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Remarks on separately subharmonic functions

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Dedicated to memory of Professor Promarz M. Tamrazov

We give some refinements to the previous results of Lelong, Avanissian, Arsove and ours, concerning the subharmonicity of separately subharmonic functions.

1. Introduction. It is well-known that a separately subharmonic function need not be subharmonic, see Wiegerinck [1], Theorem, p. 770, and also Wiegerinck and Zeinstra [2], Theorem 1, p. 246. On the other hand, Lelong [3], Théorème 1 bis, p. 315, and Avanissian [4], Théorème 9, p. 140, see also [5], Proposition 3, p. 24, and [6], Theorem, p. 31, showed that a separately subharmonic function is subharmonic provided it is locally bounded above. According to Arsove [7], Theorem 1, p. 622, it is sufficient to suppose that the function has locally an $\mathcal{L}^1$-integrable majorant. Later we gave the following improvement:

Theorem ([8], Theorem 1, p. 69). *Let* Ω *be a domain in* $\mathbb{R}^{m+n}$, $m, n \geq 2$. *Let* $u : \Omega \to [-\infty, +\infty)$ *be such that*

(a) *for each* $y \in \mathbb{R}^n$ *the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is subharmonic,

(b) *for each* $x \in \mathbb{R}^m$ *the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is subharmonic,

(c) *for some* $p > 0$ *there is a function* $v \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$ *such that* $u \leq v$.

Then u *is subharmonic in* Ω.

Our proof was based on a generalized mean value inequality for subharmonic functions, see [9], Lemma, p. 172 (for absolute values of harmonic functions), [8], Lemma, p. 69, [10], Theorem 1, p. 19, [11], Theorem, p. 188, and the references therein, see also [12], pp. 349, 350. For a different proof, based on distribution theory, see [13], Theorem 1.1, pp. 79 – 81.

Observe that still further improvements exist. Armitage and Gardiner gave a result [14], Theorem 1, pp. 255, 256, which includes all previous related results and which is even close to being sharp. With the aid of quasinearly subharmonic functions it was, however, possible to generalize and improve their result still slightly further, see [15], Theorem 4.1 and Corollary 4.5, pp. 8, 9, 13, and [16], Theorem 3.3.1 and Corollary 3.3.3, pp. e2621, e2622. See also [17], p. 184.

Since the results of Armitage and Gardiner and ours are both somewhat complicated, it is worthwhile to give further improvements also to the above concise Theorem. Recall here also that proofs of these newer results are based on Lelong's and Avanissian's result, or on Arsove's result, or on the above Theorem.

Our improvement to the above Theorem will be given in Corollary 2. We begin, however, with Theorem 1 and Theorem 2, where we generalize our previous results [18], Theorem 3.1, Corollary 3.2, Corollary 3.3, pp. 58, 59, 61 – 63, and [16], Theorem 3.2.2, Corollary 3.2.4, Corollary 3.2.5, pp. e2620, e2621, by replacing, among others, the previously used quasinearly subharmonic n.s. (quasinearly subharmonic in the narrow sense) functions with quasinearly subharmonic functions.

The methods and ideas of the proofs have their roots already in [8] and [18], see also [19] and [20]. The proofs are rather simple, especially when compared with the proofs of the already cited older results.

For the notation and the definitions used, see e.g. [19, 20, 18, 15, 21, 16, 12], and [22, 6] and [23]. For the convenience of the reader we, nevertheless, recall here some of definitions used below.

Let D be a domain in $\mathbb{R}^N$, $N \geq 2$. An upper semicontinuous function $u : D \to [-\infty, +\infty)$ is *subharmonic* if for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_N\, r^N} \int\limits_{B^N(x,r)} u(y)\, dm_N(y).$$

The function $u \equiv -\infty$ is considered subharmonic.

