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Some properties of nonhomogeneous linear differential polynomials in unit disc

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Dedicated to memory of Professor Promarz M. Tamrazov

In this paper, we investigate the complex oscillation of the nonhomogeneous linear differential polynomial $g_f = g(f, f', \dots, f^{(k)}) = \sum_{j=0}^k d_j f^{(j)} + b$, where d_j ($j = 0, 1, \dots, k$), b are analytic functions generated by solutions of the differential equation $f^{(k)} + A(z)f = 0$, $k \geq 2$, where $A(z) \not\equiv 0$ is analytic function with finite iterated p -order in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. This paper improves very recent result of Cao, Li, Tu and Xu.

1. Introduction and statement of results. In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory on the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [1 – 5]). We need to give some definitions and discussions.

Definition 1.1 [6, 7]. Let f be an analytic function in Δ , and

$$D_M(f) := \limsup_{r \rightarrow 1^-} \frac{\log^+ M(r, f)}{-\log(1-r)} = a < \infty \text{ (or } a = \infty),$$

then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r, f) = \max_{|z|=r} |f(z)|$.

Now we give the definitions of iterated order and finiteness degree of the order to classify generally the functions of fast growth in Δ as those in

\mathbb{C} (see [8, 9, 3]). Let us define inductively, for $r \in [0, 1)$, $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N}$. We also define for all r sufficiently large in $(0, 1)$, $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.2 [10, 11]. Let f be a meromorphic function in Δ . Then the iterated p -order of f is defined as

$$\rho_p(f) = \limsup_{r \rightarrow 1^-} \frac{\log_p^+ T(r, f)}{-\log(1-r)} \quad (p \geq 1 \text{ is an integer}),$$

where $\log_1^+ x = \log^+ x = \max\{\log x, 0\}$, $\log_{p+1}^+ x = \log^+ \log_p^+ x$. If f is analytic in Δ , then the iterated p -order of f is defined as

$$\rho_{M,p}(f) = \limsup_{r \rightarrow 1^-} \frac{\log_{p+1}^+ M(r, f)}{-\log(1-r)} \quad (p \geq 1 \text{ is an integer}).$$

Remark 1.1. It follows by M. Tsuji [4, p. 205]) that if f is an analytic function in Δ , then we have the inequalities

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1,$$

which are the best possible in the sense that there are analytic functions g and h such that $\rho_{M,1}(g) = \rho_1(g)$ and $\rho_{M,1}(h) = \rho_1(h) + 1$, see [11]. However, it follows by Proposition 2.2.2 in [3] that $\rho_{M,p}(f) = \rho_p(f)$ for $p \geq 2$.

Definition 1.3 [10]. The finiteness degree of the order of analytic function $f(z)$ in Δ is defined as

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is of finite degree,} \\ \min\{j \in \mathbb{N} : \rho_{M,j}(f) < +\infty\} & \text{if } f \text{ is of infinite degree,} \\ \text{and } \rho_{M,j}(f) < \infty \text{ for some } j \in \mathbb{N}, \\ +\infty, & \text{if } \rho_{M,j}(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Definition 1.4 [12]. Let f be a meromorphic function in Δ . Then the iterated exponent of convergence of the sequence of zeros of $f(z)$ is defined as

$$\lambda_p(f) = \limsup_{r \rightarrow 1^-} \frac{\log_p^+ N\left(r, \frac{1}{f}\right)}{-\log(1-r)} \quad (p \geq 1 \text{ is an integer}),$$

where $N\left(r, \frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$. Similarly, the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined as

$$\bar{\lambda}_p(f) = \limsup_{r \rightarrow 1^-} \frac{\log_p^+ \bar{N}\left(r, \frac{1}{f}\right)}{-\log(1-r)} \quad (p \geq 1 \text{ is an integer}),$$

where $\bar{N}\left(r, \frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.

