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On the angular derivatives of certain class of holomorphic functions in the unit disc

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Dedicated to memory of Professor Promarz M. Tamrazov

In this paper, a boundary version of the Schwarz lemma is investigated. We take into consideration a function f holomorphic in the unit disc, $f(0)=0$, $f'(0)=1$ and $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ for $|z|<1$, where $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$. We obtain sharp lower bounds on the angular derivative $f'(b)$ at the point b, where $|b|=1$ and $f(b)=0$.

Let f be a holomorphic function in the disc $D=\{z:|z|<1\}$, $f(0)=$ $=0$ and $|f(z)|<1$ for $|z|<1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)|\leq|z|$ and $|f'(0)|\leq 1$. Equality in these inequalities (in the first one, for $z\neq 0$) occurs only if $f(z)=cz$, $|c|=1$ (cf. [1, p. 329]).

Let $f(z)=z+c_2z^2+c_3z^3+\ldots$ be a holomorphic function in D and let $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ for $|z|<1$, where $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$.

Consider the functions $\varphi(z)=\frac{h(z)-\alpha}{\alpha}$, where $h(z)=\frac{f(z)}{\lambda f(z)+(1-\lambda)z}$, and $\phi(z)=\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)}$. $\varphi(z)$ and $\phi(z)$ are holomorphic functions in D, $|\phi(z)|<1$ for $|z|<1$ and $\phi(0)=0$. Therefore, from the Schwarz lemma, we obtain

$$|f(z)|\leq\frac{\alpha\,(1-\lambda)\,|z|\,(1+|z|)}{\alpha+(1-\alpha)\,|z|-\alpha\lambda\,(1+|z|)}\tag{1.1}$$

and

$$|c_2| \le \frac{2\alpha - 1}{\alpha\,(1-\lambda)}\,. \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if $f(z)$ is the function of the form $f(z) = \frac{\alpha(1-\lambda)z\left(1-ze^{i\theta}\right)}{\alpha(1-\alpha)ze^{i\theta}-\alpha\lambda(1-ze^{i\theta})}$, where θ is a real number.

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b| = 1$, and if $|f(b)| = 1$ and $f'(b)$ exists, then $|f'(b)| \ge 1$, which is known as the Schwarz lemma on the boundary. Applying inequality (5) in [1, p. 330] to the function $\frac{f(z)}{z}$, we arrive at the following generalization of the Schwarz lemma (cf. [1]):

$$|f(z)| \le |z|\,\frac{|z| + |f'(0)|}{1 + |z|\,|f'(0)|}\,, \quad z \in D. \tag{1.3}$$

If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial D$, $|f(b)| = 1$, then by Julia–Wolff lemma the angular derivative $f'(b)$ exists and $1 \le |f'(b)| \le \infty$ (cf. [2]). Then, passing to the angular limit in (1.3), we arrive at the boundary Schwarz lemma (cf. [3])

$$|f'(b)| \geqslant \frac{2}{1 + |f'(0)|}\,. \tag{1.4}$$

In (1.3) for real z and in the left-hand-side inequality in (1.4) for $b = 1$, the equality occurs for the function $f(z) = z\,(z+\gamma)\,/\,(1+\gamma z)\,,\ 0 \le \gamma \le 1$.

It follows that

$$|f'(b)| \geqslant 1, \tag{1.5}$$

with equality only if f is of the form $f(z) = ze^{i\theta}$, θ is real.

Moreover, if $f(z) = c_p z^p + c_{p+1} z^{p+1} \ldots$, then

$$|f'(b)| \geqslant p + \frac{1 - |c_p|}{1 + |c_p|}\,. \tag{1.6}$$

It follows that $|f'(b)| \geqslant p$, with equality only if f is of the form $f(z) = = z^p e^{i\theta}$, θ is real.

Previously, R. Osserman examined sharp Schwarz inequality at the boundary (see [3]). Afterwards, the Schwarz inequality that has been obtained by V. Dubinin is strengthened (see [4]). Some other types of strengthening inequalities are obtained in (see [5]).