We say that a function $u : D \to [-\infty, +\infty)$ is *nearly subharmonic*, if u is Lebesgue measurable, $u^+ \in \mathcal{L}^1_{\rm loc}(D)$, and for all $\overline{B^N(x,r)} \subset D$,

$$u(x) \leq \frac{1}{\nu_N\, r^N} \int\limits_{B^N(x,r)} u(y)\, dm_N(y).$$

Observe that in the standard definition of nearly subharmonic functions one uses the slightly stronger assumption that $u \in \mathcal{L}^1_{\rm loc}(D)$, see e.g. [6], p. 14. However, our above, slightly more general definition seems to be more practical. See [18], pp. 51, 52, and [16], pp. e2613, e2614.

Proceeding as in [6], proof of Theorem 1, pp. 14, 15, (and referring also to [18], Proposition 2.1 and Proposition 2.2, pp. 54, 55, or [16], Proposition 1.5.1 and Proposition 1.5.2, p. e2615) one gets:

Lemma. *Let D be a domain in $\mathbb{R}^N$, $N \geq 2$. Let $u : D \to [-\infty, +\infty)$ be Lebesgue measurable. Then u is nearly subharmonic in D if and only if there exists a function u^*, subharmonic in D such that $u^* \geq u$ and $u^* = u$ almost everywhere in D. Here u^* is the lowest upper semicontinuous majorant of u:*

$$u^*(x) = \limsup_{x' \to x} u(x').$$

u^* is called the regularized subharmonic function to u.

We say that a Lebesgue measurable function $u : D \to [-\infty, +\infty)$ is *K-quasinearly subharmonic*, if $u^+ \in \mathcal{L}^1_{\rm loc}(D)$ and if there is a constant $K = K(N,u,D) \geq 1$ such that for all $x \in D$ and $r > 0$ such that $\overline{B^N(x,r)} \subset D$, one has

$$u_M(x) \leq \frac{K}{\nu_N\, r^N} \int\limits_{B^N(x,r)} u_M(y)\, dm_N(y)$$

for all $M \geq 0$, where $u_M := \max\{u, -M\} + M$. A function $u : D \to [-\infty, +\infty)$ is *quasinearly subharmonic*, if u is K-quasinearly subharmonic in D for some $K \geq 1$.

Observe that a function $u : D \to [-\infty, +\infty)$ is 1-quasinearly subharmonic if and only if it is nearly subharmonic.

2. Results. In the proofs below we need the following result:

Proposition ([18], Proposition 3.1, pp. 57, 58, and [16], Proposition 3.2.1, p. e2620). *Let* Ω *be a domain in* $\mathbb{R}^{m+n}$, $m, n \geq 2$, *and let* $K_1, K_2 \geq 1$. *Let* $u : \Omega \to [-\infty, +\infty)$ *be a Lebesgue measurable function such that*

(a) *for each* $y \in \mathbb{R}^n$ *the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is K_1*-quasinearly subharmonic,*

(b) *for almost every* $x \in \mathbb{R}^m$ *the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is K_2*-quasinearly subharmonic,*

(c) *there exists a non-constant permissible function* $\psi : [0, +\infty) \to [0, +\infty)$ *such that* $\psi \circ u^+ \in \mathcal{L}^1_{\rm loc}(\Omega)$.

Then u *is* $\frac{4^{m+n}\nu_{m+n}K_1K_2}{\nu_m\,\nu_n}$*-quasinearly subharmonic in* Ω.

