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T. M. Osipchuk

(Institute of Mathematics of NASU, Kyiv)

Analytical conditions of local linear convexity in the space $\mathbb{H}^n_{\alpha,\beta}$

otm82@mail.ru

Dedicated to memory of Professor Promarz M. Tamrazov

Встановлено необхідні, а також достатні умови локальної лінійної опуклості областей з гладкою межею в багатовимірному узагальнено кватерніонному просторі $\mathbb{H}^n_{\alpha,\beta}$ $(n \geq 2)$.

The necessary and sufficient conditions of local linear convexity of domains with smooth boundary in multidimensional generalized quaternion space $\mathbb{H}^n_{\alpha,\beta}$ $(n \geq 2)$ are found.

The paper ([1]) presents analytical conditions of local linear convexity of domains with smooth boundaries in a multidimensional quaternion space $\mathbb{H}^n$ in terms of some special linear form of the second order. The aim of this work is to establish similar conditions for the case of generalized quaternion algebra $\mathbb{H}_{\alpha,\beta}$ ([2, 3], see also [4]).

Let $\mathbb{H}_{\alpha,\beta}$ be the algebra of generalized quaternions (further quaternions)

$$a = a_0 + a_1e_1 + a_2e_2 + a_3e_3,$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$, and $1, e_1, e_2, e_3$ satisfy the following multiplication table:

$\cdot$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\alpha$	e_3	$-\alpha e_2$
e_2	e_2	$-e_3$	$-\beta$	βe_1
e_3	e_3	αe_2	$-\beta e_1$	$-\alpha\beta$,

where α, β are real nonzero numbers.

Let us consider the vector space $\mathbb{H}^n_{\alpha,\beta} := \underbrace{\mathbb{H}_{\alpha,\beta} \times \mathbb{H}_{\alpha,\beta} \times \ldots \times \mathbb{H}_{\alpha,\beta}}_{n\geq 2}$ with elements $z := (z_1, z_2, \ldots, z_n) \in \mathbb{H}^n_{\alpha,\beta}$, where

$$z_j := x_0^j + x_1^j e_1 + x_2^j e_2 + x_3^j e_3 \in \mathbb{H}_{\alpha,\beta}, \;\; j = \overline{1,n}, \quad \|z\| = \sqrt{\sum_{j=1}^{n}\sum_{k=0}^{3}\left(x_k^j\right)^2}.$$

The neighborhood $U(w)$ of a point $w = (w_1, w_2, \ldots, w_m) \in \mathbb{H}^n_{\alpha,\beta}$ is an opened ball $U(w) = \{z : \|z - w\| < \rho_0\}$.

Since the algebra $\mathbb{H}_{\alpha,\beta}$ is non-commutative with respect to the multiplication, the equations $\sum\limits_{j=1}^{n} a_j z_j = 0$, $\sum\limits_{j=1}^{n} z_j a_j = 0$, $a_j, z_j \in \mathbb{H}_{\alpha,\beta}$, define geometrically different hyperplane of $4n-4$ dimension, in general case. In addition, we may consider $2^n - 2$ intermediate cases between two mentioned hyperplanes for which equations contain the alternated summands $a_j z_j$, $z_j a_j$. For a concreteness, in the following definition we use the case where variables are multiplied by constants from the left only.

Definition. Domain $\Omega \subset \mathbb{H}^n_{\alpha,\beta}$ is said to be *locally linearly convex* if for every boundary point $w = (w_1, w_2, \ldots, w_n)$ of the domain there is a hyperplane $\sum\limits_{j=1}^{n} a_j (z_j - w_j) = 0$, $a_j \in \mathbb{H}_{\alpha,\beta}$, $z = (z_1, z_2, \ldots, z_n) \in \mathbb{H}^n_{\alpha,\beta}$, which passes through the point $w \in \partial\Omega$ and does not intersect Ω in some neighborhood of w. If a such intersection is empty, then Ω is said to be *linearly convex*.