Our method depends on a classical results of Schwarz lemma on the boundary. We will obtain more general results at the boundary. In the following theorems, new inequalities of Schwarz inequality at the boundary are obtained and the sharpness of these inequalities is proved.

Theorem 1. *Let* $f(z) = z + c_2z^2 + c_3z^3 + \dots$ *be a holomorphic function in the disc* D *and* $\left| \frac{f(z)}{\lambda f(z)+(1-\lambda)z} - \alpha \right| < \alpha$ *for* $|z| < 1$, *where* $\frac{1}{2} < \alpha \leq \frac{1}{1+\lambda}$, $0 \leq \lambda < 1$. *Further assume that, for some* $b \in \partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b) = 0$. *Then*

$$|f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha - 1}. \tag{1.7}$$

The equality in (1.7) *holds if and only if*

$$f(z) = \frac{\alpha(1-\lambda)z\left(1 - ze^{i\theta}\right)}{\alpha(1-\alpha)ze^{i\theta} - \alpha\lambda\left(1 - ze^{i\theta}\right)},$$

where θ *is a real number.*

Proof. Let $\phi(z) = \frac{\varphi(z)-\varphi(0)}{1-\varphi(0)\varphi(z)}$. The function $\phi(z)$ is holomorphic in the unit disc D, $|\phi(z)| < 1$ for $|z| < 1$, $\phi(0) = 0$ and $|\phi(b)| = 1$ for $b \in \partial D$.

From (1.5), we obtain

$$1 \leq |\phi'(b)| = \frac{1 - |\varphi(0)|^2}{\left|1 - \overline{\varphi(0)}\varphi(b)\right|^2} |\varphi'(b)| = \frac{1 - \left(\frac{1-\alpha}{\alpha}\right)^2}{\left(1 - \frac{1-\alpha}{\alpha}(-1)\right)^2} \frac{|f'(b)|}{\alpha(1-\lambda)}$$

and

$$1 \leq |\phi'(b)| = \frac{(2\alpha - 1)}{\alpha(1-\lambda)} |f'(b)|. \tag{1.8}$$

Therefore, we have

$$|f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha - 1}.$$

If $|f'(b)| = \frac{\alpha(1-\lambda)}{2\alpha-1}$ from (1.8) and $|\phi'(b)| = 1$, we obtain

$$f(z) = \frac{\alpha(1-\lambda)z\left(1 - ze^{i\theta}\right)}{\alpha(1-\alpha)ze^{i\theta} - \alpha\lambda\left(1 - ze^{i\theta}\right)}.$$

Theorem 2. *Under the same assumptions as in Theorem 1, we have*

$$|f'(b)| \geq \frac{2\alpha(1-\lambda)}{2\alpha - 1 + \alpha(1-\lambda)|c_2|}. \tag{1.9}$$

The inequality (1.9) *is sharp with equality for the function*

$$f(z)=\frac{\alpha\left(1-\lambda\right)z\left(1-z^{2}\right)}{\alpha+\left(2\alpha-1\right)az-\left(1-\alpha\right)z^{2}-\alpha\lambda\left(1-z^{2}\right)},$$

where $a=\frac{\alpha(1-\lambda)|c_2|}{2\alpha-1}$ *is an arbitrary number from* $[0,1]$ (*see* (1.2)).

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 1. Using the inequality (1.4) for the function $\phi(z)$, we obtain

$$\begin{aligned}\frac{2}{1+|\phi'(0)|} &\leq |\phi'(b)|=\frac{(2\alpha-1)}{\alpha\left(1-\lambda\right)}\left|f'(b)\right|,\\ \frac{2}{1+\frac{\alpha(1-\lambda)|c_2|}{2\alpha-1}} &\leq \frac{(2\alpha-1)}{\alpha\left(1-\lambda\right)}\left|f'(b)\right|\end{aligned}$$

and

$$\left|f'(b)\right|\geq\frac{2\alpha\left(1-\lambda\right)}{2\alpha-1+\alpha\left(1-\lambda\right)|c_2|}.$$

