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The discrete Schredinger type hierarchies of nonlinear dynamical system and their by-Hamiltonian integrability

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Dedicated to memory of Professor Promarz M. Tamrazov

Показано как функционально-аналитические градиентно-голономные структуры могут быть использованы для анализа изоспектральной интегрируемости нелинейных динамических систем на дискретных многообразиях. Разработанный метод применен для получения подробного доказательства интегрируемости дискретных нелинейных динамических систем Шредингера, Рагниско–Ту и Римана–Бюргера.

It is shown how functional-analytic gradient-holonomic structures can be used for an isospectral integrability analysis of nonlinear dynamical systems on discrete manifolds. The approach developed is applied to obtain detailed proofs of the integrability of the discrete nonlinear Schrödinger, Ragnisco–Tu and Riemann–Burgers dynamical systems.

1. Introduction. With a fairly generous definition, a discrete nonlinear Schredinger (DNLS) equation [28, 14] is any equation that can be obtained from a nonlinear Schredinger (NLS) equation of general form

$$i\partial u/\partial t + \Delta u \ + f(\ \bar{u}u\ )u = 0, \qquad (1.1)$$
$$-i\partial \bar{u}/\partial t + \Delta \bar{u} + f(\bar{u}u)\bar{u} = 0,$$

by employing some finite-difference approximation to the differential operators acting on the space-time-dependent vector-function $(u, \bar{u})^{\intercal} \in C^2(\mathbb{R}\times \times\mathbb{R}^n; \mathbb{C}^2), n \in \mathbb{Z}_+$, where the overbar denotes the complex conjugate. In (1.1) $\Delta := <\nabla, \nabla>_{\mathbb{E}^n}$ is the Laplace operator acting in one, two, or more spatial dimensions, and $f : \mathbb{C} \to \mathbb{R}$ is a quite general function that, for most purposes, is taken to be differentiable and with $f(0) = 0$. In the most well-known case of cubic nonlinearity, $f(\bar{u}u) := 2\xi\bar{u}u$, where $\xi \in \mathbb{R}$. Equation (1.1) is often referred to as the NLS equation, and is integrable with the inverse scattering method [37], if the number of spatial dimensions is one. Here we use the term DNLS equation to denote the set of coupled ordinary differential equations resulting from discretizing all spatial variables in (1.1), while keeping the time-variable $t \in \mathbb{R}$ continuous. However, one may also consider equations with discrete time ("fully discrete NLS equations"), as well as equations with only some of the spatial dimensions discretized ("discrete-continuum NLS equations"). The former are of interest as algorithms for numerical solution of (1.1), while the latter may describe pulse propagation in arrays of coupled nonlinear optical fibers [3]. The simplest example of a DNLS equation can be formally obtained by just replacing the Laplacian operator in (1.1) with the corresponding discrete Laplacian. Thus, for the one-dimensional (1-D) case we let $u_n(t) := u(x = na; t), \bar{u}_n(t) := \bar{u}(x = na; t)$, $n \in \mathbb{Z}$, where $a \in \mathbb{R}_+$ is the lattice parameter, so that for the particular case of cubic nonlinearity the following equation is obtained:

$$\begin{pmatrix} du_n/dt \\ d\bar{u}_n/dt \end{pmatrix} = \begin{pmatrix} \delta i(u_{n+1} - 2u_n + u_{n-1}) - 2\xi i\bar{u}_n u_n^2 \\ -\delta i(2\bar{u}_n - u_{n+1} - u_{n-1}) + 2\xi i\bar{u}_n^2 u_n \end{pmatrix} =: K_n[u, \bar{u}], \quad (1.2)$$

where $\delta = 1/(2a)$ and the mapping $K \ : M_2 \to T(M_2)$ naturally defines a nonlinear dynamical system on the "discrete"manifold $M_2 := l_2(\mathbb{Z}; \mathbb{C}^2)$. This set of equations with purely diagonal ("on-site") nonlinearity is sometimes called the diagonal DNLS equation, but since it is the by far most studied example of a DNLS equation, it is most commonly referred to as simply the DNLS equation. Extensions to higher dimensions are straightforward, so that, for example, for a 2-D lattice with $x = ma, y = na$, $m, n \in \mathbb{Z}$, the DNLS equation reads

$$\begin{pmatrix} du_{m,n}/dt \\ d\bar{u}_{m,n}/dt \end{pmatrix} =$$

$$=\begin{pmatrix} \delta i(u_{m+1,n}+u_{m-1,n}+u_{m,n-1}+u_{m,n-1}-4u_{m,n})-2\xi i\bar{u}_{m,n}u_{m,n}^2 \\ -\delta i(\bar{u}_{m+1,n}+\bar{u}_{m-1,n}+\bar{u}_{m,n-1}+\bar{u}_{m,n-1}-4\bar{u}_{m,n})+2\xi i\bar{u}_{m,n}^2u_{m,n} \end{pmatrix}=:$$

$$=: K[u,\bar{u}],$$

where $K:\tilde{M}^2\in T(\tilde{M}_2)$ is the corresponding nonlinear dynamical ssytem on the "discrete"manifold $\tilde{M}^2:=l_2(\mathbb{Z}^2;\mathbb{C}^2)$. The study of the DNLS equations has a long and fascinating history, beginning in the 1950-th within solid state physics with Holstein's model for polaron motion in molecular crystals [26], reappearing in the 1970-th within biophysics with Davydov's model for energy transport in biomolecules (e.g., [45], Chapter 5.6), in the 1980-th within physical chemistry in the theory of local modes of small molecules (e.g., [45], Chapter 5.4) and within nonlinear optics modeling coupled nonlinear waveguides (e.g., [23], Chapter 1.4), and most recently around the turn of the century within matter wave physics in the description of a dilute Bose–Einstein condensate trapped in a periodic potential [46]. A brief account of experimental verifications of the validity of the DNLS description in the two latter contexts available at the time of writing was given in [14]. In addition, the D-DNLS equation has played a central role in the development of the general theory for intrinsic localized modes ("discrete breathers") in systems of coupled anharmonic oscillators during the 1990-th [17]. The reader should also note that the DNLS equation is a particular example of the more general "discrete self-trapping" (DST) systems (described under a separate entry), where the general DST dispersion matrix describing interactions between lattice sites is restricted to nearest-neighbor couplings. Thus, the general theory described for DST systems is also applicable for the DNLS equation.

The reason for the ubiquity of the DNLS equation in nonlinear lattice systems is analogous to that of the one-dimensional NLS equation for continuum systems: it takes into account dispersion as well (through the nearest neighbor interaction terms) as nonlinearity (the term $\xi(u_n\bar{u}_n)u_n, n\in\mathbb{Z}$, at the lowest order of approximation. It can be derived, for example, from a general system of coupled anharmonic oscillators using a "rotating wave approximation", where it is assumed that each oscillator approximately can be described by a complex rotating-wave amplitude [17]. Thus, this approximation assumes time-periodic solutions to have a purely harmonic time dependence, neglecting the generation of all higher harmonics. This approximation can be justified for small-amplitude oscillations in weakly coupled oscillator chains, using perturbational techniques with expansions on multiple time scales [17]. As for general DST systems, the D-DNLS equation has, in addition to the energy functional – Hamiltonian:

$$H = \delta \sum_{n\in\mathbb{Z}} (u_{n+1} - u_n)(\bar{u}_{n+1} - \bar{u}_n) - \xi u_n^2 \bar{u}_n^2),$$

A second conserved quantity, which is the excitation number $N = \sum_{n\in\mathbb{Z}} u_n \bar{u}_n$. The conservation of excitation number results (through Noether's theorem) from the invariance of the equation under infinitesimal transformations of the overall phase ($u_n \to u_n \exp(i\varepsilon)$, $u_n \to u_n \exp(-i\varepsilon)$ as $\varepsilon \to 0$, $n \in \mathbb{Z}$). As a consequence, the DNLS equation is integrable for two degrees of freedom but non-integrable for larger systems. Still, the existence of a second conserved quantity has some notable consequences, which makes the DNLS equation nongeneric among general Hamiltonian lattice systems, such as

$$\begin{pmatrix} du_n/dt \\ d\bar{u}_n/dt \end{pmatrix} = K_n[u, u^*] :=$$

$$= \begin{pmatrix} \delta i(u_{n+1} - 2u_n + u_{n-1}) - \xi i \bar{u}_n u_n (u_{n+1} + u_{n-1}) \\ -\delta i(2\bar{u}_n - u_{n+1} - u_{n-1}) + \xi i \bar{u}_n u_n (\bar{u}_{n+1} + \bar{u}_{n-1}) \end{pmatrix}. \qquad (1.3)$$

It has purely harmonic time-periodic solutions $(u_n(t), \bar{u}(t))^\intercal = (A_n \exp(-i\omega t), \bar{A}_n \exp(i\omega t))^\intercal \in M_2$ with time-independent $(A_n, \bar{A}_n)^\intercal \in \mathbb{C}^2$, $n \in \mathbb{Z}$, ("stationary solutions"). It also has continuous families of time-quasi-periodic solutions, with two incommensurate frequencies, which may be spatially exponentially localized also in infinite systems ("quasi-periodic breathers") (e.g. [14]). From a mathematical point of view, it is highly interesting that there also exist discretizations of the integrable 1-D cubic NLS equation that conserve its integrability. The most famous integrable DNLS equation is the so-called Ablowitz–Ladik AL-DNLS equation, the integrability of which was first proven by Ablowitz & Ladik [2]. The dynamical system (1.3) appears to be a bi-Hamiltonian flow on the discrete manifold M_2 with respect to non-canonical Poisson brackets (see e.g. [15]). There is, at this date, no known direct physical application of the AL-DNLS equation; however, it is commonly used as a starting point for perturbational studies of physically more relevant equations such as (1.2). A particularly interesting model allowing interpolations between Equations (1.2) and (1.3) is the so-called "*Salerno equation*", which is described under a separate entry.

Taking into account a rich analytical structure of solutions to the AL-DNLS equation (1.3), it is naturally to explain this phenomenon by means of its rich symmetry structure. Namely, in our work we study in detail the related differential-geometric and symplectic stucture of these hidden symmetries responsible for the complete integrability by means of the symplectic gradient-holonomic approach devised before in [39, 7] for the smooth nonlinear dynmamical ssytems on functional manifolds.