Definition 1.5 [12]. The finiteness degree of the convergence exponent of the sequence of zeros of analytic function $f(z)$ in Δ is defined as

$$i_\lambda(f) = \begin{cases} 0, & \text{if } N\left(r, \frac{1}{f}\right) = O\left(\log \frac{1}{1-r}\right), \\ \min \{j \in \mathbb{N} : \lambda_j(f) < +\infty\} & \text{if some } j \in \mathbb{N} \\ & \text{with } \lambda_j(f) < +\infty \text{ exists,} \\ +\infty, & \text{if } \lambda_j(f) = +\infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

Remark 1.2. Similarly, we can define the finiteness degree $i_{\bar{\lambda}}(f)$ of $\bar{\lambda}_p(f)$.

For $k \geq 2$, we consider the linear differential equation

$$f^{(k)} + A(z)f = 0, \tag{1}$$

where $A(z) \not\equiv 0$ is an analytic function in the unit disc of finite iterated p -order. It is well-known that all solutions of equation (1) are analytic functions in Δ and that there are exactly k linearly independent solutions of (1) (see, [2]). For fixed points of entire functions or meromorphic functions on the plane, there are sequences of results, see [13]. In [14], Chen firstly studied the problem on the fixed points and hyper-order of solutions of second order linear differential equations with entire coefficients. After that, there were some results which improve those of Chen, see [15 – 20]. Recently many important results have been obtained on the complex oscillation theory of solutions and differential polynomials generated by solutions of differential equations in the unit disc Δ , refer to see [21, 22, 12, 23, 24] and others. In this paper, we continue to consider this subject and investigate the complex oscillation theory of differential polynomials generated by analytic solutions of higher order linear differential equations in Δ .

Let $\mathcal{L}(\mathbf{G})$ denote a differential subfield of the field $\mathcal{M}(\mathbf{G})$ of meromorphic functions in a domain $\mathbf{G} \subset \mathbb{C}$. Throughout this paper, we simply denote \mathcal{L} instead of $\mathcal{L}(\Delta)$. Special case of such differential subfield

$$\mathcal{L}_{p+1,\rho} = \{g \text{ meromorphic in } \Delta: \rho_{p+1}(g) < \rho\},$$

where ρ is a positive constant.

In [23], Cao, Li, Tu and Xu investigated the fixed points of linear differential polynomial generated by analytic solutions of second order differential equation in the unit disc and obtained the following result.

Theorem A [23]. *Let $A(z)$ be an analytic function of infinite degree and of finite iterated order $\rho_{M,p}(A) = \rho > 0$ in the unit disc Δ , and let $f \not\equiv 0$ be a solution of the equation*

$$f'' + A(z)f = 0.$$

Moreover, let

$$P[f] = P(f, f', \dots, f^{(m)}) = \sum_{j=0}^m p_j f^{(j)}$$

be a linear differential polynomial with analytic coefficients $p_j \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients p_j does not vanish identically. If $\varphi(z) \in \mathcal{L}_{p+1,\rho}$ is a non-zero analytic function in Δ , and neither $P[f]$ nor $P[f] - \varphi$ vanishes identically, then we have

$$i_{\bar{\lambda}}(P[f] - \varphi) = i(f) = p + 1$$

and

$$\bar{\lambda}_{p+1}(P[f] - \varphi) = \rho_{M,p+1}(f) = \rho_{M,p}(A) = \rho.$$

Remark 1.3. The idea of the proof of Theorems A is borrowed from the paper of Laine and Rieppo [18] with the modifications reflecting the change from the complex plane \mathbb{C} to the unit disc Δ .

The question which arises: Can we obtain a result which generalizes Theorem A by considering equation (1)?