For the definition of permissible functions, see [19], p. 159, [20], p. 231, 232, [18], p. 54, [16], p. e2615, and [21], Lemma 1 and Remark 1, pp. 92, 93. We list here only some examples of permissible functions: $\psi_1(t) = t^p$, $p > 0$, and $\psi_2(t) = c\,t^{p\alpha}[\log(\delta + t^{p\gamma})]^{\beta}$, $c > 0$, $0 < \alpha < 1$, $\delta \geq 1$, $\beta, \gamma \in \mathbb{R}$ such that $0 < \alpha + \beta\,\gamma < 1$, and $p \geq 1$. Also functions of the form $\psi_3 = \phi \circ \varphi$, where $\phi : [0, +\infty) \to [0, +\infty)$ is a concave surjection whose inverse ϕ^{-1} satisfies the Δ_2-condition and $\varphi : [0, +\infty) \to [0, +\infty)$ is an increasing, convex function satisfying the Δ_2-condition, are permissible. Recall that a function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies the Δ_2*-condition*, if there is a constant $C = C(\varphi) \geq 1$ such that $\varphi(2t) \leq C\,\varphi(t)$ for all $t \in [0, +\infty)$.

The following result improves [18], Theorem 3.1, pp. 58, 59, and [16], Theorem 3.2.2, p. e2620:

Theorem 1. *Let* Ω *be a domain in* $\mathbb{R}^{m+n}$, $m, n \geq 2$, *and let* $K \geq 1$. *Let* $u : \Omega \to [-\infty, +\infty)$ *be a Lebesgue measurable function such that*

(a) *for each* $y \in \mathbb{R}^n$ *the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is 1*-quasinearly subharmonic,*

(b) *for almost every* $x \in \mathbb{R}^m$ *the function*

$$\Omega(x) \ni y \mapsto u(x,y) \in [-\infty, +\infty)$$

is K*-quasinearly subharmonic,*

(c) *there exists a non-constant permissible function* $\psi : [0,+\infty) \to [0,+\infty)$ *such that* $\psi \circ u^+ \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then u *is* K*-quasinearly subharmonic in* Ω.

Proof. By the above Proposition u is quasinearly subharmonic and thus locally integrable in Ω.

Take $M \geq 0$ arbitrarily, and write $u_M := \max\{u, -M\} + M$. It remains to show that for all $(a,b) \in \Omega$ and $R > 0$ such that $\overline{B^{m+n}((a,b),R)} \subset \Omega$,

$$u_M(a,b) \leq \frac{K}{\nu_{m+n} R^{m+n}} \int\limits_{B^{m+n}((a,b),R)} u_M(x,y)\, dm_{m+n}(x,y).$$

To see this, we just proceed in the following standard way, see e.g. [6], proof of Theorem a), pp. 32, 33:

$$\begin{aligned}
&\frac{K}{\nu_{m+n} R^{m+n}} \int\limits_{B^{m+n}((a,b),R)} u_M(x,y)\, dm_{m+n}(x,y) = \\
=&\frac{\nu_n}{\nu_{m+n} R^{m+n}} \int\limits_{B^m(a,R)} [(R^2 - \mid x-a \mid^2)^{\frac{n}{2}} \times \\
&\times \frac{K}{\nu_n (R^2 - \mid x-a \mid^2)^{\frac{n}{2}}} \int\limits_{B^n(b,\sqrt{R^2-|x-a|^2})} u_M(x,y)\, dm_n(y)]\, dm_m(x) \geq \\
\geq&\frac{\nu_n}{\nu_{m+n} R^{m+n}} \int\limits_{B^m(a,R)} (R^2 - \mid x-a \mid^2)^{\frac{n}{2}} u_M(x,b)\, dm_m(x) \geq u_M(a,b).
\end{aligned}$$

Above we have used, in addition to the fact that, for almost every $x \in \mathbb{R}^m$, the functions $u_M(x,\cdot)$ are K-quasinearly subharmonic, also the following lemma. (Observe that the proof of the Lemma, see [6], proof of Theorem 2 a), p. 15, works also in our slightly more general situation.)

