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Sums of reciprocal eigenvalues in doubly connected domains

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Dedicated to the memory of Professor Promarz M. Tamrazov

For the fixed membrane eigenvalues, G. Pólya and M. Schiffer [3] showed that the sum of the first n reciprocal eigenvalues is minimal for the disk among all domains which are images of the unit disk under normalized conformal mappings. The aim of this paper is an analogous result for doubly connected domains.

1. Introduction. Let $D \subset R^2$ be a bounded doubly connected domain. We consider the following eigenvalue problem

$$\begin{aligned} \Delta u + \lambda u &= 0 \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D. \end{aligned} \tag{1}$$

It is well-known that there exists an infinity of eigenvalues with finite multiplicity [1, 2]

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \ .$$

Each eigenvalue is counted as many times as its multiplicity. For the corresponding eigenfunctions

$$u_1, u_2, u_3, \dots$$

hold

$$\int_D u_i u_j \, dA_z = \delta_{ij}, \quad i, j = 1, 2, \dots \ . \tag{2}$$

For simply connected planar domains with the maximal conformal radius 1 it was proven by G. Pólya and M. Schiffer [3] that for the eigenvalues of (1) for any n holds

$$\sum_{k=1}^{n} \frac{1}{\lambda_k} \geq \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}}\,, \tag{3}$$

where $\lambda^{(o)}$ are the eigenvalues of the unit disc.

Many authors dealt with this problem among others J. Hersch, C. Bandle, R. Laugesen and C. Morpurgo [1, 4, 2, 5, 6]. A new approach to this problem was introduced in [5]. The main idea is to use the Green's function with Dirichlet boundary conditions for the variational characterization.

For the free membrane eigenvalues (3) was proven in [7, 8]. In [6] a sharper of (3) is given for a simply connected domain with maximal conformal radius 1. The aim of this paper is to prove such an inequality for doubly connected domains in the plane. Let $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ be an univalent conformal mapping in the annulus U_{1R} with $f(U_{1R}) = D$, then the main result is the following

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} \geq \sum_{j=1}^{n} \frac{1}{\lambda_j^{(o)}} \int_{U_{1R}} {u_j^{(o)}}^2 |f'(z)|^2 \, dA_z =$$

$$= \sum_{j=1}^{n} \frac{1}{\lambda_j^{(o)}} \int_{U_{1R}} u_j^{(o)2} \sum_{n=-\infty}^{\infty} n^2 |a_n|^2 |z|^{2n-2} \, dA_z \geq |a_1|^2 \sum_{j=1}^{n} \frac{1}{\lambda_j^{(o)}}$$

with equality if and only if $f(z) = a_0 z + a_1 z$, where here are $\lambda_j^{(o)}$ the eigenvalues of the annulus and $u_j^{(o)}$ are the eigenfunctions. According to [5] we consider also

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2}$$

and can see that the annulus is a minimizer among all doubly connected domains with the same modulus.

2. Annulus. By the Riemann mapping theorem, all simply connected domains with more than one boundary point are conformally equivalent to each other. An ideal standard domain in this situation is the interior of a circle because all eigenfunction are known. It is well-known that no exact equivalent of the Riemann mapping theorem holds in the multiply connected case and that any doubly connected domain can be conformally mapped onto an annulus $U_{rR} = \{r < |z| < R\}$. The ratio of the radii R/r is called the modulus of the doubly connected domain and two doubly connected domains are conformally equivalent if and only if the modulus is the same. We choose for doubly connected domains the standard domain U_{1R}. The eigenfunctions of (1) for the annulus U_{1R} are

$$\Big(A_{mn}J_n(d_{mn}r) + B_{mn}Y_n(d_{mn}r)\Big)\sin n\varphi,$$
$$\Big(A_{mn}J_n(d_{mn}r) + B_{mn}Y_n(d_{mn}r)\Big)\cos n\varphi,$$