Let us consider the following quaternions:

$$\begin{array}{lcl} z_j & := & x_0^j + x_1^j e_1 + x_2^j e_2 + x_3^j e_3\,, \\ z_j^1 & := & x_0^j + x_1^j e_1 - x_2^j e_2 - x_3^j e_3\,, \\ z_j^2 & := & x_0^j - x_1^j e_1 + x_2^j e_2 - x_3^j e_3\,, \\ z_j^3 & := & x_0^j - x_1^j e_1 - x_2^j e_2 + x_3^j e_3\,. \end{array}$$

We express the real variables x_k^j in terms of the quaternions z_j^l with respect to non-commutativity of generalized quaternions. So, we get that

$$\begin{array}{ll} x_0^j & = 4^{-1}\left(z_j + z_j^1 + z_j^2 + z_j^3\right), \\ x_1^j & = 4^{-1}\alpha^{-1} e_1 \left(-z_j - z_j^1 + z_j^2 + z_j^3\right), \\ x_2^j & = 4^{-1}\beta^{-1} e_2 \left(-z_j + z_j^1 - z_j^2 + z_j^3\right), \\ x_3^j & = 4^{-1}(\alpha\beta)^{-1} e_3 \left(-z_j + z_j^1 + z_j^2 - z_j^3\right) \end{array} \tag{1}$$

or

$$\begin{aligned} x_0^j &= 4^{-1}\left(z_j + z_j^1 + z_j^2 + z_j^3\right), \\ x_1^j &= 4^{-1}\alpha^{-1}\left(-z_j - z_j^1 + z_j^2 + z_j^3\right) e_1, \\ x_2^j &= 4^{-1}\beta^{-1}\left(-z_j + z_j^1 - z_j^2 + z_j^3\right) e_2, \\ x_3^j &= 4^{-1}(\alpha\beta)^{-1}\left(-z_j + z_j^1 + z_j^2 - z_j^3\right) e_3. \end{aligned} \tag{2}$$

Let $\Omega = \{z : \rho(z) < 0\}$ be a domain of the space $\mathbb{H}^n_{\alpha,\beta}$ with the boundary $\partial\Omega = \{z : \rho(z) = 0\}$, where $\rho(z) = \rho(z, z^1, z^2, z^3) : \mathbb{H}^n_{\alpha,\beta} \to \mathbb{R}$ is a twice continuously differentiable real-valued function in the neighborhood $U(\partial\Omega)$, $z^l = (z_1^l, z_2^l, \ldots, z_n^l) \in \mathbb{H}^n_{\alpha,\beta}$, $l = 1, 2, 3$, and $\operatorname{grad}\rho \neq 0$ everywhere on $\partial\Omega$. Let us consider the total differential of the function ρ as real function of independent variables $\{x_l^j\}_{j=1}^n$, $l = \overline{0,4}$ at the point $w \in U(\partial\Omega)$:

$$d\rho(w) = \sum_{j=1}^{n}\sum_{l=0}^{3} \frac{\partial\rho(w)}{\partial x_l^j}\, dx_l^j\,. \tag{3}$$

Replacing the variables x_l^j and z_j^l in (1) and (2) with their differentials dx_l^j and dz_j^l, respectively, and substituting in (3) the expressions of dx_l^j, we obtain (here and everywhere $z_j^0 := z_j$)

$$d\rho(w) = \sum_{j=1}^{n}\sum_{l=0}^{3} \frac{\partial\rho(w)}{\partial z_j^l}\, dz_j^l$$

or

$$d\rho(w) = \sum_{j=1}^{n}\sum_{k=0}^{3} dz_j^k\, \frac{\partial\rho(w)}{\partial z_j^k}\,,$$

where formal derivatives $\dfrac{\partial\rho(w)}{\partial z_j^l}$ are calculated with the formulas

$$\begin{aligned} \frac{\partial\rho(w)}{\partial z_j^0} &= \frac{1}{4}\left(\frac{\partial\rho(w)}{\partial x_0^j} - \frac{1}{\alpha}\frac{\partial\rho(w)}{\partial x_1^j}e_1 - \frac{1}{\beta}\frac{\partial\rho(w)}{\partial x_2^j}e_2 - \frac{1}{\alpha\beta}\frac{\partial\rho(w)}{\partial x_3^j}e^3\right), \\ \frac{\partial\rho(w)}{\partial z_j^1} &= \frac{1}{4}\left(\frac{\partial\rho(w)}{\partial x_0^j} - \frac{1}{\alpha}\frac{\partial\rho(w)}{\partial x_1^j}e_1 + \frac{1}{\beta}\frac{\partial\rho(w)}{\partial x_2^j}e_2 + \frac{1}{\alpha\beta}\frac{\partial\rho(w)}{\partial x_3^j}e_3\right), \\ \frac{\partial\rho(w)}{\partial z_j^2} &= \frac{1}{4}\left(\frac{\partial\rho(w)}{\partial x_0^j} + \frac{1}{\alpha}\frac{\partial\rho(w)}{\partial x_1^j}e_1 - \frac{1}{\beta}\frac{\partial\rho(w)}{\partial x_2^j}e_2 + \frac{1}{\alpha\beta}\frac{\partial\rho(w)}{\partial x_3^j}e_3\right), \\ \frac{\partial\rho(w)}{\partial z_j^3} &= \frac{1}{4}\left(\frac{\partial\rho(w)}{\partial x_0^j} + \frac{1}{\alpha}\frac{\partial\rho(w)}{\partial x_1^j}e_1 + \frac{1}{\beta}\frac{\partial\rho(w)}{\partial x_2^j}e_2 - \frac{1}{\alpha\beta}\frac{\partial\rho(w)}{\partial x_3^j}e_3\right). \end{aligned} \tag{4}$$