2. Preliminary notions and definitions. Consider an infinite-dimensional discrete manifold $M_m \subset l^2(\mathbb{Z};\mathbb{C}^m)$ for some integer $m \in \mathbb{Z}_+$ and a nonlinear dynamical system of the form

$$dw/dt = K[w], \tag{2.1}$$

where $w \in M_m$ and $K : M_m \to T(M_m)$ is a Fréchet smooth nonlinear local mapping of M_m into its tangent space $T(M_m)$ and $t \in \mathbb{R}$ is the evolution parameter. As examples of the dynamical system (2.1) at $m = 2$ on a discrete manifold $M_2 \subset l_2(\mathbb{Z};\mathbb{C}^2)$, one can consider the well-known [1, 34] discrete nonlinear Schrödinger equation (1.3) (also known as the Ablowitz–Ladik AL-DNLS equation):

$$\begin{pmatrix} du_n/dt \\ d\bar{u}_n/dt \end{pmatrix} = K_n[u,\bar{u}] :=$$

$$:= \begin{pmatrix} i(u_{n+1} - 2u_n + u_{n-1}) - i\bar{u}_n u_n(u_{n+1} + u_{n-1}) \\ -i(2\bar{u}_n - u_{n+1} - u_{n-1}) + i\bar{u}_n u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) \end{pmatrix}, \tag{2.2}$$

where we put, for brevity, $w = (u,u)^\intercal$, $\delta = 1 = \xi$, (the overbar denotes, as before, the complex conjugate) and the so-called Ragnisco–Tu [42] equation:

$$\begin{pmatrix} du_n/dt \\ dv_n/dt \end{pmatrix} = \tilde{K}_n[u,v] := \begin{pmatrix} u_{n+1} - u_n^2 v_n \\ -v_{n-1} + u_n v_n^2 \end{pmatrix} \tag{2.3}$$

on the funbctional manifold $M_2 \subset l^2(\mathbb{Z};\mathbb{R}^2)$, where we put $w = (u,v)^\intercal \in$ $\in M_2$, which have interesting applications [13, 28, 14] in a wide range of plasma physics problems.

To analyze the integrability properties of the differential-difference dynamical system (2.1), we shall develop a gradient-holonomic scheme related to those devised in [39, 25, 33, 7] for nonlinear dynamical systems defined on spatially one-dimensional functional manifolds and extended in [40] to include discrete manifolds.

Denote by $(\cdot,\cdot)$ the standard bilinear form (or pairing) on the space $T^*(M_m)\times T(M_m)$ naturally induced by the inner product in the Hilbert space $l^2(\mathbb{Z};\mathbb{C}^m)$. We define $\mathcal{D}(M_m)$ to be the space of smooth functionals on M_m, so for any $\gamma\in\mathcal{D}(M_m)$ one can define the gradient grad $\gamma[w]\in$ $\in T^*(M_m)$ as

$$\text{grad }\gamma[u,\bar{u}] := \gamma^{\prime,*}[w]\cdot 1, \tag{2.4}$$

where the prime denotes the Fréchet derivative and "$*$" represents the conjugation with respect to the standard bracket on $T(M_m)\times T^*(M_m)$.

Definition 2.1. *A linear smooth operator $\vartheta : T^*(M_m)\to T(M_m)$ is called Poissonian on the manifold M_m, if the bilinear bracket*

$$\{\cdot,\cdot\}_\vartheta := (\text{grad }(\cdot), \vartheta\text{grad }(\cdot))$$

satisfies [4, 5, 39, 8, 18] *the Jacobi identity on the space $\mathcal{D}(M_m)$ of all smooth functionals on M_m.*

This means, in particular, that the bracket (2.4) satisfies the standard Jacobi identity on $\mathcal{D}(M_m)$.

Definition 2.2. *A linear smooth operator $\vartheta : T^*(M_m)\to T(M_m)$ is called Nötherian* [8, 18, 39] *with respect to the nonlinear dynamical system* (2.1) *if*

$$L_K\vartheta = \vartheta' K - \vartheta K^{\prime,*} - K'\vartheta = 0 \tag{2.5}$$

holds identically on the manifold M_m, where L_K is the Lie-derivative along the vector field $K : M_m\to T(M_m)$.

If the mapping $\vartheta : T^*(M_m)\to T(M_m)$ is invertible with inverse mapping $\vartheta^{-1} := \Omega : T(M_m)\to T^*(M_m)$, it is called *symplectic*. It then follows easily from (2.5) that

$$L_K\Omega = \Omega' K + \Omega K' + K^{\prime,*}\Omega = 0 \tag{2.6}$$

hold identically on M_m. Having now assumed that the manifold $M_m\subset$ $\subset l^2(\mathbb{Z};\mathbb{C}^2)$ is endowed with a smooth Poissonian structure $\vartheta : T^*(M_m)\to$ $T(M_m)$, one can define the Hamiltonian system

$$dw/dt := -\vartheta\text{ grad }H[w], \tag{2.7}$$

corresponding to a Hamiltonian function $H\in\mathcal{D}(M_m)$. It follows directly from the definition (2.7) that the dynamical system $dw/dt = K[w] :=$ $:= -\vartheta$ grad $H[w]$ satisfies the Nötherian conditions (2.5). We are studying the integrability [5, 37, 8, 7] of the discrete dynamical system (2.1).

Accordingly we need to construct invariants with respect to it functions, called conservation laws, which are mutually commuting with respect to the Poisson bracket (2.4). The following Lax criterion [31, 39, 7] proves to be very useful.

Lemma 2.3. *Any smooth solution $\varphi \in T^*(M_m)$ to the Lax equation*

$$L_K\ \varphi = d\varphi/dt + K^{\prime,*}\varphi = 0, \tag{2.8}$$

satisfying the symmetry condition $\varphi' = \varphi^{\prime,*}$, with respect to bracket $(\cdot,\cdot)$, is related to the conservation law

$$\gamma := \int_0^1 d\lambda(\varphi[w\lambda], w). \tag{2.9}$$

Proof. The expression (2.9) follows easily from the well-known Volterra homology equalities

$$\gamma = \int_0^1 \frac{d\gamma[w\lambda]}{d\lambda} d\lambda = \int_0^1 d\lambda(1, \gamma'[w\lambda]\cdot w) =$$

$$= \int_0^1 d\lambda(\gamma^{\prime,*}[w\lambda]\cdot 1, w) = \int_0^1 d\lambda(\text{grad}\ \gamma[w\lambda], w)$$

and $(\text{grad}\ \gamma[w])' = (\text{grad}\ \gamma[w])^{\prime,*}$, holding identically on M_m. Whence, one finds that there exists a function $\gamma \in \mathcal{D}(M_m)$ such that $L_K\gamma = 0$, $\text{grad}\ \gamma[w] = \varphi[w]$ for any $w \in M_m$. Lemma is proved.

This result of Lax lemma is a direct consequence of the following generalized Nöther type result.

Lemma 2.4. *Let a smooth element $\psi \in T^*(M_m)$ satisfy the Nöther condition*

$$L_K\psi = d\psi/dt + K^{\prime,*}\psi = \text{grad}\ \mathcal{L}_\psi \tag{2.10}$$

for some smooth functional $\mathcal{L}_\psi \in \mathcal{D}(M_m)$. Then the following Hamiltonian representation $K = -\vartheta$ grad H_ϑ holds, where $\vartheta := \psi' - \psi^{\prime,*}$ and the Hamiltonian function is $H_\vartheta = (\psi, K) - \mathcal{L}_\psi$.

It is easy to see that Lemma 2.3 follows from Lemma 2.4, if the conditions $\psi' = \psi^{\prime,*}$ and $\mathcal{L}_\psi = 0$ are imposed on (2.10).

Assume now that equation (2.10) allows an additional (non-symmetric) smooth solution $\phi \in T^*(M_m)$:

$$L_K\phi = d\phi/dt + K^{\prime,*}\phi = \text{grad}\ \mathcal{L}_\phi\,. \tag{2.11}$$

This means that our system (2.1) is bi-Hamiltonian: $-\vartheta\ \mathrm{grad} H_\vartheta = K =$ $= -\eta\ \mathrm{grad}\ H_\eta$, where, by definition,

$$\eta := \phi^{'} - \phi^{'*}, \qquad H_\eta = (\phi, K) - \mathcal{L}_\phi\,. \tag{2.12}$$

Definition 2.5. *One says that two Poissonian structures* ϑ, η : $T^*(M_m) \to T(M_m)$ *on* M_m *are compatible* [32, 18, 39, 8], *if for any* $\lambda, \mu \in \mathbb{R}$ *the linear combination* $\lambda\vartheta + \mu\eta : T^*(M_m) \to T(M_m)$ *will be also Poissonian on* M_m.

It is easy to see that this condition is satisfied if, for instance, there exists an inverse $\vartheta^{-1} : T(M_m) \to T^*(M_m)$ and the composite map $\eta(\vartheta^{-1}\eta) : T^*(M_m) \to T(M_m)$ is also Poissonian on M_m.

Concerning the complete integrability of the infinite-dimensional dynamical system (2.1) on the discrete manifold M_m it is, in general, necessary, but not sufficient [37, 39, 7], to prove the existence of an infinite hierarchy of mutually commuting conservation laws with respect to the Poissonian structure (2.4).

Since in the case of Lax integrability of (2.1) there exist compatible Poissonian structures and related hierarchies of conservation laws, we shall focus our analysis by devising an integrability algorithm under the *a priori* assumption that the nonlinear dynamical system (2.1) on the manifold M_m is Lax integrable. This means that it possesses a Lax representation in the following general form:

$$\Delta f_n := f_{n+1} = l_n[w;\lambda] f_n\,, \tag{2.13}$$

where $f := \{f_n \in \mathbb{C}^r : n \in \mathbb{Z}\} \subset l^2(\mathbb{Z}; \mathbb{C}^r)$ for some integer $r \in \mathbb{Z}_+$ and the matrices $l_n[w;\lambda] \in End\mathbb{C}^r$, $n \in \mathbb{Z}$, in (2.13) are local matrix-valued functionals on M_m, depending on the "spectral" parameter $\lambda \in \mathbb{C}$, invariant with respect to our dynamical system (2.1).