for $n, m = 0, 1, \dots$, where J_n and Y_n are Bessel functions of the first and second type, respectively [9, 10], d_{mn}^2 are the eigenvalues and A_{mn}, B_{mn} are appropriate coefficients such that (2) is realized. This is mentioned in [11, p. 91] without any details. A complete discussion of (1) for the annulus seems not appear in the literature although the structure of the eigenfunctions is used often. We will give some of the essential details here. In order to see this we recognize that for the annulus U_{1R} an infinity set of eigenfunctions u_n and eigenvalues λ_n exists with finite multiplicity satisfying (1). Let $G(z,\zeta)$ be Green's function of the Laplacian with Dirichlet boundary conditions. It is a classical result that the Green's function of the annulus is given by ϑ-functions [12]. The eigenvalue problem (1) is equivalent to the eigenvalue problem of the integral equation with the Green's function as the kernel and from this follows that all eigenfunctions u_n are in $C^2(U_{1R})$ because Green's function of the annulus satisfies all needed conditions [13] and therefore we obtain

$$u_n(r,\varphi) = \frac{a_{n0}}{2} + \sum_{j=1}^{\infty} a_{nj}\cos j\varphi + b_{nj}\sin j\varphi, \ \text{for} \ \ 1 < r < R, \ 0 \le \varphi \le 2\pi,$$

where the Fourier series converges for all r. It is

$$a_{nj}(r) = \frac{1}{2\pi}\int_0^{2\pi} u_n(r,\varphi)\cos j\varphi \ d\varphi, j = 0, 1, \dots,$$
$$b_{ni}(r) = \frac{1}{2\pi}\int_0^{2\pi} u_n(r,\varphi)\sin i\varphi \ d\varphi, i = 1, 2, \dots,$$

from which we see $(a_{nj}(r\varphi))_{rr} = \frac{1}{2\pi}\int_0^{2\pi}(u_n(r,\varphi))_{rr}\cos j\varphi \ d\varphi$ and an analog for b_{ni}. In the case $D = U_{1R}$ we have for (1)

$$r^2 u_{rr} + u_{\varphi\varphi} + r u_r + \lambda r^2 u = 0.$$

Multiplying by $\cos j\varphi$ or $\sin i\varphi$, we obtain after integration with respect to φ from 0 to 2π

$$\begin{aligned}
&r^2(a_{nj})_{rr} + r(a_{nj})_r + a_{nj}(r^2\lambda - j^2) = 0, \;\; 1 < r < R, \; j = 0, 1, \dots,\\
&r^2(b_{ni})_{rr} + r(b_{ni})_r + b_{ni}(r^2\lambda - i^2) = 0, \;\; 1 < r < R, \; i = 1, 2, \dots,\\
&a_{nj}(1) = a_{nj}(R) = 0, \quad b_{ni}(1) = b_{ni}(R) = 0,
\end{aligned}$$

where we have used integration by parts for the calculation of $\int_0^{2\pi} u_{\varphi\varphi} \cos j\varphi d\varphi$.

The differential equation above is Bessel's equation and the general solution is given as a linear combination of the Bessel functions $J_j(r\sqrt{\lambda})$, $Y_j(r\sqrt{\lambda})$ [9, 10] and it follows from the boundary conditions in (1)

$$A_{mn} J_n(d_{mn}) + B_{mn} Y_n(d_{mn}) = 0, \quad A_{mn} J_n(d_{mn} R) + B_{mn} Y_n(d_{mn} R) = 0,$$

with $d_{mn}^2 = \lambda$ must be chosen to make the eigenfunctions vanish for $r = 1, r = R$. A consequence of the existence of the set of eigenfunctions is that a set of solutions d_{mn} exists. What is left is to show the orthogonality condition (2). For the same eigenvalue it follows from the orthogonality of the trigonometrical functions and for different eigenvalues from a well-known property of two solutions of Bessel's equation. What we need in the following is that **for every eigenvalue either the eigenfunction is radial or the sum of the square of the two eigenfunctions belonging to the same eigenvalue is radial**.