The main purpose of this paper is to investigate the complex oscillation of the linear differential polynomial $g_f = g(f, f', \dots, f^{(k)}) = \sum_{j=0}^k d_j f^{(j)} + b$, where d_j ($j = 0, 1, \dots, k$), b are analytic functions generated by solutions of equation (1). The method used in the proofs of

our theorems is simple, and different, from the method in Laine and Rieppo [18] and Cao, Li, Tu and Xu [23]. Before we state our results, we define the sequence of functions $\alpha_{i,j}$ ($j = 0, 1, \dots, k-1$) by

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \text{for all } i = 1, 2, \dots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases}$$

and

$$\alpha_{i,0} = \begin{cases} d_i, & \text{for all } i = 1, 2, \dots, k-1, \\ d_0 - d_k A, & \text{for } i = 0. \end{cases} \quad (2)$$

We define also h by

$$h = \begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix}$$

and $\psi(z)$ by

$$\psi(z) = C_0(\varphi - b) + C_1(\varphi' - b') + \dots + C_{k-1}(\varphi^{(k-1)} - b^{(k-1)}),$$

where $C_j \in \mathcal{L}_{p+1,\rho}$ ($j = 0, 1, \dots, k-1$) are meromorphic functions depending on $\alpha_{i,j}$ and $\varphi(z) (\not\equiv 0) \in \mathcal{L}_{p+1,\rho}$ is analytic function. We obtain:

Theorem 1.1. *Let $A(z)$ be an analytic function of infinite degree and of finite iterated order $\rho_{M,p}(A) = \rho > 0$ in the unit disc Δ , and let $f \not\equiv 0$ be a solution of equation (1). Moreover, let*

$$g_f = g(f, f', \dots, f^{(k)}) = \sum_{j=0}^k d_j f^{(j)} + b \quad (3)$$

be a linear differential polynomial with analytic coefficients $d_j \in \mathcal{L}_{p+1,\rho}$, $b \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients d_j does not vanish identically such that $h \not\equiv 0$. Let $\varphi(z) (\not\equiv 0) \in \mathcal{L}_{p+1,\rho}$ be an analytic function such that $\psi(z) \not\equiv 0$. Then the differential polynomial g_f satisfies

$$i_{\bar{\lambda}}(g_f - \varphi) = i_{\lambda}(g_f - \varphi) = i(f) = p + 1$$

and

$$\bar{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A) = \rho.$$

Theorem 1.2. *Let H be a set of complex numbers satisfying $\overline{\text{dens}}_{\Delta}\{|z| : z \in H \subseteq \Delta\} > 0$, and let $A(z) \not\equiv 0$ be an analytic function in the unit disc Δ such that $\rho_p(A) = \rho < +\infty$ and for real number $\alpha > 0$, we have for all $\varepsilon > 0$ sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1} \left\{ \alpha \left(\frac{1}{1-|z|} \right)^{\rho-\varepsilon} \right\}$$

as $z \rightarrow 1^-$ for $z \in H$, and let $f \not\equiv 0$ be a solution of equation (1). Let be the linear differential polynomial (3) with analytic coefficients $d_j \in \mathcal{L}_{p+1,\rho}$, $b \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients d_j does not vanish identically such that $h \not\equiv 0$. Let $\varphi(z) (\not\equiv 0) \in \mathcal{L}_{p+1,\rho}$ be an analytic function such that $\psi(z) \not\equiv 0$. Then the differential polynomial g_f satisfies

$$i_{\bar{\lambda}}(g_f - \varphi) = i_{\lambda}(g_f - \varphi) = i(f) = p + 1$$

and $\rho_p(A) \leq \bar{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \rho_{M,p}(A)$.

2. Some lemmas. We need the following lemmas in the proofs of our theorems.

Lemma 2.1 [10]. *If f and g are meromorphic functions in Δ , $p \geq 1$ is an integer, then we have*

- (i) $\rho_p(f) = \rho_p(1/f)$, $\rho_p(a.f) = \rho_p(f)$ ($a \in \mathbb{C} - \{0\}$);
- (ii) $\rho_p(f) = \rho_p(f')$;
- (iii) $\max\{\rho_p(f+g), \rho_p(fg)\} \leq \max\{\rho_p(f), \rho_p(g)\}$;
- (iv) if $\rho_p(f) < \rho_p(g)$, then $\rho_p(f+g) = \rho_p(g)$, $\rho_p(fg) = \rho_p(g)$.