Now, we shall show that the inequality (1.9) is sharp. Choose an arbitrary $a\in[0,1]$. Let

$$f(z)=\frac{\alpha\left(1-\lambda\right)z\left(1-z^{2}\right)}{\alpha+\left(2\alpha-1\right)az-\left(1-\alpha\right)z^{2}-\alpha\lambda\left(1-z^{2}\right)}.$$

Then

$$\begin{aligned}f'(z)=\alpha\left(1-\lambda\right)&\frac{\left(1-3z^{2}\right)\left(\alpha+\left(2\alpha-1\right)az-\left(1-\alpha\right)z^{2}-\alpha\lambda\left(1-z^{2}\right)\right)}{\left(\alpha+\left(2\alpha-1\right)az-\left(1-\alpha\right)z^{2}-\alpha\lambda\left(1-z^{2}\right)\right)^{2}}-\\ &-\alpha\left(1-\lambda\right)\frac{\left(\left(2\alpha-1\right)a-2\left(1-\alpha\right)z+2\alpha\lambda z\right)\left(z-z^{3}\right)}{\left(\alpha+\left(2\alpha-1\right)az-\left(1-\alpha\right)z^{2}-\alpha\lambda\left(1-z^{2}\right)\right)^{2}}\end{aligned}$$

and

$$f'(1)=-2\alpha\frac{(1-\lambda)}{(1+a)\left(2\alpha-1\right)}.$$

Since $a=\frac{\alpha(1-\lambda)|c_2|}{2\alpha-1}$, (1.9) is satisfied with equality.

If $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\ldots$, $p\geq1$, is a holomorphic function in D and $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right|<\alpha$ for $|z|<1$, where $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$, then

$$|f(z)|\leq\frac{\alpha\left(1-\lambda\right)|z|\left(1+|z|^{p}\right)}{\alpha+\left(1-\alpha\right)|z|^{p}-\alpha\lambda\left(1+|z|^{p}\right)}$$

and

$$|c_{p+1}| \leq \frac{2\alpha - 1}{\alpha(1-\lambda)}. \tag{1.10}$$

Theorem 3. *Let $z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\ldots$, $p \geq 1$, be a holomorphic function in the disc D and $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z} - \alpha\right| < \alpha$ for $|z| < 1$, where $\frac{1}{2} < \alpha \leq \frac{1}{1+\lambda}$, $0 \leq \lambda < 1$. Further assume that, for some $b \in \partial D$, f has an angular limit $f(b)$ at b, $f(b) = 0$. Then*

$$|f'(b)| \geq p\frac{\alpha(1-\lambda)}{2\alpha - 1}, \tag{1.11}$$

with equality in (1.11) *if and only if $f(z)$ is a function of the form*

$$f(z) = \frac{\alpha(1-\lambda)z\left(1 - z^p e^{i\theta}\right)}{\alpha(1-\alpha)z^p e^{i\theta} - \alpha\lambda(1 - z^p e^{i\theta})},$$

where θ is a real number.

Proof. Using the inequality $|f'(b)| \geq p$ for the function $\phi(z)$, we obtain $|\phi'(b)| \geq p$. So,

$$p \leq |\phi'(b)| = \frac{1 - |\varphi(0)|^2}{\left|1 - \overline{\varphi(0)}\varphi(b)\right|^2}|\varphi'(b)| = \frac{1 - \left(\frac{1-\alpha}{\alpha}\right)^2}{\left(1 - \frac{1-\alpha}{\alpha}(-1)\right)^2}\frac{|f'(b)|}{\alpha(1-\lambda)}$$

and

$$p \leq |\phi'(b)| = \frac{(2\alpha - 1)|f'(b)|}{\alpha(1-\lambda)}. \tag{1.12}$$