3. The sum of all reciprocal eigenvalues. The eigenvalue problem of the fixed membrane (1) in a doubly connected domain D in the plane is conformally equivalent to the following problem in the annulus $U_{1R} = \{z : 1 < |z| < R\}$ with an appropriate R

$$\begin{aligned}
&\Delta u + \lambda u |f'(z)|^2 = 0 \text{ in } U_{1R}\,,\\
&u\big|_{\partial U_{1R}} = 0\,,\\
&\int_{U_{1R}} u_i u_j |f'(z)|^2 \, dA_z = \delta_{ij}, \quad i, j = 1, 2, \dots\;,
\end{aligned}$$

where $f(z)$ maps U_{1R} onto D, u_i denote the eigenfunctions and δ_{ij} the Kronecker delta. We use the same notation for the transplanted eigenfunctions. The eigenfunctions $\{u_j\}_1^\infty$ of (1) are the eigenfunctions of the Green's function of the domain D and now it follows that $G(z, \zeta)|f'(z)||f'(\zeta)|$, where

$G(z,\zeta)$ denotes Green's function of the annulus U_{1R}, has the eigenfunction $\{u_j|f'(z)|\}_1^\infty$ and following [5] we have

Theorem 1. *Let $G(z,\zeta)$ be the Green's function of the annulus U_{1R} and let $f(z)$ maps U_{1R} conformally onto the bounded doubly connected domain D with the eigenvalues $\lambda_1, \lambda_2, \dots$, then*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} = \int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2 \, dA_z \, dA_\zeta \,. \tag{4}$$

The proof of the next lemma runs analogous to the proof of Lemma 2.2 in [5].

Lemma 1. *$G^2(z,\zeta)$ is positive definite, that is*

$$\int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)h(z)h(\zeta) \, dA_z \, dA_\zeta \ \geq 0$$

for all $h \in C^2(U_{1R}) \cap C(\overline{U_{1R}})$ and equality holds if and only if $h \equiv 0$.

Taking $h \equiv |f'(z)|^2$ we obtain from the proof the following corollary.

Corollary 1. *Under the assumptions of Theorem 1 holds*

$$\int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2 \, dA_z \, dA_\zeta = \sum_{j=1}^{\infty} \frac{\int_{U_{1R}} (\nabla G_j)^2 dA}{\lambda_j^{(o)}}$$

with

$$G_j(\zeta) = \int_{U_{1R}} G(z,\zeta) u_j^{(o)} |f'(z)|^2 \, dA_z,$$

where $u_j^{(o)}, \lambda_j^{(o)}$ denote the eigenfunctions and eigenvalues of the annulus U_{1R}.

The next lemma follows by the same method as in [5].

Lemma 2. *Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be an univalent conformal mapping in U_{1R} with $f(U_{1R}) = D$. For a radial eigenfunction $u_j^{(o)}$, we have for any conformal mapping,*

$$\int_{U_{1R}} (\nabla G_j)^2 dA \geq -\frac{|a_1|^4}{\lambda_j^{(o)}} + \frac{2|a_1|^2}{\lambda_j^{(o)}} \int_{U_{1R}} {u_j^{(o)}}^2 |f'(z)|^2 dA = \frac{|a_1|^4}{\lambda_j^{(o)}} +$$

$$+ \frac{2|a_1|^2}{\lambda_j^{(o)}} \int_{U_{1R}} {u_j^{(o)}}^2 \Big(\sum_{n=2}^{\infty} n^2 |a_n|^2 |z|^{2n-2} + \sum_{n=1}^{\infty} n^2 |a_{-n}|^2 |z|^{2(-n-1)} \Big) dA \geq \frac{|a_1|^4}{\lambda_j^{(o)}} \,.$$