Lemma 2.2 ([12], Lemma 2.5). *Let $p \geq 1$ be an integer, and let $f(z)$ be a meromorphic solution in the unit disc Δ of the differential equation*

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \dots + A_1(z) f' + A_0(z) f = F,$$

where $A_0(z), \dots, A_{k-1}(z)$ and $F \not\equiv 0$ are meromorphic functions in Δ such that $\max\{\rho_{p+1}(F), \rho_{p+1}(A_j) \ (j = 0, \dots, k-1)\} < \rho_{p+1}(f) < +\infty$. Then $i_{\bar{\lambda}}(f) = i_{\lambda}(f) = i(f) = p + 1$ and $\bar{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{p+1}(f)$.

Lemma 2.3 [10]. *Let $p \geq 1$ be an integer, and let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in Δ such that $i(A_0) = p$. If*

$$\max\{i(A_j) : j = 1, \dots, k-1\} < p$$

or

$$\max\{\rho_{M,p}(A_j) : j = 1, \dots, k-1\} < \rho_{M,p}(A_0),$$

then every solution $f \not\equiv 0$ of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0$$

satisfies $i(f) = p+1$ and $\rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A_0)$.

Lemma 2.4. Let $A(z)$ be an analytic function of infinite degree and of finite iterated order $\rho_{M,p}(A) = \rho > 0$ in the unit disc Δ , and let $f \not\equiv 0$ be a solution of equation (1). Moreover, let

$$g_f = g\left(f, f', \dots, f^{(k)}\right) = \sum_{j=0}^k d_j f^{(j)} + b \quad (4)$$

be a linear differential polynomial with analytic coefficients $d_j \in \mathcal{L}_{p+1,\rho}$, $b \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients d_j does not vanish identically such that $h \not\equiv 0$. Then, the differential polynomial g_f satisfies

$$i(g_f) = i(f) = p+1, \quad \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A) = \rho.$$

Proof. Suppose that $f \not\equiv 0$ is a solution of (1). Then by Lemma 2.3, we have $i(f) = p+1$, $\rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A) = \rho$. Substituting $f^{(k)} = -Af$ into (4), we obtain

$$\begin{aligned} g_f - b &= d_k f^{(k)} + d_{k-1} f^{(k-1)} + \dots + d_0 f = \\ &= d_{k-1} f^{(k-1)} + \dots + (d_0 - d_k A) f. \end{aligned} \quad (5)$$

We can rewrite (5) as

$$g_f - b = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \quad (6)$$

where $\alpha_{i,0}$ are defined in (2). Differentiating both sides of equation (6) and

replacing $f^{(k)}$ by $f^{(k)} = -Af$, we obtain

$$\begin{aligned}
 g'_f - b' &= \sum_{i=0}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \\
 &= \sum_{i=0}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=1}^k \alpha_{i-1,0} f^{(i)} = \\
 &= \alpha'_{0,0} f + \sum_{i=1}^{k-1} \alpha'_{i,0} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)} = \\
 &= \alpha'_{0,0} f + \sum_{i=1}^{k-1} (\alpha'_{i,0} + \alpha_{i-1,0}) f^{(i)} - \alpha_{k-1,0} Af = \\
 &= \sum_{i=1}^{k-1} (\alpha'_{i,0} + \alpha_{i-1,0}) f^{(i)} + (\alpha'_{0,0} - \alpha_{k-1,0} A) f. \quad (7)
 \end{aligned}$$

We can rewrite (7) as

$$g'_f - b' = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \quad (8)$$

where

$$\alpha_{i,1} = \begin{cases} \alpha'_{i,0} + \alpha_{i-1,0}, & \text{for all } i = 1, 2, \dots, k-1, \\ \alpha'_{0,0} - A\alpha_{k-1,0}, & \text{for } i = 0. \end{cases} \quad (9)$$