As the Lax representation (2.13) is 'local' with respect to the discrete variable $n \in \mathbb{Z}$, we shall assume for convenience that our manifold $M_m :=$ $:= M_m^{(N)} \subset l^\infty(\mathbb{Z}/N\mathbb{Z}; \mathbb{C}^m)$ is periodic with respect to the discrete index $n \in \mathbb{Z}_N$, that is for any $n \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ and $\lambda \in \mathbb{C}$

$$l_n[w;\lambda] = l_{n+N}[w;\lambda] \tag{2.14}$$

for some integer $N \in \mathbb{Z}_+$. In this case the smooth functionals on $M_m^{(N)}$ can be represented as $\gamma := \sum_{n\in\mathbb{Z}_N} \gamma_n[w]$ for some local Fréchet smooth densities $\gamma_n : M_m^{(N)} \to \mathbb{C}$, $n \in \mathbb{Z}_N$.

3. Integrability analysis: the gradient-holonomic scheme. Consider the representation (2.13) and define its fundamental solution $F_{m,n}(\lambda) \in Aut(\mathbb{C}^r)$, $m, n \in \mathbb{Z}_N$, satisfying the equation

$$F_{m+1,n}(\lambda) = l_m[w;\lambda] F_{m,n}(\lambda)$$

and the condition $F_{m,n}(\lambda)|_{m=n} = \mathbf{1}$ for all $\lambda \in \mathbb{C}$ and $n \in \mathbb{Z}_N$. Then the matrix function

$$S_n(\lambda) := F_{n+N,n}(\lambda) \tag{3.1}$$

is called the *monodromy* matrix for the linear equation (2.14) and satisfies for all $n \in \mathbb{Z}_N$ the Novikov–Lax relationship

$$S_{n+1}(\lambda) l_n = l_n S_n(\lambda). \tag{3.2}$$

It easy to compute that $S_n(\lambda) := \prod_{k=0}^{N-1} l_{n+k}[u;\lambda]$ owing to the periodicity condition (2.14). Construct now the generating functional

$$\bar{\gamma}(\lambda) := \mathrm{tr} S_n(\lambda), \tag{3.3}$$

where tr is the standard trace map, having the asymptotic expansion

$$\bar{\gamma}(\lambda) \sim \sum_{j \in \mathbb{Z}_+} \bar{\gamma}_j \lambda^{j_0 - j} \tag{3.4}$$

as $\lambda \to \infty$ for some fixed $j_0 \in \mathbb{Z}_+$. Then, owing to the obvious condition $D_n \bar{\gamma}(\lambda) = 0$ for all $n \in \mathbb{Z}_N$, where we have introduced the 'discrete' derivative $D_n := \Delta - 1$, we find that all functionals $\bar{\gamma}_j \in \mathcal{D}(M_m^{(N)})$, $j \in \mathbb{Z}_+$, are independent of the discrete index $n \in \mathbb{Z}_N$ and are simultaneously conservation laws for the dynamical system (2.1).

We now make an additional natural assumption, namely that the gradient vector

$$\bar{\varphi}(\lambda) := \mathrm{grad}\ \bar{\gamma}(\lambda)[w] = \mathrm{tr} l_n'^{,*}(S_n(\lambda) l_n^{-1}), \tag{3.5}$$

solving the Lax determining equation (2.8), satisfies, owing to (3.2), for all $\lambda \in \mathbb{C}$,

$$z(\lambda) \vartheta\ \bar{\varphi}(\lambda) = \eta\ \bar{\varphi}(\lambda), \tag{3.6}$$

where $z : \mathbb{C} \to \mathbb{C}$ is a meromorphic function, and ϑ and $\eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$ are compatible Poissonian operators on the manifold $M_m^{(N)}$ that are Nötherian with respect to the dynamical system (2.1). Then it follows at once that the generating functional $\bar{\gamma}(\lambda) \in \mathcal{D}(M_m^{(N)})$ satisfies the commutation relationships

$$\{\bar{\gamma}(\lambda), \bar{\gamma}(\mu)\}_\vartheta = 0 = \{\bar{\gamma}(\lambda), \bar{\gamma}(\mu)\}_\eta \tag{3.7}$$

for all $\lambda, \mu \in \mathbb{C}$. Consequently, if we define on $M_{(N)}$ a generating dynamical system $dw/d\tau := -\vartheta \text{ grad } \bar{\gamma}(\lambda)[w]$ as $\lambda \to \infty$, it follows from (3.7) that the hierarchy of functionals defined by the coefficients in (3.4) comprise its conservation laws.

With the importance of invariants and Poissonian structures related to the linear spectral problem (2.13) firmly in mind, we now describe its main Lie-algebraic properties and connections with the whole hierarchy of integrable differential-difference dynamical systems on the manifold M_m. More precisely, we sketch the Lie-algebraic aspects [36, 15, 43, 44] of the differential-difference dynamical systems associated with the Lax linear difference spectral problem (2.13). In this process we shall assume that $l_n := l_n[w;\lambda] \in G_n := GL^2(\mathbb{C}) \otimes \mathbb{C}(\lambda, \lambda^{-1})$ for $n \in \mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ as $\lambda \to \infty$. To describe the related Lax integrable dynamical systems, we first define first the matrix product-group $G^N := \otimes_{j=1}^N G_j$ and its action $G^N \times M_G^{(N)} \to M_G^{(N)}$ on the phase space $M_G^{(N)} := \{l_n \in G_n : n \in \mathbb{Z}_N\}$, given as $\{g_n \in G_n : n \in \mathbb{Z}_N\} \times \{l_n \in G_n : n \in \mathbb{Z}_N\} = \{g_n l_n g_{n+1}^{-1} \in G_n : n \in \mathbb{Z}_N\}$. A functional $\gamma \in \mathcal{D}(M_G^{(N)})$ is invariant for this action iff the following discrete relationship

$$\text{grad}\gamma(l_n) l_n = l_{n+1} \text{grad}\gamma(l_{n+1}) \tag{3.8}$$

holds for all $n \in \mathbb{Z}_N$.

We assume further that the matrix group G^N is identified with its tangent spaces $T_l(G^N)$, $l \in G^N$, which is locally isomorphic to the Lie algebra $\mathcal{G}^{(N)}$, where $\mathcal{G}^{(N)}$ is the corresponding Lie algebra of the Lie group G^N, which is isomorphic to the tangent space $T_e(G^N)$ at the group unity $e \in G^N$. With any element $l \in G^N$ there are associated, respectively, the left $\eta_l : \mathcal{G}^{(N)} \to T_l(G^N)$ and right $\rho_l : \mathcal{G}^{(N)} \to T_l(G^N)$ differentials of the left and right translations on the Lie group G^N, and their adjoint mappings $\rho_l^* : T_l^*(G^N) \to \mathcal{G}^{(N),*}$ and $\eta_l^* : T_l^*(G^N) \to \mathcal{G}^{(N),*}$, where

$$(\rho_l^* \text{grad}\gamma(l), X) = (\text{grad}\gamma(l), Xl) = (l \text{ grad}\gamma(l), X) := \text{Tr}(l \text{ grad}\gamma(l) X),$$

$$(\eta_l^* \text{grad}\gamma(l), X) = (\text{grad}\gamma(l), lX) = (\text{grad}\gamma(l) l, X) := \text{Tr}(\text{grad}\gamma(l) l X) \tag{3.9}$$

for any $X \in \mathcal{G}^{(N)}$ and smooth functional $\gamma \in \mathcal{D}(G^N)$. Here $\mathrm{Tr} : G^N \to \mathbb{C}$ is the trace operation on the group G^N defined as

$$\mathrm{Tr}A := \mathrm{res}_{\lambda=\infty} \sum_{j\in\mathbb{Z}_N} SpA_j[u, \bar{u}; \lambda]$$

for any $A \in G^N$. By virtue of (3.8) and (3.9), we can define the set $\{\Phi_n = = \mathrm{grad}\gamma(l_n)l_n \in \mathcal{G}_n^* := T_e^*(G),\ n \in \mathbb{Z}_N\}$ belonging to the space $\mathcal{G}^{(N),*} \simeq \simeq T_e^*(G^N)$ and satisfying the following invariance property:

$$\Phi_{n+1} = Ad_{l_n}^* \Phi_n(\lambda) = l_n^{-1}\Phi_n(\lambda)l_n \tag{3.10}$$

for any $n \in \mathbb{Z}_N$. The relationship (3.10) allows to define a function $\varphi : G^N \to \mathbb{C}$ invariant with respect to the adjoint action $G_n \times G_n \ni (g, S_n(\lambda)) \to ad_g S_n(\lambda) = gS_n(\lambda)g^{-1} \in G_n$ for any $n \in \mathbb{Z}_N$ and such that

$$\gamma(l) = \varphi[S_N(\lambda)],\ \Phi_N = \mathrm{grad}\varphi[S_N(\lambda)]S_N(\lambda), \tag{3.11}$$

where, by definition, the expression

$$S_N(\lambda) = \prod_{j=1}^{N} l_j[u, \bar{u}; \lambda] \tag{3.12}$$

coincides exactly with the proper monodromy matrix for the linear spectral problem (2.13). Owing to (3.10), the matrices $\Phi_n = \mathrm{grad}\varphi[S_n(\lambda)]S_n(\lambda) \in \in \mathcal{G}_n^*, n \in \mathbb{Z}_N$, can be reconstructed from (3.12). Therefore, we have [15, 44] the following Poissonian flow on the matrices $S_n(\lambda) \in G_n, n \in \mathbb{Z}_N$:

$$dS_n(\lambda)/dt = [\mathcal{R}(\mathrm{grad}\varphi[S_n(\lambda)]S_n(\lambda)), S_n(\lambda)] \tag{3.13}$$

with respect to the invariant Casimir function $\varphi \in I(\mathcal{G}_n^*)$ and the quadratic Poissonian structure

$$\{\gamma_1, \gamma_2\} := (l, [\mathrm{grad}\gamma_1(l), \mathcal{R}(l\ \mathrm{grad}\gamma_2(l))] + [\mathcal{R}(l\ \mathrm{grad}\gamma_1(l)), \mathrm{grad}\gamma_2(l)]) \tag{3.14}$$

for any functionals $\gamma_1, \gamma_2 \in \mathcal{D}(G^N)$, which is constructed by means of a skew-symmetric $\mathcal{R}$-structure $\mathcal{R} : \mathcal{G}^{(N),*} \to \mathcal{G}^{(N)}$. In particular, the equality $[\mathrm{grad}\varphi(S_n), S_n] = 0$ holds for all $n \in \mathbb{Z}_N$.