Let $u_j^{(o)}$ and $u_{j+1}^{(o)}$ be the eigenfunctions belonging to the same eigenvalue $\lambda_j^{(o)}$, such that $(u_j^{(o)})^2 + (u_{j+1}^{(o)})^2$ is radial. Then

$$\int_{U_{1R}} (\nabla G_j)^2 dA + \int_{U_{1R}} (\nabla G_{j+1})^2 \, dA \geq$$

$$\geq -\frac{2|a_1|^4}{\lambda_j^{(o)}} + \frac{2|a_1|^2}{\lambda_j^{(o)}} \int_{U_{1R}} ({u_j^{(o)}}^2 + {u_{j+1}^{(o)}}^2)|f'(z)|^2 \, dA = \frac{2|a_1|^4}{\lambda_j^{(o)}} + \frac{2|a_1|^2}{\lambda_j^{(o)}} \times$$

$$\times \int_{U_{1R}} ({u_j^{(o)}}^2 + {u_{j+1}^{(o)}}^2) \Big(\sum_{n=2}^{\infty} n^2 |a_n|^2 |z|^{2n-2} + \sum_{n=1}^{\infty} n^2 |a_{-n}|^2 |z|^{2(-n-1)} \Big) dA \geq$$

$$\geq \frac{2|a_1|^4}{\lambda_j^{(o)}} .$$

Equality occurs in both inequalities if and only if $f(z) = a_0 + a_1 z$.

The main result of this chapter is the following theorem.

Theorem 2. *Let D be a doubly connected bounded domain in the plane with the fixed membrane eigenvalues $\lambda_1, \lambda_2, \dots$ and let $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ be an univalent conformal mapping in U_{1R} with $f(U_{1R}) = D$. Then*

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} \geq -|a_1|^4 \sum_{j=1}^{\infty} \frac{1}{{\lambda_j^{(o)}}^2} + 2|a_1|^2 \int_{U_{1R}} \sum_{j=1}^{\infty} \frac{{u_j^{(o)}}^2(z)}{{\lambda_j^{(o)}}^2} \Big(\sum_{n=2}^{\infty} n^2 |a_n|^2 |z|^{2n-2} +$$

$$+ \sum_{n=1}^{\infty} n^2 |a_{-n}|^2 |z|^{2(-n-1)} \Big) dA_z \geq |a_1|^4 \sum_{j=1}^{\infty} \frac{1}{{\lambda_j^{(o)}}^2} .$$

Equality occurs if and only if $f(z) = a_0 + a_1 z$.

We give two different proofs of this theorem. The **first proof** follows [5] and used Corollary 1 and Lemma 2. The **second proof** used only Theorem 1 and Lemma 1 and is much more elementary. This proof runs in the following way. We have with Lemma 1

$$\int_{U_{1R}} \int_{U_{1R}} G^2(z, \zeta) h(z) h(\zeta) \, dA_z \, dA_\zeta \; \geq 0$$

and obtain using $h = |f'(z)|^2 - |a_1|^2$

$$\int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)\big(|f'(z)|^2 - |a_1|^2\big)\big(|f'(\zeta)|^2 - |a_1|^2\big)\, dA_z\, dA_\zeta =$$

$$= \int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)|f'(z)|^2|f'(\zeta)|^2\, dA_z\, dA_\zeta +$$

$$+ |a_1|^4 \int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)\, dA_z\, dA_\zeta -$$

$$-2\,|a_1|^2 \int_{U_{1R}} \int_{U_{1R}} G^2(z,\zeta)|f'(z)|^2\, dA_z\, dA_\zeta \geq 0\,.$$

It follows with Theorem 1

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j^2} \geq -|a_1|^4 \sum_{1}^{\infty} \frac{1}{{\lambda_j^{(o)}}^2} + 2|a_1|^2 \int_{U_{1R}} |f'(z)|^2 \int_{U_{1R}} G^2(z,\zeta)\, dA_\zeta dA_z\,.$$

The function $\int_{U_{1R}} G^2(z,\zeta) dA_\zeta$ is a radial function in U_{1R} because the Green's function of the annulus is radial which follows from the uniqueness of the Green's function, another argument is given by (4) and

$$\int_{U_{1R}} G^2(z,\zeta)\, dA_\zeta = \sum_{j=1}^{\infty} \frac{{u_j^{(o)}}^2(z)}{{\lambda_j^{(o)}}^2}\,.$$

This makes the proof complete.