If $|f'(b)| = p\frac{\alpha(1-\lambda)}{2\alpha-1}$ from (1.12) and $|\phi'(b)| = p$, we obtain

$$f(z) = \frac{\alpha(1-\lambda)z\left(1 - z^p e^{i\theta}\right)}{\alpha(1-\alpha)z^p e^{i\theta} - \alpha\lambda(1 - z^p e^{i\theta})}.$$

Theorem 4. *Under the same assumptions as in Theorem 3, we have*

$$|f'(b)| \geq \frac{\alpha(1-\lambda)}{2\alpha - 1}\left(p + \frac{2\alpha - 1 - \alpha(1-\lambda)|c_{p+1}|}{2\alpha - 1 + \alpha(1-\lambda)|c_{p+1}|}\right). \tag{1.13}$$

The inequality (1.13) *is sharp with equality for the function*

$$f(z) = \frac{\alpha(1-\lambda)z\left(1 + cz - z^{p+1} - cz^p\right)}{\alpha(1+cz) - (1-\alpha)(z^p + cz^p) - \alpha\lambda(1 + cz - z^{p+1} - cz^p)},$$

where $c = \frac{\alpha(1-\lambda)|c_{p+1}|}{2\alpha-1}$ *is an arbitrary number from* $[0,1]$ (*see* (1.10)).

Proof. Using the inequality (1.6) for the function $\phi(z)$, we obtain

$$p + \frac{1-|a_p|}{1+|a_p|} \leq |\phi'(b)| = \frac{(2\alpha-1)\,|f'(b)|}{\alpha\,(1-\lambda)},$$

where $|a_p| = \frac{\left|\phi^{(p)}(0)\right|}{p!} = \frac{\alpha(1-\lambda)|c_{p+1}|}{2\alpha-1}$.

Therefore, we take

$$\begin{aligned} p + \frac{1-\frac{\alpha(1-\lambda)|c_{p+1}|}{2\alpha-1}}{1+\frac{\alpha(1-\lambda)|c_{p+1}|}{2\alpha-1}} &\leq \frac{(2\alpha-1)\,|f'(b)|}{\alpha\,(1-\lambda)}, \\ p + \frac{2\alpha-1-\alpha\,(1-\lambda)\,|c_{p+1}|}{2\alpha-1+\alpha\,(1-\lambda)\,|c_{p+1}|} &\leq \frac{(2\alpha-1)\,|f'(b)|}{\alpha\,(1-\lambda)} \end{aligned}$$

and

$$|f'(b)| \geq \frac{\alpha\,(1-\lambda)}{2\alpha-1}\left(p + \frac{2\alpha-1-\alpha\,(1-\lambda)\,|c_{p+1}|}{2\alpha-1+\alpha\,(1-\lambda)\,|c_{p+1}|}\right).$$

The equality in (1.13) is obtained for the function

$$f(z) = \frac{\alpha\,(1-\lambda)\,z\left(1+cz-z^{p+1}-cz^p\right)}{\alpha\,(1+cz)-(1-\alpha)\,(z^p+cz^p)-\alpha\lambda\,(1+cz-z^{p+1}-cz^p)},$$

as simple calculations show.

Theorem 5. *Let* $z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\ldots$, $p\geq 1$, *be a holomorphic function in the disc* D *and* $\left|\frac{f(z)}{\lambda f(z)+(1-\lambda)z}-\alpha\right| < \alpha$ *for* $|z|<1$, *where* $\frac{1}{2}<\alpha\leq\frac{1}{1+\lambda}$, $0\leq\lambda<1$. *Further assume that, for some* $b\in\partial D$, f *has an angular limit* $f(b)$ *at* b, $f(b)=0$. *Let* $a_1,a_2,\ldots,a_n$ *be fixed points of* $f(z)$ *in* D *that are different from zero. Then we have the inequality*

$$|f'(b)| \geq \frac{\alpha\,(1-\lambda)}{2\alpha-1}\left(p + \sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2} + \frac{(2\alpha-1)\prod_{k=1}^{n}|a_k| - \alpha\,(1-\lambda)\,|c_{p+1}|}{(2\alpha-1)\prod_{k=1}^{n}|a_k| + \alpha\,(1-\lambda)\,|c_{p+1}|}\right). \tag{1.14}$$