Taking into account (3.11), one can rewrite (3.13) as $dS_n/dt = = [\mathcal{R}(\mathrm{grad}\gamma(l_n)l_n), S_n]$ for all $n \in \mathbb{Z}_N$. This together with (3.10) makes it possible to retrieve [27, 43] the related evolution of elements $l_n \in G_n$, $n \in \mathbb{Z}_N$:

$$dl_n/dt = p_{n+1}(l)l_n - l_n p_n(l), \tag{3.15}$$
$$p_n(l) := \mathcal{R}(\text{grad}\gamma(l_n)l_n)$$

from the relationships

$$S_n(\lambda) = \psi_n(l) S_N(\lambda) \psi_n^{-1}(l)\,, \quad \psi_n(l) = \prod_{j=1}^{n} l_j[u, v; \lambda]\,.$$

The solution $f \in l^\infty(\mathbb{Z}, \mathbb{C}^2)$ to the linear spectral problem (2.13) satisfies the associated temporal evolution equation

$$df_n/dt = p_n(l) f_n \tag{3.16}$$

for any $n \in \mathbb{Z}$. It is easy to check that the compatibility condition for the linear equations (2.13) and (3.16) is equivalent to the discrete Lax representation (3.15), which upon reduction on the group manifold M_G, gives rise to the corresponding nonlinear Lax integrable dynamical system on the discrete manifold $M_m^{(N)}$. Hence, all Casimir invariant functions, when reduced on the manifold M_G, are in involution [43, 44, 16] with respect to the Poisson bracket (3.14).

Since the existence of an infinite hierarchy of mutually commuting conservation laws is a characteristic of the Lax integrability of the nonlinear dynamical system (2.1), this property can be effectively implemented into the scheme of our analysis. Namely, we have the following result.

Proposition 3.1. *The determiming Lax equation* (2.8) *allows the following asymptotic* $(as\ \lambda \to \infty)$ *periodic solution* $\varphi(\lambda) \in T^*(M_m^{(N)})$:

$$\varphi_n(\lambda) \sim a_n(\lambda) \exp[\omega(t; \lambda)] \prod_{j=0}^{n} \sigma_j(\lambda), \tag{3.17}$$

where for all $n \in \mathbb{Z}$

$$a_n(\lambda) := (1, a_{(1),n}[w; \lambda], a_{(2),n}[w; \lambda], ..., a_{(m-1),n}[w; \lambda])^\tau, \tag{3.18}$$
$$a_{(k),n}(\lambda) \sim \sum_{s \in \mathbb{Z}_+} a_{(k),n}^{(s)}[w] \lambda^{-s+\tilde{a}}, \sigma_j(\lambda) \sim \sum_{s \in \mathbb{Z}_+} a_j^{(s)}[w] \lambda^{-s+\tilde{\sigma}},$$

$1 \leq k \leq m-1$ *and* $\omega(t; \cdot) : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}$, *is a dispersion function. Moreover, the functional* $\gamma(\lambda) := \sum_{n \in \mathbb{Z}_N} \ln(\lambda^{-\tilde{\sigma}} \sigma_n[w; \lambda]) \in \mathcal{D}(M_m^{(N)})$ *is a generating function of conservation laws for the dynamical system* (2.1).

Proof. Lemma 2.3 and relationship (3.5) imply that the functional (3.3) is a conservation law for our dynamical system (2.1). Whence, expression (3.1) and equation (2.13) lead to the solution representation (3.17) for the Lax equation (2.8). Now, making use of the periodicity of the manifold $M_m^{(N)}$, it follows from the period translation of (3.17) that the functional

$$\gamma(\lambda) := \sum_{n\in\mathbb{Z}_N} \ln(\lambda^{-\tilde{\sigma}}\sigma_n[w;\lambda]) \sim \sum_{j\in\mathbb{Z}_+} \gamma_j\lambda^{-j} \tag{3.19}$$

generates an infinite hierarchy of conservation laws to (2.1), which completes the proof.

Thus, if we start the Lax integrability analysis of a given nonlinear dynamical system (2.1), it is necessary, as the first step, to study the asymptotic solutions (3.17) to the corresponding Lax equation (2.8). These solutions are then used to construct a related hierarchy of conservation laws in the functional form (3.19), taking into account expansions (3.18).

Remark 3.2. It is easy to observe that, owing to the arbitrariness of the period $N \in \mathbb{Z}_+$ of the manifold $M_m^{(N)}$, all of the finite-sum expressions obtained above can be generalized to the corresponding infinite-dimensional manifold $M_m \subset l^2(\mathbb{Z};\mathbb{C}^m)$, if the associated infinite series are convergent.

Since our dynamical system (2.1) induces a bi-Hamiltonian flow on the manifold $M_{(N)}$ under the above circumstances, the next step is to analyze the related compatible Poissonian or symplectic structures, satisfying, respectively, either equality (2.5) or equality (2.6). Before doing this, we shall need the following useful result.

Lemma 3.3. *All functionals* $\gamma_j \in \mathcal{D}(M_m^{(N)})$ *in the expansion* (3.19) *are mutually with respect to both Poissonian structures* $\vartheta, \eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$ *satisfying the gradient relationship* (3.20).

Proof. It follows from the representations (3.17) and (3.5) that the followinf asymptotic (as $\lambda \to \infty$) relationship holds:

$$\ln\bar{\gamma}(\lambda) \simeq \gamma(\lambda). \tag{3.20}$$

Since the generating function $\bar{\gamma}(\lambda) \in \mathcal{D}(M_m^{(N)})$ satisfies the commutation relationships (3.7), the same also holds, owing to (3.20), for the generating function $\gamma(\lambda) \in \mathcal{D}(M_m^{(N)})$. Thus, the proof is complete.

We proceed now with the construction of the Poissonian structures $\vartheta, \eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$ for the dynamical system (2.1). Note that these Poissonian structures are also Nötherian for the whole hierarchy of dynamical systems

$$dw/dt_j := -\vartheta \text{ grad } \gamma_j[w], \qquad t_j \in \mathbb{R}, j \in \mathbb{Z}_+, \tag{3.21}$$

are the corresponding evolution parameters, and which, owing to (3.7), commute with each other on the manifold $M_m^{(N)}$. Therefore, it possible to apply Lemma 2.4 to any one of the dynamical systems (3.21) if the related vector fields commuting with (2.1) are assumed known.

To solve equation (2.10) for an element $\varphi \in T^*(M_m^{(N)})$ one can, in the case of a polynomial dynamical system (2.1), make use of the well-known asymptotic small parameter method [39, 33]. When applying this approach, it is necessary to take into account the following expansions at zero — element $(u, \bar{u})^\intercal = 0 \in M_m^{(N)}$ with respect to the small parameter $\mu \to 0$:

$$\begin{aligned}
w := \mu w^{(1)},\ \varphi[w^{(1)}] &= \varphi^{(0)} + \mu\varphi^{(1)}[w^{(1)}] + \mu^2\varphi^{(2)}[w^{(1)}] + \dots,\\
d/dt &= d/dt_0 + \mu d/dt_1 + \mu^2 d/dt_2 + \dots,\\
K[w^{(1)}] &= \mu K^{(1)}[w^{(1)}] + \mu^{(2)}K^{(2)}[w^{(1)}] + \dots,\\
K'[w^{(1)}] &= K'_0 + \mu K'_1[w^{(1)}] + \mu^2 K'_2[w^{(1)}] + \dots,\\
\text{grad } \mathcal{L}[w^{(1)}] &= \text{grad } \mathcal{L}^{(0)} + \mu\text{grad } \mathcal{L}^{(1)}[w^{(1)}] + \mu^2\text{grad } \mathcal{L}^{(2)}[w^{(1)}] + \dots\ .
\end{aligned}$$

After solving the corresponding set of linear nonuniform functional equations

$$\begin{aligned}
&d\varphi^{(0)}/dt_0 + K_0'^{*}\varphi^{(0)} = \text{grad } \mathcal{L}^{(0)},\\
&d\varphi^{(1)}/dt_0 + K_0'^{*}\varphi^{(1)} = \text{grad } \mathcal{L}^{(1)} - K_0'^{*}\varphi^{(0)},\\
&d\varphi^{(2)}/dt_0 + K_0'^{*}\varphi^{(2)} = \text{grad } \mathcal{L}^{(2)} - K_1'^{*}\varphi^{(1)} - K_2'^{*}\varphi^{(0)}
\end{aligned}$$

and so on, using Fourier transforms applied to the suitable N-periodic functions, one can obtain the related Poissonian structure in the series form $\vartheta^{-1} = \varphi^{(0),\prime} - \varphi^{(0),\prime *} + \mu(\varphi^{(1),\prime} - \varphi^{(1),\prime *}) + \dots$ and finally set $\mu = 1$.

Another direct way of obtaining a Poissonian operator $\vartheta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$ for (2.1) is the following: First reduce the Nötherian equation (2.5) to the set of linear nonuniform equations

$$\begin{aligned}
&\frac{d}{dt_0}(\vartheta_0\varphi^{(0)}) = K'_0(\vartheta_0\varphi^{(0)})\,,\\
&\frac{d}{dt_0}(\vartheta_1\varphi^{(0)}) = K'_0(\vartheta_1\varphi^{(0)}) + \vartheta_0 K_1^{\prime,*}\varphi^{(0)} + K'_1\vartheta_0\varphi^{(0)},
\end{aligned}$$

$$\frac{d}{dt_0}(\vartheta_2\varphi^{(0)}) = K_0'(\vartheta_2\varphi^{(0)}) - \varphi^{(0)\prime}K^1 + \vartheta_0 K_2^{\prime,*}\varphi^{(0)} + \\ + \vartheta_1 K_1^{\prime,*}\varphi^{(0)} + \vartheta_2 K_0^{\prime,*}\varphi^{(0)} + K_1'\vartheta_1\varphi^{(0)} + K_2'\vartheta_0\varphi^{(0)},$$

and then solve using the above small parameter asymptotics. The analytical expressions for actions $\vartheta_j : \varphi^{(0)} \to \vartheta_j\varphi^{(0)}$, $j \in \mathbb{Z}_+$ can now be used to retrieve them in operator form from the expansion $\vartheta = \vartheta_0 + \mu\vartheta_1 + \mu^2\vartheta_2 + + \dots$, by setting $\mu = 1$ at the end of the calculations. Similarly one can also construct the second Poissonian operator $\eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$ for the nonlinear dynamical system (2.1).

Now the next result follows directly from all of the above analysis.

