mathbb{R}^n, \quad \overline{r} = r\left(x^{-1}_{\overline{1,n}D}(b'), x^{-1}_{\overline{1,n}D}(b'')\right). \tag{3.14}$$

Simplest examples of anisotropic metric spaces (3.14) are the above described abstract surfaces.

3.3. Let D be a domain in an anisotropic metric space $\mathcal{X}$ with an anisotropic metric r. Let S be a k-dimensional surface in $\mathbb{R}^n$, $1 \leq k < n$, given by a bi-Lipschitz mapping $U \to \mathbb{R}^n$ of an open set $U \subset \mathbb{R}^k$. Suppose that there exists a system of coordinate foliations $x_{\overline{1,n}D} = (x_{1D}, x_{2D}, \ldots, x_{nD})$, $1 \leq n < \infty$, such that the mapping

$$x_{\overline{1,n}D} : M \in D \to (x_{1D}, x_{2D}, \ldots, x_{nD}) \in S, \quad x_{\overline{1,n}D}(D) = S, \tag{3.15}$$

is one-to-one. Here investigation of the geometric structure of (D, r) is reducing to investigation of the anisotropic metric space

$$(S, \overline{r}), \quad \overline{r} = r\left(x^{-1}_{\overline{1,n}D}(b'), x^{-1}_{\overline{1,n}D}(b'')\right).$$

3.4. Introduce the concept of derivatives of functions on an anisotropic metric space. Let D be an nonempty subset of $\mathcal{X}$, $f : D \to \mathbb{R}^1$ be a function and let $x_D = \{\gamma\}$ be a foliation of D. *The derivative* of the function f with respect to x_D at the point $a \in D$ is the following quantity

$$\frac{\partial f}{\partial x_D}(a) = \lim_{a'\to a} \frac{f(a') - f(a)}{\vec{r}(a', a)}\,,$$

where a' strives to a along the arc $\gamma \in x_D$, $\gamma \ni a$, and the quantity $\vec{r}(a', a) = r(a', a)$ if the point a' follows the point a on γ, and $\vec{r}(a', a) = -r(a', a)$ if a' precides a.

In the special case, where D is a domain on $\mathbb{R}^n$ and the foliation x_D is a collection of intervals parallel (and equally directed) to the coordinate axis $0x$ in $\mathbb{R}^n$, the introduced quantity is the partial derivative of the function f with respect to the variable x.

If x_{1D}, x_{2D} are foliations of D, then we put

$$\frac{\partial^2 f}{\partial x_{1D} \partial x_{2D}}(a) = \frac{\partial}{\partial x_{1D}} \left[\frac{\partial f}{\partial x_{2D}} \right] (a).$$

By standard way we define (partial) derivatives of higher orders:

$$\frac{\partial^k f}{\partial x_{1D} \partial x_{2D} \dots \partial x_{kD}}\,,$$

where $2 < k < \infty$ is an integer and $x_{1D}, x_{2D}, \dots, x_{kD}$ are some foliations.

In the case, if

$$x_{1D} = x_{2D} = \dots = x_{kD} = x_D\,,$$

we will use the short notation

$$\frac{\partial^k f}{\partial x_D^k} = \frac{\partial^k f}{\partial x_{1D} \partial x_{2D} \dots \partial x_{kD}}\,.$$

There exist analogs of partial differential equations. For example, the following (formal) generalization of the Laplace equation, corresponding to the pair of foliations x_D and y_D, has the form

$$\frac{\partial^2 f}{\partial {x_D}^2} + \frac{\partial^2 f}{\partial {y_D}^2} = 0\,.$$

3.5. Let D be a domain on $\mathcal{X}$ and let $x_D = \{\gamma\}$ be a foliation of D. *The differential of the foliation* at a point $a \in D$ is defined as the quantity

$$dx_D \equiv \vec{r}(a', a),$$

where a' are points, belonging to the layer $\gamma \in x_D$, $\gamma \ni a$.

If $f : D \to \mathbb{R}^1$ is a function and $x_D = \{\gamma\}$ is a foliation, then the *differential of the function* f at a point $a \in D$ with respect to the foliation x_D is, by definition, the quantity

$$df(a) = \frac{\partial f}{\partial x_D}(a)\, dx_D.$$

Let $x_{\overline{1,n}D} = \{x_{1D}, x_{2D}, \dots, x_{nD}\}$, $1 < n < \infty$, be a system of foliations of D and let $a \in D$ be a point. For an arbitrary $1 \le k \le n$ by $\gamma_k(a)$ we denote the layer $x_k(D)$, containing the point a. We will call a function $f : D \to \mathbb{R}^1$ is *differentiable at the point* $a \in D$ with respect to the system of foliations $x_{\overline{1,n}D}$, if there exist constants $c_1, c_2, \dots, c_n$ such that the function's increment is representable in the form

$$f(a') - f(a) = \sum_{k=1}^{n} c_k\, \vec{r}(a'_k, a) + o\Bigg(\sum_{k=1}^{n} \vec{r}^{\,2}(a'_k, a)\Bigg)^{1/2}, \qquad a' \to a, \quad (3.16)$$

where $a'_k = (x_{1D}(a), \dots, x_{(k-1)D}(a), x_{kD}(a'), x_{(k+1)D}(a), \dots, x_{nD}(a))$, $k = 1, 2, \dots, n$, and for an arbitrary $l = 1, 2, \dots, n$ the following relations hold

$$\vec{r}(a'_l, a) = o(\vec{r}(a'_k, a)) \quad \text{as} \quad a' \to a\,, \;\; a' \in \gamma_k(a), \;\; k \ne l. \qquad (3.17)$$

(Because D is a domain and a its inner point, then for a' sufficiently near to a, the points a'_k belong to D.)