Differentiating both sides of equation (8) and replacing $f^{(k)}$ by $f^{(k)} = -Af$, we obtain

$$\begin{aligned}
 g''_f - b'' &= \sum_{i=0}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \\
 &= \sum_{i=0}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^k \alpha_{i-1,1} f^{(i)} = \\
 &= \alpha'_{0,1} f + \sum_{i=1}^{k-1} \alpha'_{i,1} f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)} = \\
 &= \alpha'_{0,1} f + \sum_{i=1}^{k-1} (\alpha'_{i,1} + \alpha_{i-1,1}) f^{(i)} - \alpha_{k-1,1} Af =
 \end{aligned}$$

$$= \sum_{i=1}^{k-1} (\alpha'_{i,1} + \alpha_{i-1,1}) f^{(i)} + (\alpha'_{0,1} - \alpha_{k-1,1}A) f \quad (10)$$

which implies that

$$g''_f - b'' = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \quad (11)$$

where

$$\alpha_{i,2} = \begin{cases} \alpha'_{i,1} + \alpha_{i-1,1}, & \text{for all } i = 1, 2, \dots, k-1, \\ \alpha'_{0,1} - A\alpha_{k-1,1}, & \text{for } i = 0. \end{cases} \quad (12)$$

By the same method as above we can easily deduce that

$$g_f^{(j)} - b^{(j)} = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \quad j = 0, 1, \dots, k-1, \quad (13)$$

where

$$\alpha_{i,j} = \begin{cases} \alpha'_{i,j-1} + \alpha_{i-1,j-1}, & \text{for all } i = 1, 2, \dots, k-1, \\ \alpha'_{0,j-1} - A\alpha_{k-1,j-1}, & \text{for } i = 0 \end{cases} \quad (14)$$

and

$$\alpha_{i,0} = \begin{cases} d_i, & \text{for all } i = 1, 2, \dots, k-1, \\ d_0 - d_k A, & \text{for } i = 0. \end{cases} \quad (15)$$

By (6) – (13) we obtain the system of equations

$$\begin{cases} g_f - b = \alpha_{0,0}f + \alpha_{1,0}f' + \dots + \alpha_{k-1,0}f^{(k-1)}, \\ g'_f - b' = \alpha_{0,1}f + \alpha_{1,1}f' + \dots + \alpha_{k-1,1}f^{(k-1)}, \\ g''_f - b'' = \alpha_{0,2}f + \alpha_{1,2}f' + \dots + \alpha_{k-1,2}f^{(k-1)}, \\ \dots \\ g_f^{(k-1)} - b^{(k-1)} = \alpha_{0,k-1}f + \alpha_{1,k-1}f' + \dots + \alpha_{k-1,k-1}f^{(k-1)}. \end{cases}$$

Since $h \neq 0$, then by Cramer's rule, we have

$$f = \frac{\begin{vmatrix} g_f - b & \alpha_{1,0} & \dots & \alpha_{k-1,0} \\ g'_f - b' & \alpha_{1,1} & \dots & \alpha_{k-1,1} \\ g''_f - b'' & \alpha_{1,2} & \dots & \alpha_{k-1,2} \\ \vdots & \vdots & \dots & \vdots \\ g_f^{(k-1)} - b^{(k-1)} & \alpha_{1,k-1} & \dots & \alpha_{k-1,k-1} \end{vmatrix}}{h}.$$

Then

$$f = C_0 (g_f - b) + C_1 (g'_f - b') + \dots + C_{k-1} (g_f^{(k-1)} - b^{(k-1)}), \quad (16)$$

where C_j are meromorphic functions depending on $\alpha_{i,j}$ with $\rho_{p+1}(C_j) < \rho_{M,p}(A)$, where $\alpha_{i,j}$ are defined in (14) and (15).