Proposition 3.4. *Let a nonlinear dynamical system* (2.1) *on a discrete manifold $M_m^{(N)}$ admit both a nontrivial symmetric solution $\varphi \in T^*(M_m^{(N)})$ to the Lax equation* (2.8) *in the asymptotic as form* (3.17) *as $\lambda \to \infty$, generating an infinite hierarchy of nontrivial functionally independent conservation laws* (3.19), *and compatible nonsymmetric solutions ψ and $\phi \in T^*(M_m^{(N)})$ to the Nöther equations* (2.10) *and* (2.11), *respectively. Then this dynamical system is a Lax integrable bi-Hamiltonian flow on $M_m^{(N)}$ with respect to two compatible Poissonian structures $\vartheta, \eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$, whose adjoint Lax representation*

$$d\Lambda/dt = [\Lambda, K^{\prime,*}], \tag{3.22}$$

where $\Lambda := \vartheta^{-1}\eta$, is the so-called recursion operator. This operator can be transformed, in virtue of the gradient relationship (3.6), *to the standard discrete Lax form $dl_n/dt = [p_n(l), l_n] + (D_n p_n(l))l_n$ for some matrix $p_n(l) \in End\mathbb{C}^r$ describing the temporal evolution $df_n/dt = p_n(l)f_n$ related to* (2.13), *for $f \in l^\infty(\mathbb{Z}; \mathbb{C}^r)$.*

Remark 3.5. *Inasmuch as all Hamiltonian flows* (3.21) *commute with each other and the dynamical system* (2.1), *and since they possess the same Poissonian and compatible (ϑ, η)-pair, the analytical algorithm described above can also be applied to any other flow commuting with* (2.1).

Solutions to the discrete linear Lax problem (2.13) can be constructed by means of the gradient-holonomic algorithm devised in [39, 25, 7] for studying the integrability of nonlinear dynamical systems on functional manifolds. More specifically, by making use of the preliminary analytical expressions for the related compatible Poissonian structures $\vartheta, \eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$ on the manifold $M_m^{(N)}$ and using the fact that the recursion operator $\Lambda := \vartheta^{-1}\eta : T^*(M_m^{(N)}) \to T^*(M_m^{(N)})$ satisfies the dual Lax commutator equality (3.22), one can retrieve the standard Lax representation for it in terms of algebraic formulas. As a corollary of Proposition 3.4, one has the existence of a nontrivial asymptotic (as $\lambda \to \infty$) solution to the Lax equation (2.8), which provides an effective Lax integrability criterion for a dynamical system (2.1) on the manifold $M_m^{(N)}$.

4. The Bogoyavlensky–Novikov finite-dimensional reduction. In this section, we assume that our dynamical system (2.1) on the periodic manifold $M_m^{(N)}$ is Lax integrable and possesses two compatible Poissonian structures $\vartheta, \eta : T^*(M_m^{(N)}) \to T(M_m^{(N)})$. Thus, we have the nonlinear finite-dimensional dynamical system

$$dw/dt := K_n[w] = -\vartheta \text{ grad } H_n[w] \tag{4.1}$$

for indices $n \in \mathbb{Z}_N$, owing to its N-periodicity. The finite-dimensional dynamical system (4.1) can be equivalently considered as that on the finite-dimensional space $M_m^{(N)} \simeq (\mathbb{C}^m)^N$ parameterized by an integer index $n \in \mathbb{Z}_N$. The Liouville integrability of this system is our next concern. To study the flow (4.1) on the manifold $M_{(N)}$, we shall make use of the Bogoyavlensky–Novikov [37, 11] reduction scheme [37, 39, 8, 40].

Let $\Lambda(M_m^{(N)}) := \otimes_{j=0}^N \Lambda^j(M_m^{(N)})$ be the standard finitely generated Grassmann algebra [5, 39, 7] of differential forms on the manifold $M_{(N)}$. Then the differential complex

$$\Lambda^0(M_m^{(N)}) \xrightarrow{d} \Lambda^1(M_m^{(N)}) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^j(M_m^{(N)}) \xrightarrow{d} \Lambda^{j+1}(M_m^{(N)}) \xrightarrow{d} \cdots,$$

where $d : \Lambda(M_m^{(N)}) \to \Lambda(M_m^{(N)})$ is the exterior differentiation, is finite and exact. Since the discrete "derivative"$D_n := \Delta - 1$ commutes with the differentiation $d : \Lambda(M_m^{(N)}) \to \Lambda(M_m^{(N)})$, $[D_n, d] = 0$ for all $n \in \mathbb{Z}_N$, and for any element $a \in \Lambda^0(M_m^{(N)})$

$$\text{grad}\Big(\sum_{n\in\mathbb{Z}_N} D_n a_n[w]\Big) = 0, \tag{4.2}$$

one can formulate the following Gelfand–Dikiy type [19] result.

Lemma 4.1. *Let $\mathcal{L}[w] \in \Lambda^0(M_m^{(N)})$ be a Fréchet smooth local Lagrangian functional on the manifold $M_m^{(N)}$. Then there exists a differential*

1-form $\alpha^{(1)} \in \Lambda^1(M_m^{(N)})$, *such that the equality*

$$d\mathcal{L}_n[w] =< \text{grad } \mathcal{L}_n[w], d(w)^{\intercal} > + D_n\alpha_n^{(1)}[w] \tag{4.3}$$

holds for all $n \in \mathbb{Z}_N$.

Proof. One can easily see that

$$d\mathcal{L}_n[w] = \sum_{j=0}^{N-1} < \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, dw_{n+j} >= \sum_{j=0}^{N-1} < \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, \Delta^j dw_n >=$$

$$=< \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}, dw_n >= D_n\Big(\sum_{j=0}^{N-1}(< p_j, dw_{n+j} >\Big),$$

where $p_k := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j+k+1}}$ for $k = 0, \ldots, N-1$. Having defined the expression grad $\mathcal{L}_n[w] := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[w]}{\partial w_{n+j}}$, one obtains the result (4.3), where

$$\alpha_n^{(1)}[w] := \sum_{j=0}^{N-1} < p_j, dw_{n+j} > \tag{4.4}$$

is the corresponding differential 1-form on the manifold $M_m^{(N)}$, thereby concluding the proof.

Exterior differentiating expression (4.3), we obtain that

$$-D_n\omega_n^{(2)}[w] =< d \text{ grad } \mathcal{L}_n[w], \wedge dw > \tag{4.5}$$

for any $n \in \mathbb{Z}$, where the 2-form

$$\omega^{(2)}[w] := d\alpha^{(1)}[w] \tag{4.6}$$

is nondegenerate on $M_m^{(N)}$ if the Hessian $\partial_n^2 \mathcal{L}[w]/\partial^2 w$ is also nondegenerate.

Consider the manifold

$$\bar{M}_m^{(N)} := \left\{ \text{grad } \mathcal{L}_n^{(\tilde{N})}[w] = 0;\ w \in M_m^{(N)} \right\}, \tag{4.7}$$

where the Lagrangian functional is defined as

$$\mathcal{L}^{(\bar{N})} := -\gamma_{\bar{N}} + \sum_{j=0}^{\bar{N}-1} c_j \gamma_j \tag{4.8}$$

with $\gamma_j \in \mathcal{D}(M_m^{(N)})$, $j = 0, \ldots, \bar{N}-1$, for some $\bar{N} \in \mathbb{Z}_+$, being suitable nontrivial conservation laws for the dynamical system (2.1) as constructed above. Here $c_j \in \mathbb{C}$, $\leq j \leq \bar{N}-1$, are arbitrary but fixed constants. It follows from (4.7) and (4.5) that the closed 2-form $\omega^{(2)} \in \Lambda^2(M_m^{(N)})$ is invariant with respect to the index $n \in \mathbb{Z}_N$ on the manifold $\bar{M}_m^{(N)}$. Moreover, the submanifold (4.7) is also invariant both with respect to the index $n \in \mathbb{Z}_N$ and the evolution parameter $t \in \mathbb{R}$. In fact, for any $n \in \mathbb{Z}_N$ the Lie derivative $L_K \mathrm{grad}\, \mathcal{L}^{(\bar{N})} = (\mathrm{grad}\, \mathcal{L}^{(\bar{N})})' K + K'^{,*}(\mathrm{grad}\, \mathcal{L}^{(\bar{N})}) = 0$, since the functional $\mathcal{L}_n^{(\bar{N})}[w] \in \mathcal{D}(\bar{M}_m^{(N)})$ is a sum of conservation laws for the dynamical system (2.1), whose gradients satisfies the Lax condition (2.8). In addition, it is easy to see that if the Lie derivative $L_K \mathrm{grad} \mathcal{L}_n^{(\bar{N})}[w] = 0, n \in \mathbb{Z}_N$, at $t = 0$, then grad $\mathcal{L}_n^{(\bar{N})}[w] = 0$ for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}_N$. Thus, the Bogoyavlensky–Novikov reduction of the dynamical system (2.1) upon the invariant submanifold $\bar{M}_m^{(N)}$ is completely invariantly defined.

At this point there is a natural question to ask: What is the relationship between the dynamical system (2.1) restricted to the submanifold $M_m^{(N)}$ and the dynamical system (2.1) reduced on the finite-dimensional submanifold $\bar{M}_m^{(N)} \subset M_m^{(N)}$? To further analyze the reduction, we consider the equation

$$< \mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w], K_n[w] >= -D_n h_n^{(t)}[w], \tag{4.9}$$

for a local functional $h^{(t)}[w] \in \Lambda^0(M_m)$, which follows from the conditions (4.2) and (2.8):

$$\begin{aligned} &\mathrm{grad} < \mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w], K_n[w] >= \\ &= (\mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w])'^{,*} K_n[w] + K_n'^{,*}[w] \mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w] = \\ &= (\mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w])' K_n[w] + K_n'^{,*}[w] \mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w] = L_K \mathrm{grad}\ \mathcal{L}_n^{(\bar{N})}[w] = 0, \end{aligned}$$

Since on the submanifold $\bar{M}_m^{(N)}$ the gradient grad $\mathcal{L}_n^{(\tilde{N})}[w] = 0$ for all $n \in \mathbb{Z}_N$, we deduce from (4.9) that the local functional $h^{(t)}[w] \in \Lambda^0(\bar{M}_m^{(N)})$ does not depend on index $n \in \mathbb{Z}_N$.