Lemma ([6], Theorem 2 a), p. 15). *Let* v *be nearly subharmonic (in the generalized sense, defined above) in a domain* U *of* $\mathbb{R}^N$, $N \geq 2$, $\psi \in$

$\in \mathcal{L}^\infty(\mathbb{R}^N)$, $\psi \geq 0$, $\psi(x) = 0$ *when* $\mid x \mid \geq \alpha$ *and* $\psi(x)$ *depends only on* $\mid x \mid$. *Then* $\psi \star v \geq v$ *and* $\psi \star v$ *is subharmonic in* U_α, *provided* $\int \psi(x) dm_N(x) = 1$, *where* $U_\alpha = \{x \in U : \overline{B^N(x, \alpha)} \subset U\}$.

Corollary 1 ([18], Corollary 3.1, p. 59, and [16], Corollary 3.2.3, pp. e2620, e2621). *Let* Ω *be a domain in* $\mathbb{R}^{m+n}$, $m, n \geq 2$. *Let* $u : \Omega \to [-\infty, +\infty)$ *be a Lebesgue measurable function such that*

(a) *for each* $y \in \mathbb{R}^n$ *the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is nearly subharmonic,

(b) *for almost every* $x \in \mathbb{R}^m$ *the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is nearly subharmonic,

(c) *for some* $p > 0$ *there is a function* $v \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$ *such that* $u \leq v$.

Then u *is nearly subharmonic in* Ω.

Then an improvement to [18], Corollary 3.2 and Corollary 3.3, p. 61, and [16], Corollary 3.2.4 and Corollary 3.2.5, p. e2621:

Theorem 2. *Let* Ω *be a domain in* $\mathbb{R}^{m+n}$, $m, n \geq 2$, *and let* $K_1, K_2 \geq 1$. *Let* $u : \Omega \to [-\infty, +\infty)$ *be such that*

(a) *for each* $y \in \mathbb{R}^n$ *the function*

$$\Omega(y) \ni x \mapsto u(x, y) \in [-\infty, +\infty)$$

is K_1*-quasinearly subharmonic, and, for almost every* $y \in \mathbb{R}^n$, *subharmonic,*

(b) *for each* $x \in \mathbb{R}^m$ *the function*

$$\Omega(x) \ni y \mapsto u(x, y) \in [-\infty, +\infty)$$

is upper semicontinuous, and, for almost every $x \in \mathbb{R}^m$, K_2*-quasinearly subharmonic,*

(c) *there exists a non-constant permissible function* $\psi : [0,+\infty) \to [0,+\infty)$ *such that* $\psi \circ u^+ \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$.

Then for each $(a,b) \in \Omega$,

$$\limsup_{(x,y)\to(a,b)} u(x,y) \le K_1 K_2\, u^+(a,b).$$

Proof. By [18], Lemma, pp. 59, 60, u is measurable. By the above Proposition u and thus also u^+ are quasinearly subharmonic and thus locally bounded above. Clearly u^+ satisfies the assumptions of the theorem. It is sufficient to show that for any $(a,b) \in \Omega$,

$$\limsup_{(x,y)\to(a,b)} u^+(x,y) \le K_1 K_2\, u^+(a,b).$$

Take $(a,b) \in \Omega$ and $R_1 > 0$ and $R_2 > 0$ arbitrarily such that $\overline{B^m(a,R_1) \times B^n(b,R_2)} \subset \Omega$. Choose an arbitrary $\lambda \in \mathbb{R}$ such that $u^+(a,b) < \lambda$. Since $u^+(a,\cdot)$ is upper semicontinuous, we find R_2', $0 < R_2' < R_2$, such that

$$\frac{1}{\nu_n R_2'^n} \int\limits_{B^n(b,R_2')} u^+(a,y)dm_n(y) < \lambda.$$

Using the fact that, for almost every $y \in \mathbb{R}^n$, the function $u^+(\cdot,y)$, is subharmonic, we get

$$\frac{1}{\nu_m r^m} \int\limits_{B^m(a,r)} u^+(x,y)dm_m(x) \to u^+(a,y) \ \text{ as } r \to 0.$$