By (6) and Lemma 2.1, we have $\rho_{p+1}(g_f) \leq \rho_{p+1}(f)$ and by (16), Lemma 2.1 we get that $\rho_{p+1}(f) \leq \rho_{p+1}(g_f)$. Hence $i(g_f) = i(f) = p + 1$ and $\rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A) = \rho$. Lemma is proved.

Remark 2.1. In Lemma 2.4, if we do not have the condition $h \neq 0$, then the conclusion of Lemma 2.4 cannot holds. For example, if we take $d_k = 1, d_0 = A$ and $d_j \equiv 0$ ($j = 1, 2, \dots, k - 1$) then $h \equiv 0$. It follows that $g_f \equiv b$ and $\rho_{p+1}(g_f) = \rho_{p+1}(b) < \rho_{M,p}(A) = \rho_{p+1}(f) = \rho_{M,p+1}(f)$.

Lemma 2.5 [22]. *Let H be a set of complex numbers satisfying $\overline{\text{dens}}_\Delta\{|z| : z \in H \subseteq \Delta\} > 0$, and let $A(z) \neq 0$ be an analytic function in the unit disc Δ such that $\rho_p(A) = \rho < +\infty$ and for real number $\alpha > 0$, we have for all $\varepsilon > 0$ sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1} \left\{ \alpha \left(\frac{1}{1 - |z|} \right)^{\rho - \varepsilon} \right\}$$

as $z \rightarrow 1^-$ for $z \in H$. Then every solution $f \neq 0$ of equation (1) satisfies $\rho_p(f) = +\infty$ and $\rho_{M,p}(A) \geq \rho_{p+1}(f) = \rho_{M,p+1}(f) \geq \rho$.

Lemma 2.6. *Let H be a set of complex numbers satisfying $\overline{\text{dens}}_\Delta\{|z| : z \in H \subseteq \Delta\} > 0$, and let $A(z) \neq 0$ be an analytic function in the unit disc Δ such that $\rho_p(A) = \rho < +\infty$ and for real number $\alpha > 0$, we have for all $\varepsilon > 0$ sufficiently small,*

$$T(r, A(z)) \geq \exp_{p-1} \left\{ \alpha \left(\frac{1}{1 - |z|} \right)^{\rho - \varepsilon} \right\}$$

as $z \rightarrow 1^-$ for $z \in H$, and let $f \neq 0$ be a solution of equation (1). Let be the linear differential polynomial (4) with analytic coefficients $d_j \in \mathcal{L}_{p+1,\rho}$, $b \in \mathcal{L}_{p+1,\rho}$, assuming that at least one of the coefficients d_j does not vanish identically such that $h \neq 0$. Then, the differential polynomial g_f satisfies

$$i(g_f) = p + 1, \quad \rho_p(A) \leq \rho_{p+1}(g_f) = \rho_{p+1}(f) \leq \rho_{M,p}(A).$$

Proof. Suppose that $f \not\equiv 0$ is a solution of (1). Then by Lemma 2.5, we have

$$i(f) = p + 1, \rho_p(A) \leq \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \rho_{M,p}(A) = \rho.$$

By using similar arguments as in the proof of Lemma 2.4, we obtain Lemma 2.6.

3. Proofs of Theorems.

Proof of Theorem 1.1. Suppose that $f \not\equiv 0$ is a solution of (1). Then by Lemma 2.4, we have

$$i(g_f) = p + 1, \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A).$$

Set $w(z) = g_f - \varphi$. Since $\rho_{p+1}(\varphi) < \rho_{M,p}(A)$, then by Lemma 2.1 we have

$$\rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A).$$

To prove $\bar{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f)$ we need to prove $\bar{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(f)$. Substituting $g_f = w + \varphi$ into (16), we get

$$f = C_0 w + C_1 w' + \cdots + C_{k-1} w^{(k-1)} + \psi(z), \quad (17)$$

where

$$\psi(z) = C_0(\varphi - b) + C_1(\varphi' - b') + \cdots + C_{k-1}(\varphi^{(k-1)} - b^{(k-1)})$$