The properties of the manifold $\bar{M}_m^{(N)}$ described above, make it possible to consider it as a symplectic manifold endowed with the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_m^{(N)})$ given by expressions (4.4) and (4.6). From this point of view we can study the integrability properties of the dynamical system (2.1) reduced on the invariant finite-dimensional manifold $\bar{M}_m^{(N)} \subset M_m^{(N)}$.

First, we observe that the vector field d/dt on $\bar{M}_{(N)}$ is canonically Hamiltonian [4, 5, 37] with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2(\tilde{M}_{(N)})$, i.e.

$$-i_{\frac{d}{dt}}\omega^{(2)}(u,p) = dh^{(t)}(u,p), \tag{4.10}$$

where $h^{(t)}(w,p) := h^{(t)}[w], \omega^{(2)}(w,p) := \omega^{(2)}[w]$ and $(w,p)^\intercal \in \bar{M}_m^{(N)}$ are canonical variables induced on the manifold $\bar{M}_m^{(N)}$ by the Liouville 1-form (4.4). More specifically, from expression (4.9) one obtains that

$$di_{\frac{d}{dt}} < \text{grad } \mathcal{L}_n^{(\bar{N})}[w], dw_n >= -D_n dh_n^{(t)}[w],$$

which together with the identity (4.5) in the form

$$i_{\frac{d}{dt}} d < \text{grad } \mathcal{L}_n^{(\bar{N})}[w], dw_n >= -D_n i_{\frac{d}{dt}}\omega_n^{(2)}[w],$$

leads to

$$\frac{d}{dt} < \text{grad } \mathcal{L}_n^{(\bar{N})}[w], dw_n >= -D_n(dh_n^{(t)}[w] + i_{\frac{d}{dt}}\omega_n^{(2)}[w]). \tag{4.11}$$

Since grad $\mathcal{L}^{(\bar{N})}[w] = 0 = L_K$ grad $\mathcal{L}[w]$ identically on $\bar{M}_m^{(N)}$, from (4.11) one obtains the result (4.10).

The same is true of any of the Hamiltonian systems (3.21) commuting with (2.1) on the manifold M_m. Moreover, owing to the functional independence of invariants $\gamma_j \in \mathcal{D}(M_m^{(N)})$, $0 \leq j \leq N-1$, in the Lagrangian functional (4.8), we can construct a set of functionally independent functions $h^{(j)} \in \mathcal{D}(\bar{M}_m^{(N)})$, $j = 0, \ldots, \bar{N}-1$, as follows: $<$ grad $\mathcal{L}_n^{(\bar{N})}[w], \vartheta$ grad $\gamma_{j,n}[w] >= D_n h_n^{(j)}[w]$. It is easy to check that these functions $h^{(j)} \in \mathcal{D}(\bar{M}_m^{(N)}), 0 \leq j \leq \bar{N}-1$, are invariant with respect to indices $n \in \mathbb{Z}_N$ and commute with each other and the Hamiltonian function $h^{(t)} \in \mathcal{D}(\bar{M}_m^{(N)})$ with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_m^{(N)})$. Thus, if the dimension $\dim \tilde{M}_{(N)} = 2\tilde{N}$, the discrete dynamical system (2.1) reduced upon the finite-dimensional submanifold $\bar{M}_m^{(N)} \subset M_m^{(N)}$ is Liouville integrable. If the set of conservation laws $\gamma_j \in \mathcal{D}(M_m^{(N)})$, $j = 0, \ldots, N-1$, is functionally dependent on $M_m^{(N)}$, the scheme can be modified using the Dirac reduction technique [4, 8, 39] for determining a regular symplectic structure $\bar{\omega}^{(2)}[w] \in \Lambda^2(\bar{M}_m^{(N)})$ on an invariant nonsingular submanifold $\bar{M}_m^{(N)}$.

5. Examples: differential-difference nonlinear Schrödinger and Ragnisco–Tu dynamical systems and their integrability.

5.1. The discrete nonlinear Schrödinger dynamical system. The discrete nonlinear Schrödinger dynamical system (2.2) is defined on the periodic manifold $M_2 \subset l^{\infty}(\mathbb{Z};\mathbb{C}^2)$. Its Lax type integrability was proved in [1, 34, 10] making use of the simplest discretization of the standard Zakharov–Shabat spectral problem for the well-known nonlinear Schrödinger equation. We begin this section by applying the gradient-holonomic integrability analysis described above to the discrete dynamical system (2.2). First, we shall show the existence of an infinite hierarchy of functionally independent conservation laws obtained by solving the determining Lax equation (2.8) in the asymptotic form (3.17). The following is a key result for our analysis.

Lemma 5.1. *The functional expression*

$$\varphi_n := \begin{pmatrix} 1 \\ a_n(\lambda) \end{pmatrix} \exp[it(2-\lambda-\lambda^{-1})] \prod_{j=0}^{n} \sigma_j(\lambda), \tag{5.1}$$

where

$$\sigma_j(\lambda) \sim \frac{\lambda}{h_j[u,\bar{u}]}(1-\sum_{s\in\mathbb{Z}_+} \sigma_j^{(s)}[u,\bar{u}]\lambda^{-s-1}), \tag{5.2}$$
$$a_n(\lambda) \sim \sum_{s\in\mathbb{Z}_+} a_n^{(s)}[u,\bar{u}]\lambda^{-s},$$

is an asymptotic solution to the determining Lax equation

$$d\varphi_n/dt + K'^{,*}\varphi_n = 0 \tag{5.3}$$

as $\lambda \to \infty$ for all $n \in \mathbb{Z}_N$ with the operator $K'^{,*} : T^*(M_2) \to T^*(M_2)$ of the form:

$$K_n'^{,*} = \begin{pmatrix} i\Delta^{-1}D_n^2 - i\bar{u}_n(u_{n+1}+u_{n-1}) - \\ \quad -i(\Delta+\Delta^{-1})\cdot\bar{u}_n u_n & i\bar{u}_n(\bar{u}_{n+1}+\bar{u}_{n-1}) \\ -iu_n(u_{n+1}+u_{n-1}) & -i\Delta^{-1}D_n^2 + iu_n(\bar{u}_{n+1}+\bar{u}_{n-1}) + \\ & \quad +i(\Delta+\Delta^{-1})\cdot\bar{u}_n u_n \end{pmatrix}. \tag{5.4}$$

Proof. It suffices to find the corresponding coefficients of the asymptotic expansions (5.2). To do this, we consider the following two equations that can be easily obtained from (5.3), (5.4) and (5.1):

$$
\begin{aligned}
&D_n^{-1}\tfrac{d}{dt}[-\ln h_n+\ln(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})]+\\
&+i\lambda[h_{n+1}^{-1}(1-\bar u_n u_n)(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})-1]+\\
&+\tfrac{i}{\lambda}\Big[(1-\bar u_{n-1}u_{n-1})h_n(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})^{-1}-1\Big]-\\
&-i\bar u_n(u_{n+1}+u_{n-1})+i\bar u_n(\bar u_{n+1}+\bar u_{n-1})\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
&(\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s})D_n^{-1}\tfrac{d}{dt}[-\ln h_n+\ln(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})]+4i(\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s})+\\
&+\Big[i\lambda h_{n+1}(\bar u_{n+1}u_{n+1}-1)(\sum_{s\in\mathbb{Z}_+}a_{n+1}^{(s)}\lambda^{-s})(\sum_{s\in\mathbb{Z}_+}a_{n+1}^{(s)}\lambda^{-s})-\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}\Big]+\\
&+\tfrac{i}{\lambda}\Big[(\bar u_{n-1}u_{n-1}-1)(\sum_{s\in\mathbb{Z}_+}a_{n+1}^{(s)}\lambda^{-s})h_n(1-\sum_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})^{-1}-\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}\Big]+\\
&+\tfrac{d}{dt}\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}-iu_n(u_{n+1}+u_{n-1})+iu_n(\bar u_{n+1}+\bar u_{n-1})\sum_{s\in\mathbb{Z}_+}a_n^{(s)}\lambda^{-s}.
\end{aligned}
$$

Now equating the coefficients of (5.5) at the same degrees of the parameter $\lambda\in\mathbb{C}$, we recursively obtain the functional expression expression for $h_n,\sigma_n^{(s)}$ and $a_n^{(s)}, n\in\mathbb{Z}, s\in\mathbb{Z}_+$; namely,

$$
\begin{aligned}
h_n &= (1-u_n^*u_n), a_n^{(0)}=0, a_n^{(1)}=\beta,\\
\sigma_n^{(0)} &= u_{n-1}^*(u_n+u_{n-2})-i\Delta^{-1}D_n^2(\ln h_n)_t,\\
\sigma_n^{(1)} &= i\frac{d}{dt}\sigma_{n-1}^{(0)}+(h_{n-1}h_{n-2}-1)+a_{n-1}^{(1)}u_{n-1}^*(u_n+u_{n-2}),\\
a_n^{(2)} &= -3a_{n-1}^{(1)}+i\frac{d}{dt}\sigma_{n-1}^{(1)}-ia_{n-1}^{(1)}D_n^{-1}(\ln h_{n-1})_t+\\
&\quad +a_n^{(1)}\sigma_n^{(0)}-u_{n-1}(u_n^*+u_{n-2}^*)a_{n-1}^{(1)},\\
dh_n/dt &= iD_n(u_{n-1}^*u_n-u_n^*u_{n-1}),\dots,
\end{aligned}
$$

whence $\sigma_n^{(0)}=-(u_n^*u_{n-1}+u_{n-1}^*u_{n-2})$, $\sigma_n^{(1)}=$

$$
=i\frac{d}{dt}\sigma_{n-1}^{(0)}+(1-u_{n-1}^*u_{n-1})(1-u_{n-2}^*u_{n-2})+\beta u_{n-1}^*(u_n+u_{n-2}),\dots,
$$

and so on. Thus, the corresponding recursion formulas are solvable for all $s\in\mathbb{Z}_+$, so it follows that the expression (5.1) is a true asymptotic solution to the Lax equation (5.3), and the proof is complete.