In addition, the equality in (1.14) *occurs for the function*

$$f(z)=\frac{\alpha\,(1-\lambda)\,z\Big(1-z^{p}\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\Big)}{\alpha-(1-\alpha)\,z^{p}\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}-\alpha\lambda\Big(1-z^{p}\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\Big)},$$

where $a_1,a_2,\ldots,a_n$ *are positive real numbers.*

Proof. Consider the functions

$$\phi(z)=\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)},\quad B(z)=\prod_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}.$$

$\phi(z)$ and $B(z)$ are holomorphic functions in D, and $|\phi(z)|<1$, $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in D$, we have $|\phi(z)|\leq$ $\leq|B(z)|$. The auxiliary function

$$\Psi(z)=\frac{\phi(z)}{B(z)}=\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)}\frac{1}{\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}}$$

is holomorphic in D, and $|\Psi(z)|<1$ for $|z|<1$, $\Psi(0)=0$ and $|\Psi(b)|=1$ for $b\in\partial D$.

Moreover, the geometric meaning of the derivative and the inequality $|\phi(z)|\leq|B(z)|$ imply the relations

$$\frac{b\phi'(b)}{\phi(b)}=|\phi'(b)|\geq|B'(b)|=\frac{bB'(b)}{B(b)}.$$

Besides, with simple calculations, we take

$$|B'(b)|=\frac{bB'(b)}{B(b)}=\sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2}.$$

Using the inequality (1.6) for the function $\Psi(z)$, we obtain

$$p+\frac{1-|k_p|}{1+|k_p|}\leq|\Psi'(b)|=\left|\frac{b\phi'(b)}{\phi(b)}-\frac{bB'(b)}{B(b)}\right|=\{|\phi'(b)|-|B'(b)|\},$$

where $|k_p|=\frac{\left|\Psi^{(p)}(0)\right|}{p!}$.

Since $|k_p| = \frac{\left|\Psi^{(p)}(0)\right|}{p!} = \frac{\alpha(1-\lambda)|c_{p+1}|}{(2\alpha-1)\prod\limits_{k=1}^{n}|a_k|}$, we may write

$$p + \frac{1 - \frac{\alpha(1-\lambda)|c_{p+1}|}{(2\alpha-1)\prod\limits_{k=1}^{n}|a_k|}}{1 + \frac{\alpha(1-\lambda)|c_{p+1}|}{(2\alpha-1)\prod\limits_{k=1}^{n}|a_k|}} \leq \frac{1-\left(\frac{1-\alpha}{\alpha}\right)^2}{\left(1-\frac{1-\alpha}{\alpha}(-1)\right)^2}\frac{|f'(b)|}{\alpha(1-\lambda)} - \sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2},$$

$$p + \frac{(2\alpha-1)\prod\limits_{k=1}^{n}|a_k| - \alpha(1-\lambda)|c_{p+1}|}{(2\alpha-1)\prod\limits_{k=1}^{n}|a_k| + \alpha(1-\lambda)|c_{p+1}|} \leq \frac{1-\left(\frac{1-\alpha}{\alpha}\right)^2}{\left(1-\frac{1-\alpha}{\alpha}(-1)\right)^2}\frac{|f'(b)|}{\alpha(1-\lambda)} -$$

$$-\sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2} = \frac{(2\alpha-1)}{\alpha(1-\lambda)}|f'(b)| - \sum_{k=1}^{n}\frac{1-|a_k|^2}{|b-a_k|^2}.$$

Therefore, we take the inequality (1.14).

The equality in (1.14) is obtained for the function

$$f(z) = \frac{\alpha(1-\lambda)z\left(1 - z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)}{\alpha - (1-\alpha)z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z} - \alpha\lambda\left(1 - z^p\prod\limits_{k=1}^{n}\frac{z-a_k}{1-\overline{a_k}z}\right)},$$

as show simple calculations.

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