and $\rho_{p+1}(\psi) < \rho_{M,p}(A)$. Substituting (17) into (1), we obtain

$$C_{k-1} w^{(2k-1)} + \sum_{j=0}^{2k-2} \phi_j w^{(j)} = -(\psi^{(k)} + A(z)\psi) = H,$$

where C_{k-1}, ϕ_j ($j = 0, 1, \dots, 2k-2$) are meromorphic functions with $\rho_{p+1}(C_{k-1}) < \rho_{M,p}(A)$, $\rho_{p+1}(\phi_j) < \rho_{M,p}(A)$. Since $\rho_{p+1}(\psi) < \rho_{M,p}(A)$ and $\psi(z) \not\equiv 0$, it follows from Lemma 2.3 that $H \not\equiv 0$. Obviously, there holds

$$\begin{aligned} \max\{\rho_{p+1}(C_{k-1}), \rho_{p+1}(\phi_j) \ (j = 0, 1, \dots, 2k-2), \rho_{p+1}(H)\} < \\ < \rho_{M,p}(A) = \rho_{p+1}(w). \end{aligned}$$

Then by Lemma 2.2, we obtain $i_{\bar{\lambda}}(w) = i_{\lambda}(w) = i(w) = p + 1$ and $\bar{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w)$, i.e. $i_{\bar{\lambda}}(g_f - \varphi) = i_{\lambda}(g_f - \varphi) = i(g_f - \varphi) = i(f) = p + 1$ and $\bar{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_{M,p+1}(f) = \rho_{M,p}(A)$.

Proof of Theorem 1.2. Suppose that $f \not\equiv 0$ is a solution of (1). Then by Lemma 2.6, we have

$$i(g_f) = p + 1, \rho_p(A) \leq \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \rho_{M,p}(A).$$

Set $w(z) = g_f - \varphi$. Since $\rho_{p+1}(\varphi) < \rho_p(A)$, then by Lemma 2.1 we have

$$\rho_p(A) \leq \rho_{p+1}(w) = \rho_{p+1}(g_f) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \rho_{M,p}(A).$$

Substituting $g_f = w + \varphi$ into (16) and using a similar reasoning as in the proof of Theorem 1.1, we get

$$C_{k-1}w^{(2k-1)} + \sum_{j=0}^{2k-2} \phi_j w^{(j)} = -\left(\psi^{(k)} + A(z)\psi\right) = H,$$

where C_{k-1}, ϕ_j ($j = 0, 1, \dots, 2k - 2$) are meromorphic functions with $\rho_{p+1}(C_{k-1}) < \rho_p(A), \rho_{p+1}(\phi_j) < \rho_p(A)$. Since $\rho_{p+1}(\psi) < \rho_p(A)$ and $\psi(z) \not\equiv 0$, it follows from Lemma 2.5 that $H \not\equiv 0$. Obviously, there holds

$$\begin{aligned} \max\{\rho_{p+1}(C_{k-1}), \rho_{p+1}(\phi_j) \ (j = 0, 1, \dots, 2k - 2), \rho_{p+1}(H)\} < \\ < \rho_p(A) \leq \rho_{p+1}(w). \end{aligned}$$

Then by Lemma 2.2, we obtain $i_{\bar{\lambda}}(w) = i_{\lambda}(w) = i(w) = p + 1$ and $\bar{\lambda}_{p+1}(w) = \lambda_{p+1}(w) = \rho_{p+1}(w)$, i.e. $i_{\bar{\lambda}}(g_f - \varphi) = i_{\lambda}(g_f - \varphi) = i(g_f - \varphi) = i(f) = p + 1$ and $\rho_p(A) \leq \bar{\lambda}_{p+1}(g_f - \varphi) = \lambda_{p+1}(g_f - \varphi) = \rho_{p+1}(f) = \rho_{M,p+1}(f) \leq \rho_{M,p}(A)$.

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