Recalling now that the expression

$$\gamma(\lambda) := -\sum_{n=0}^{N-1} \ln h_n + \sum_{n=0}^{N-1} \ln(1 - \sum_{s\in\mathbb{Z}_+} \sigma_n^{(s)} \lambda^{-s-1})$$

as $\lambda \to \infty$ is a generating function of conservation laws for the dynamical system (2.2), one finds that functionals

$$\bar{\gamma}_0 = \sum_{n=0}^{N-1} \ln(1 - \bar{u}_n u_n),\ \gamma_0 = -\sum_{n=0}^{N-1} \sigma_n^{(0)},\ \gamma_1 = -\sum_{n=0}^{N-1} (\sigma_n^{(1)} + \frac{1}{2}\sigma_n^{(0)}\sigma_n^{(0)}),$$

$$\gamma_2 = -\sum_{n=0}^{N-1} (\sigma_n^{(2)} + \frac{1}{3}\sigma_n^{(0)}\sigma_n^{(0)}\sigma_n^{(0)} + \sigma_n^{(0)}\sigma_n^{(1)}), \dots ,$$

and so on, make up an infinite hierarchy of exact conserved quantities for the discrete nonlinear Schrödinger dynamical system (2.2).

A few remarks are in order concerning the complete integrability of the discrete nonlinear Schrödinger dynamical system (2.2). First, we can easily show using the standard asymptotic small parameter approach [39, 25, 7] that the Nöther equation (2.5) on the manifold $M_2^{(N)}$ possesses [40, 34] the exact Poissonian operator solution

$$\vartheta_n = \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix}, \tag{5.6}$$

for $n \in \mathbb{Z}_N$, subject to which the dynamical the dynamical system (2.2) is Hamiltonian via $\frac{d}{dt}(u, u^*)^\intercal = -\vartheta \text{ grad } H_\vartheta[u, u^*]$ on the periodic manifold $M_2^{(N)}$, where the Hamiltonian function is

$$H_\vartheta := \sum_{n=0}^{N} \ln h_n^2 - \sum_{n=0}^{N} (\bar{u}_n u_{n+1} - \bar{u}_n u_{n+1}) = 2\ln|\gamma_0| - \frac{1}{2}(\gamma_0 + \bar{\gamma}_0).$$

Similar, but more cumbersome, calculations can be employed to find a second Poissonian operator solution to the Nöther equation (2.5) in the matrix form:

$$\eta = \begin{pmatrix} (h_n - u_n D_n^{-1} u_n)\Delta & (u_n^2 + u_n D_n^{-1} u_n)\Delta^{-1} \\ u_n^* D_n^{-1} u_n^* \Delta & -(1 + u_n^* D_n^{-1} u_n)\Delta^{-1} \end{pmatrix} \times$$

$$\times \begin{pmatrix} u_n D_n^{-1} u_n & (h_n - u_n D_n^{-1} u_n^* \\ 1 + u_n^* D_n^{-1} u_n & -(u_n^* + u_n^* D_n^{-1} u_n^*) \end{pmatrix}, \tag{5.7}$$

where the operation $D_n^{-1}(\cdot) := (1/2)[\sum_{k=0}^{n-1}(\cdot)_k - \sum_{k=n}^{N-1}(\cdot)_k]$ is quasi-skew-symmetric with respect to the usual bilinear form on $T^*(M_2^{(N)}) \times T(M_2^{(N)})$, satisfying the operator identity $(D_n^{-1})^* = -\Delta^{-1}D_n^{-1}\Delta$, $n \in \mathbb{Z}$.

The Poissonian operators (5.6) and (5.7) are compatible, so we can obtain the related Lax representation for the dynamical system (2.2) by means of the algebraic gradient-holonomic algorithm. The corresponding result is as follows: the discrete linear spectral problem

$$\Delta f_n = l_n[u, u^*; \lambda] f_n, \tag{5.8}$$

where $f \in l^\infty(\mathbb{Z}; \mathbb{C}^2)$ and for $n \in \mathbb{Z}$

$$l_n[u, u^*; \lambda] = \begin{pmatrix} \lambda & u_n \\ u_n^* & \lambda^{-1} \end{pmatrix},$$

allows the linear Lax isospectral evolution

$$df_n/dt = p_n(l) f_n \tag{5.9}$$

for some matrix $p_n(l) \in \mathrm{End}\ \mathbb{C}^2, n \in \mathbb{Z}$, which is equivalent to the Hamiltonian flow

$$df_n/dt = \{H_\vartheta, f_n\}_\vartheta\ , \tag{5.10}$$

where $\{\cdot,\cdot\}_\vartheta$ is the Poissonian structure on the manifold $M_2^{(N)}$ corresponding to (5.6). The equivalence of (5.6) and (5.10) can be easily demonstrated by constructing the monodromy matrix $S_n(\lambda), n \in \mathbb{Z}_N$, for all $\lambda \in \mathbb{C}$ corresponding to (5.8) and calculating the Hamiltonian evolution

$$\frac{d}{dt} S_n(\lambda) = \{H_\vartheta, S_n(\lambda)\}_\vartheta = [p_n(l), S_n(\lambda)],$$

giving rise to the same matrix $p_n(l) \in End\ \mathbb{C}^2, n \in \mathbb{Z}$, as in equation (5.9).

Thus, we have shown that the nonlinear discrete Schrödinger dynamical system (2.2) is a Lax integrable bi-Hamiltonian flow on the manifold $M_2^{(N)}$. Since the solution $\varphi(\lambda) \in T^*(M_2^{(N)})$ constructed above satisfies the gradient-like relationship $\lambda\vartheta\ \varphi(\lambda) = \eta\ \varphi(\lambda)$ for all for $\lambda \in \mathbb{C}$, we showed that the conservation laws are mutually commuting with respect to both Poisson brackets $\{\cdot,\cdot\}_\vartheta$ and $\{\cdot,\cdot\}_\eta$. From whence follows the classical Liouville integrability [5, 33] of the discrete nonlinear Schrödinger dynamical system (2.2) on the periodic manifold $M_2^{(N)}$. A detailed analysis of the integrability procedure via the Bogoyavlensky–Novikov reduction [11, 37] and an explicit construction of solutions to the dynamical system (2.2) are planned for a later paper.

5.2. The discrete nonlinear Ragnisco–Tu dynamical system. We now consider the Ragnisco–Tu differential-difference dynamical system (2.3) defined on the periodic manifold $M_2^{(N)} \subset l^\infty(\mathbb{Z};\mathbb{R}^2)$, and construct first the corresponding asymptotic solution to the Lax equation (2.8). The following result is quite useful.

Lemma 5.2. *The functional expression*

$$\varphi_n := \binom{a_n(\lambda)}{1} \exp(\lambda t) \prod_{j=1}^{n} \sigma_j(\lambda), \tag{5.11}$$

is an asymptotic (*as* $\lambda \to \infty$) *solution to the determining Lax equation* (2.8) *for all* $n \in \mathbb{Z}_N$ *with the operator* $K'^{,*} : T^*(M_2^{(N)}) \to T^*(M_2^{(N)})$ *of the form:*

$$K_n'^{,*} = \begin{pmatrix} \Delta^{-1} - 2u_n v_n & v_n^2 \\ -u_n^2 & -\Delta + 2u_n v_n \end{pmatrix},$$

where, by definition,

$$\sigma_n(\lambda) \sim \lambda(1 - \sum_{s\in\mathbb{Z}_+} \sigma_n^{(s)}[u,v]\lambda^{-s}), \tag{5.12}$$

$$a_n(\lambda) \sim \sum_{s\in\mathbb{Z}_+} a_n^{(s)}[u,v]\lambda^{-s},$$

and the following analytical expressions

$$\begin{aligned} &\sigma_n^{(0)} = 0, a_n^{(0)} = 0; \sigma_n^{(1)} = -2u_{n-1}v_{n-1}, a_n^{(1)} = -v_n^2; \\ &\sigma_n^{(2)} = 2u_{n-1}v_{n-2} - u_{n-1}^2 v_{n-1}^2, a_n^{(2)} = 2v_n(v_{-n-1} - v_n^2 u_n); \\ &\sigma_n^{(3)} = -2u_{n-1}v_{n-2} - D_n^{-1}(d\sigma_n^{(2)}/dt + \sigma_n^{(1)} d\sigma_n^{(1)}/dt), \\ &a_n^{(3)} = -da_n^{(2)}/dt - 2(u_{n-1}v_{n-2}v_n^2 - u_n v_n v_{n-1}^2), ..., \end{aligned} \tag{5.13}$$

and so on, hold.

Proof. It is easy to calculate that local σ- and a-functionals on $M_{(N)}$ satisfy the following functional equations:

$$\lambda(1 - \sigma_n(\lambda)) + D_n^{-1} \tfrac{d}{dt} \ln \sigma_n(\lambda) - u_n^2 a_n(\lambda) + 2u_n v_n = 0,$$

$$da_n(\lambda)/dt + \lambda a_n(\lambda) + a_n(\lambda) D_n^{-1} \tfrac{d}{dt} \ln \sigma_n(\lambda) -$$

$$-2u_n v_n \lambda^{-1} a_{n-1}(\lambda)\sigma_n(\lambda)^{-1} + v_n^2 = 0,$$

which allow the asymptotic (as $\lambda \to \infty$) solutions in the form (5.12). Then, solving the corresponding recurrence relations inductively, one obtains the exact analytical expressions (5.13). Taking now into account that for each $n \in \mathbb{Z}_+$ there exists a local functional $\rho_n(\lambda)$ such that the expression $\frac{d}{dt} \ln \sigma_n(\lambda) = D_n \rho_n(\lambda)$ holds on $M_2^{(N)}$, we obtain the functional expression (5.11) solving the Lax equation (2.8), which proves the lemma.

As a simple corollary of Lemma 5.2, we find that the expression

$$\gamma(\lambda) := \sum_{n=1}^{N} \ln(1 - \sum_{s \in \mathbb{Z}_+} \sigma_n^{(s)} \lambda^{-s-1}) \sim \sum_{j \in \mathbb{Z}_+} \gamma_j \lambda^{-j} \tag{5.14}$$

is a generating functional for the infinite hierarchy of conservation laws $\gamma_j \in D(M_2^{(N)}), j \in \mathbb{Z}_+$, of the Ragnisco-Tu differential-difference dynamical system (2.3).