Let us rewrite the formulas (4) as

$$\frac{\partial\rho(w)}{\partial z_j^l} = \frac{1}{4}\sum_{p=0}^{3}\gamma_{lp}\frac{\partial\rho(w)}{\partial x_p^j}e_p, \quad l=\overline{0,3},$$

where γ_{lp} are the elements of matrix

$$\Gamma = \begin{pmatrix} 1 & -\frac{1}{\alpha} & -\frac{1}{\beta} & -\frac{1}{\alpha\beta} \\ 1 & -\frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\alpha\beta} \\ 1 & \frac{1}{\alpha} & -\frac{1}{\beta} & \frac{1}{\alpha\beta} \\ 1 & \frac{1}{\alpha} & \frac{1}{\beta} & -\frac{1}{\alpha\beta} \end{pmatrix}.$$

And let

$$\frac{\partial^2\rho(w)}{\partial z_i^k\partial z_j^l} := \frac{1}{16}\sum_{p,g=0}^{3}\gamma_{lp}\gamma_{kg}\frac{\partial^2\rho(w)}{\partial x_g^i\partial x_p^j}e_pe_g, \ j,i=\overline{1,n}, \ l,k=\overline{0,3}.$$

Then

$$d^2\rho(w) = \sum_{i,j=1}^{n}\sum_{k,l=0}^{3}\frac{\partial^2\rho(w)}{\partial x_k^i\partial x_l^j}\,dx_l^j\,dx_k^i = \sum_{i,j=1}^{n}\sum_{k,l=0}^{3}\frac{\partial^2\rho(w)}{\partial z_i^k\partial z_j^l}\,dz_j^l\,dz_i^k =$$
$$= \sum_{i,j=1}^{n}\sum_{k,l=0}^{3}dz_i^k\,\frac{\partial^2\rho(w)}{\partial z_i^k\partial z_j^l}\,dz_j^l = \sum_{i,j=1}^{n}\sum_{k,l=0}^{3}dz_j^l\,dz_i^k\,\frac{\partial^2\rho(w)}{\partial z_i^k\partial z_j^l}. \quad (5)$$

We pay attention to the order of multiplication of quaternions in (5).

Theorem. *If the domain Ω is left locally linearly convex, then for every point $w\in\partial\Omega$ and for all vectors $s=(s_1,s_2,\dots,s_n)\in\mathbb{H}^n_{\alpha,\beta}$, $\|s\|=1$, which satisfy the equation*

$$\sum_{j=1}^{n}\frac{\partial\rho(w)}{\partial z_j^0}s_j = 0, \quad (6)$$

the following inequalities are true:

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3}\frac{\partial^2\rho(w)}{\partial z_i^k\partial z_j^l}s_j^ls_i^k \geq 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_j^l \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} s_i^k \geq 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_j^l s_i^k \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} \geq 0.$$

If for every point $w \in \partial\Omega$ and the same vectors s at least one of the inequalities is true

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} s_j^l s_i^k > 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_j^l \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} s_i^k > 0,$$

$$\sum_{i,j=1}^{n}\sum_{k,l=0}^{3} s_j^l s_i^k \frac{\partial^2\rho(w)}{\partial z_i^k \partial z_j^l} > 0\,,$$

then the domain Ω is locally linearly convex.

The proof of Theorem is similar to that of Theorem 1 from [1] and Theorem from [5]. We note only that to proof it we need to consider Taylor series in a quaternion form. In addition, the equation (6) defines a hyperplane of the definition for the domain Ω.

References

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