Since u^+ is locally bounded above, one can use Lebesgue Dominated Convergence Theorem. Thus we find R_1', $0 < R_1' < R_1$, such that

$$\frac{1}{\nu_n R_2'^n} \int\limits_{B^n(b,R_2')} [\frac{1}{\nu_m R_1'^m} \int\limits_{B^m(a,R_1')} u^+(x,y)dm_m(x)]dm_n(y) < \lambda.$$

Choose r_1, $0 < r_1 < R_1'$, and r_2, $0 < r_2 < R_2'$, arbitrarily. Then for each $(x,y) \in B^m(a,r_1) \times B^n(b,r_2)$,

$$u^+(x,y) \leq \frac{K_1}{\nu_m(R_1'-r_1)^m} \int\limits_{B^m(x,R_1'-r_1)} u^+(\xi,y)dm_m(\xi) \leq$$

$$\leq \frac{K_1}{\nu_m(R_1'-r_1)^m} \int\limits_{B^m(x,R_1'-r_1)} \left[\frac{K_2}{\nu_n(R_2'-r_2)^n} \int\limits_{B^n(y,R_2'-r_2)} u^+(\xi,\eta)dm_n(\eta)\right] dm_m(\xi) \leq$$

$$\leq \frac{K_2}{\nu_n(R_2'-r_2)^n} \int\limits_{B^n(y,R_2'-r_2)} \left[\frac{K_1}{\nu_m(R_1'-r_1)^m} \int\limits_{B^m(x,R_1'-r_1)} u^+(\xi,\eta)dm_m(\xi)\right] dm_n(\eta) \leq$$

$$\leq \left(\frac{R_1'}{R_1'-r_1}\right)^m \left(\frac{R_2'}{R_2'-r_2}\right)^n K_1K_2 \frac{1}{\nu_n R_2'^n} \times$$

$$\times \int\limits_{B^n(b,R_2')} \left[\frac{1}{\nu_m R_1'^m} \int\limits_{B^m(a,R_1')} u^+(\xi,\eta)dm_m(\xi)\right] dm_n(\eta) <$$

$$< \left(\frac{R_1'}{R_1'-r_1}\right)^m \left(\frac{R_2'}{R_2'-r_2}\right)^n K_1K_2\,\lambda\,.$$

Sending then $r_1 \to 0$, $r_2 \to 0$, one gets

$$\limsup_{(x,y)\to(a,b)} u^+(x,y) \leq K_1K_2\,\lambda\,,$$

concluding the proof.

Corollary 2. *Let Ω be a domain in $\mathbb{R}^{m+n}$, $m,n \geq 2$. Let $u : \Omega \to [-\infty,+\infty)$ be such that*

(a) *for each $y \in \mathbb{R}^n$ the function*

$$\Omega(y) \ni x \mapsto u(x,y) \in [-\infty,+\infty)$$

is nearly subharmonic, and, for almost every $y \in \mathbb{R}^n$, subharmonic,

(b) *for each $x \in \mathbb{R}^m$ the function*

$$\Omega(x) \ni y \mapsto u(x,y) \in [-\infty,+\infty)$$

is upper semicontinuous, and, for almost every $x \in \mathbb{R}^m$, (nearly) subharmonic,

(c) *for some $p > 0$ there is a function $v \in \mathcal{L}^p_{\mathrm{loc}}(\Omega)$ such that $u \leq v$.*

Then u is upper semicontinuous and thus subharmonic in Ω.

Proof. It is easy to see that for each $M \geq 0$, the function $u_M :=$ $:= \max\{u, -M\} + M$ satisfies the assumptions of Theorem 2. Thus u_M is upper semicontinuos. Since by Corollary 1, u_M is anyway nearly subharmonic, it is in fact subharmonic. Using then e.g. [6], a), p. 8, one sees that u is subharmonic and thus also upper semicontinuos.

Remark. Observe that Corollary 2 is partially related to the result [6], Proposition 2, pp. 34, 35: Though our assumptions are slightly stronger, our proof is, on the other hand, different and shorter.

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