Now we show that the Ragnisco–Tu differential-difference dynamical system (2.3) is a bi-Hamiltonian dynamical system on the functional manifold $M_2^{(N)}$. To this end, we observe that it follows from Lemma 2.4 that the element $\psi := \frac{1}{2}(v_n, -u_n)^\intercal \in T^*(M_2^{(N)})$ satisfies the functional equation (2.10):

$$d\psi/dt + K^{',*}\psi = \text{grad}\, \mathcal{L}, \quad \mathcal{L} = -\frac{1}{2} \sum_{k=0}^{N-1} u_n^2 v_n^2,$$

giving rise to the first Poissonian structure

$$\vartheta_n := \psi_n' - \psi_n'^{;*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.15}$$

on the manifold $M_2^{(N)}$ with respect to which the differential-difference dynamical system (2.3) is Hamiltonian. In particular, $\frac{d}{dt}(u_n, v_n)^\intercal = -\vartheta_n \text{ grad } H_{\vartheta,n}[u,v]$, where the Hamiltonian function, owing to the relationship (2.12), equals

$$H_\vartheta := (\psi, K) - \mathcal{L}_\psi) = \sum_{k=0}^{N-1} (u_n^2 v_n^2/2 - u_n v_{n-1}) = -\frac{1}{2} \sum_{k=0}^{N-1} \sigma_n^{(2)}.$$

In the same way one can find the second compatible with (5.15) Poissonian operator

$$\eta_n := \begin{pmatrix} -u_n^2 + 2u_n D_n^{-1} \Delta u_n & \Delta - 2u_n D_n^{-1} \Delta v_n \\ -\Delta^{-1} + 2u_n v_n - 2v_n D_n^{-1} \Delta u_n & -v_n^2 + 2v_n D_n^{-1} \Delta v_n \end{pmatrix}, \tag{5.16}$$

for which $\frac{d}{dt}(u_n, v_n)^\intercal = -\eta_n$ grad $H_{\eta,n}[u,v]$, where the Hamiltonian function is

$$H_\eta := -\sum_{k=1}^{N} u_n v_n = \frac{1}{2}\sum_{k=1}^{N} \sigma_{n+1}^{(1)}.$$

We claim that the hierarchy of conservation laws (5.14) satisfies as $\lambda \to \infty$ the gradient relationship

$$\lambda\vartheta \text{ grad } \gamma(\lambda) = \eta \text{ grad } \gamma(\lambda), \tag{5.17}$$

their mutual commutation with respect to both Poissonian structures (5.15) and (5.16). Accordingly the Ragnisco–Tu differential-difference dynamical system (2.3) is a completely integrable bi-Hamiltonian dynamical system on the manifold $M_2^{(N)}$.

The gradient relationship (5.17) gives rise to the following 'adjoint' Lax representation $d\Lambda/dt = [\Lambda, K^{\prime,*}]$, where, by definition, the expression $\Lambda := \vartheta^{-1}\eta : T^*(M_2^{(N)}) \to T^*(M_2^{(N)})$ is called a *recursion* operator. Based on the gradient relationship (5.17) and expression (3.5), we conclude using the gradient holonomic approach that the Ragnisco–Tu differential-difference dynamical system (2.3) is also Lax integrable, with an associated standard linear shift Lax spectral problem of the form

$$\Delta f_n = l_n[u,v;\lambda] f_n, \qquad l_n[u,v;\lambda] = \begin{pmatrix} \lambda + u_n v_n & u_n \\ v_n & 1 \end{pmatrix},$$

for all $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, where $(u,v) \in M_2^{(N)}$ and $f \in l^\infty(\mathbb{Z};\mathbb{C}^2)$.

6. Conclusion. The gradient-holonomic scheme for studying Lax integrability of differential-difference nonlinear dynamical systems devised here appears to be effective for applications in the one-dimensional case similar to that of nonlinear dynamical systems defined on spatially one-dimensional functional manifolds [39, 25, 7, 33, 14]. The algorithm, which was suggested in [40, 34], makes it possible to readily construct an infinite hierarchy of conservation laws as well as to calculate their compatible co-symplectic structures. As was also shown, the Bogoyavlensky–Novikov reduction to integrable Hamiltonian dynamical systems on the corresponding invariant periodic submanifolds generates finite-dimensional Liouville integrable Hamiltonian systems with respect to the canonical Gelfand–Dikiy type symplectic structures. As an example, an almost complete integrability analysis of the nonlinear discrete Schrödinger dynamical system was presented in detail.

As for different indirect approaches to studying the integrability of differential-difference dynamical systems on discrete manifolds, it is worth mentioning the works [29, 12, 35, 6, 9, 24] based on the inverse spectral transform and related Lie-algebraic methods, where *a priori* Lax integrable Hamiltonian flows possessing infinite hierarchies of conservation laws are constructed. Many important analytical properties of these other approaches were constructively incorporated into the algorithmic gradient-holonomic scheme presented above.

In this vein, the interesting differential-algebraic approaches [41, 22, 47] proposed for analyzing the integrability both of differential and differential-difference dynamical systems should also be noted. For example, in [41, 20, 21] these types of differential-algebraic tools were used to study the integrability of a generalized (owing to D. Holm and M. Pavlov) Riemann hydrodynamical hierarchy of dynamical systems of the form

$$D_t^s u = 0, \quad D_t := \partial/\partial t + uD_x, D_x := \partial/\partial x, \tag{6.1}$$

on a smooth functional manifold $M \subset C^\infty(\mathbb{R};\mathbb{R})$ for any integer $s \in \mathbb{Z}_+$. It was proved that these systems are Lax integrable and possess a bi-Hamiltonian structure. By replacing the spatial differentiation D_x, $x \in \mathbb{R}$, by its discrete analog $D_n = \Delta - 1$, $n \in \mathbb{Z}$, in these systems, one can similarly construct a generalized Riemann type hierarchy of the following discrete dynamical systems

$$\mathfrak{D}_t^s u_n = 0, \quad \mathfrak{D}_t := \partial/\partial t + u_n(D_n + D_{n-1})/2, \tag{6.2}$$

for any integer $s \in \mathbb{Z}_+$ on a suitable discrete manifold $M \subset l^2(\mathbb{Z};\mathbb{R})$. And like their counterparts analyzed above, the integrability properties of (6.2) are important for several practical applications. Naturally, it would be interesting to apply our direct gradient-holonomic integrability approach to the hierarchy (6.2) and find its differential-difference analog using the known [41, 21, 38] corresponding Lax representations. As one can easily check, one of the discrete analogs of the corresponding linear Lax "spectral" problem for (6.1) for $s = 2$ has the form

$$\Delta f_n = l_n[u, z; \lambda] f_n, \quad l_n[u, z; \lambda] := \begin{pmatrix} 1 - \lambda D_n u_n & -D_n z_n \\ 2\lambda^2 & 1 + \lambda D_n u_n \end{pmatrix}, \tag{6.3}$$

where $z_n := \mathfrak{D}_n u_n$ for any $n \in \mathbb{Z}$. Unfortunately, the strongly singular nature of the spectral problem (6.3)does not seem to allow the construction of the related Poissonian structures in a reasonable closed form. On the other hand, this not the case for the following inviscid discrete Riemann–Burgers dynamical system (6.2) for $s = 1$:

$$\mathfrak{D}_t w_n = 0 \Rightarrow dw_n/dt = -(w_{n+1} - w_{n-1})/2 := K_n[w], \qquad (6.4)$$

which is defined on an N-periodic discrete manifold $M \subset l^\infty(\mathbb{Z}_N;\mathbb{R})$. Following the gradient-holonomic scheme developed for the earlier examples, we first show the existence of an infinite hierarchy of conservation laws and the corresponding bi-Hamiltonian formulation for (6.4) .

From Proposition 3.1 we have the determining equation (2.8)

$$d\varphi_n/dt + [(\Delta - \Delta^{-1})w_n/2 + (w_{n+1} - w_{n-1})/2]\varphi_n = 0$$

and its asymptotic solution $\varphi \in T^*(M)$ in the form (3.17):

$$\varphi_n = \prod_{j=0}^{n-1} \sigma_j[w;\lambda],$$

where $n \in \mathbb{Z}$ and the local functionals $\sigma_j[w;\lambda], j \in \mathbb{Z}_+$, possess as $\lambda \to \infty$ the expansions $\sigma_j[w;\lambda] \sim \sum_{s\in\mathbb{Z}_+} \sigma_j^{(s)}[w]\lambda^{-s}$. Upon recursively solving the resulting functional equations

$$D_n^{-1}(\ln\sigma_n)_t - (w_{n-1}/\sigma_{n-1} - w_{n+1}\sigma_n)/2 - (w_{n+1} - w_{n-1}) = 0,$$

one easily obtains the infinite hierarchy (5.14) of conservations laws

$$\gamma_0 = \sum_{n=0}^{N-1}(w_n + w_{n-1}), \gamma_1 = 0,$$
$$\gamma_2 = \sum_{n=0}^{N-1}[(w_n + w_{n-1})^2 + w_n(w_{n-1} + w_{n+1})], \ldots, \gamma_{2j+1} = 0$$

for all $j \in \mathbb{Z}_+$. Then, applying to the hierarchy of conservation laws the approach of Lemma 2.4, one can finds by straightforward but lengthy calculations the following pair $\vartheta, \eta : T^*(M) \to T(M)$ of compatible Poissonian operators on the manifold M:

$$\vartheta_n := w_n(\Delta - \Delta^{-1})w_n,$$
$$\eta_n := (w_n w_{n+1}\Delta^2 - w_n w_{n-1}\Delta^{-2})(w_n + w_{n-1}\Delta^{-1}). \qquad (6.5)$$

In particular, the Hamiltonian representation of the Riemann–Burgers system (6.4) is easily seen to be

$$dw_n/dt = -\vartheta_n \mathrm{grad}\ H_\vartheta,\ H_\vartheta := -\sum_{n=0}^{N-1}(w_n + w_{n-1})/2.$$

Moreover, the first Poissonian structure of (6.5) allows the continuous limit $\lim_{\substack{\Delta x\to 0\\ n\to\infty}} w_n := w(x)$, if $n\Delta x := x \in \mathbb{R}$, to the well-known [30] correct continuum form $\vartheta := (w\partial + \partial w)(w + \partial^{-1} w\partial)/2$. Making use of the Poissonian pair (6.5), one can use the gradient holonomic scheme to find a Lax representation related to the inviscid discrete Riemann–Burgers dynamical system (6.4), whose l-operator is given by the matrix expression

$$l_n[w;\lambda] = \begin{pmatrix} \lambda & -w_n \\ 1 & 0 \end{pmatrix}$$

for $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$. It should be noted that the higher flows generated by the inviscid Riemann–Burgers dynamical system (6.4), have nothing to do with the generalized Riemann hydrodynamic systems (5.11) and their discrete approximations. Thus, it is necessary to develop a different approach to constructing their integrable discrete Lax representations so that they are compatible with the related continuous limits.

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