Theorem 3.1. *If a function $f : D \to \mathbb{R}^1$ is differentiable at a point $a \in D$ with respect to a system of foliations $x_{\overline{1,n}D}$ and exist the derivatives*

$$\frac{\partial f}{\partial x_{kD}}(a), \;\; 1 \le k \le n,$$

then for the constants $c_1, c_2, \dots, c_n$ of (3.16) *the following relations hold*

$$c_k = \frac{\partial f}{\partial x_{kD}}(a), \quad k = 1, 2, \dots, n.$$

Proof. Fix arbitrarily $k \in \overline{1,n}$ and the corresponding layer $\gamma_k(a)$ of the foliation x_{kD}. By (3.16) for $a' \to a$, $a' \in \gamma_k(a)$, we have

$$\frac{f(a') - f(a)}{\vec{r}(a'_k, a)} = c_k + \sum_{\substack{j=1 \\ j \ne k}}^{n} c_j\, \frac{\vec{r}(a'_j, a)}{\vec{r}(a'_k, a)} + \varepsilon(a', a),$$

where $\varepsilon(a', a) \to 0$ as $a' \to a$.

The supposition (3.17) implies

$$\frac{f(a')-f(a)}{\vec{r}(a'_k,a)} = c_k + \varepsilon_1(a',a) \quad \text{as} \quad a' \to a,$$

and since the derivative $\frac{\partial f}{\partial x_{kD}}(a)$ exists, then

$$\lim_{a'\to a} \frac{f(a')-f(a)}{\vec{r}(a'_k,a)} = \frac{\partial f}{\partial x_{k,D}}(a)$$

and the statement is proved.

Some sufficient conditions for existence of the total differential of functions in anisotropic metric spaces see [8].

4. Points of local extremum. As in the case of the metric space, we define the concept of the local extremum of a function on domains of anisotropic metric spaces.

The following statement holds

Theorem 4.1. *If $a \in D$ is a point of local extremum of a function $f : D \to \mathbb{R}^1$, the function f is differentiable at the point a with respect to a system of foliations $x_{\overline{1,n}D}$, there exist derivatives $\frac{\partial f}{\partial x_{kD}}(a)$, $1 \le k \le n$ and the point a is the inner point of a layer $\gamma_k(a)$, then*

$$\frac{\partial f}{\partial x_{kD}}(a) = 0\,. \tag{4.18}$$

Proof. Fix foliations $x_{1D}, x_{2D}, \ldots, x_{nD}$, $1 \le n < \infty$, of the domain D and corresponding layers $\gamma_k(a)$. For an arbitrary $k = 1, 2, \ldots, n$ and $a' \to a,\ a' \in \gamma_k(a)$ we have

$$f(a') - f(a) = \frac{\partial f}{\partial x_{kD}}(a)\,\vec{r}(a'_k,a) + \sum_{\substack{i=1\\ i\neq k}}^{n} \frac{\partial f}{\partial x_{iD}}(a)\,\vec{r}(a'_i,a) + $$

$$+ \bar{o}\left(\sum_{i=1}^{n} \vec{r}^{\,2}(a'_i,a)\right)^{1/2}.$$

From here, as in the proof of Theorem 3.1, we prove that

$$f(a') - f(a) = \frac{\partial f}{\partial x_{kD}}(a)\,\vec{r}(a'_k,a) + o(\vec{r}(a'_k,a)) \tag{4.19}$$

as $\quad a' \to a,\ a' \in \gamma_k(a)$.

Suppose that a is the point of a local minimum of the function f, however the derivative $\frac{\partial f}{\partial x_{kD}}(a) \neq 0$. Then by (4.19) we conclude, that

$$\frac{\partial f}{\partial x_{kD}}(a)\, \vec{r}(a'_k, a) + o(\vec{r}(a'_k, a)) \geq 0$$

for all $a' \in \gamma_k(a)$, sufficiently near to a. This is impossible, because a is the inner point of $\gamma_k(a)$, and the quantity $\vec{r}(a'_k, a)$ changes its sign if a' passes over the point a on $\gamma_k(a)$.

We indicate a simple counterexample.

Example 4.1. Let $D = \mathbb{R}^2$ be the plane with coordinates (x_1, x_2) and the Euclidean distance. Define the foliation x_D as in Example 3.1.

Consider the function

$$x(x_1, x_2) = \begin{cases} 0 & \text{if} \quad (x_1, x_2) \in \Pi_1, \\ x_1 & \text{if} \quad (x_1, x_2) \in \Pi_2. \end{cases}$$

Here at every point $a \in (\partial\Pi_1 \cap \partial\Pi_2)$ we have

$$\frac{\partial f}{\partial x_D}(a) = 0\,,$$

however these points are not points of a local extremum.

Concerning setting of this problem and similar questions see [9].

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