4. The sums of finite many reciprocal eigenvalues. The goal of this chapter is an result analogous to the inequality given by G. Pólya and M. Schiffer. We give first two lemmas and follow the main ideas in [7, 6]. We remind that $G(z,\zeta)|f'(z)||f'(\zeta)|$, where $G(z,\zeta)$ denotes Green's function of the annulus U_{1R}, has the eigenfunctions $\{u_j|f'(z)|\}_{j=1}^{\infty}$ and following [7] we obtain a variational characterization for the eigenvalue λ_n of the domain D.

Lemma 3.

$$\max \int_{U_{1R}} \int_{U_{1R}} G(z,\zeta)|f'(z)|h(z)|f'(\zeta)|h(\zeta)\, dA_z\, dA_\zeta \;= \frac{1}{\lambda_n}\,,$$

where the maximum is taken over all $h \in L^2(U_{1R})$ *with* $\int_{U_{1R}} h^2 dA = 1$ *and*

$$\int_{U_{1R}} u_j(z)|f'(z)|h(z)dA_z = 0, \quad j = 1, 2, \ldots, n-1.$$

Equality occurs if $h = u_n|f'(z)|$.

From this lemma follows easily

Lemma 4.

$$\sum_1^n \frac{1}{\lambda_j} = \max_{L_n} \sum_{i=1}^n \int_{U_{1R}} \int_{U_{1R}} G(z,\zeta)|f'(z)|h_i(z)|f'(\zeta)|h_i(\zeta)\, dA_z\, dA_\zeta,$$

where $\{h_i\}_{i=1}^n$ *is a basis of* L_n *with* $\int_{U_{1R}} h_i h_j dA_z = \delta_{ij}$ *and* L_n *ranges over all* n*-dimensional subspace of* $L^2(U_{1R})$. *Equality occurs for* $L_n = \{u : u = \sum_{i=1}^n c_i u_i |f'(z)|\}$.

Now we are in a position to prove the main theorem.

Theorem 3. *Let* D *be a doubly connected domain in the plane with the eigenvalues* $\{\lambda_j\}_{j=1}^\infty$. *Then, for any* $n \geq 1$, *we have*

$$\sum_{j=1}^n \frac{1}{\lambda_j} \geq \sum_{j=1}^n \frac{1}{\lambda_j^{(o)}} \int_{U_{1R}} {u_j^{(o)}}^2 |f'(z)|^2 dA_z =$$

$$= \sum_{j=1}^n \frac{1}{\lambda_j^{(o)}} \int_{U_{1R}} {u_j^{(o)}}^2 \sum_{n=-\infty}^{\infty} n^2 |a_n|^2 |z|^{2n-2}\, dA_z \geq |a_1|^2 \sum_{j=1}^n \frac{1}{\lambda_j^{(o)}}\,.$$

Equality occurs for $f(z) = a_0 + a_1 z$.

Proof. We will see that a set of functions $\{h_j\}_{j=1}^n$ exists with $\int_{U_{1R}} h_i h_j |f'(z)|^2\, dA = \delta_{ij}$ and

$$h_j = \sum_{i=1}^j c_{ji} u_i^{(o)}, \quad c_{ii} \neq 0,\ i = 1, 2, \ldots, n,$$

where we have replaced $h_i|f'(z)|$ instead of h_i in Lemma 4.

First we show that a set of functions $\{h_j\}_{j=1}^n$ exists with the orthogonality conditions mentioned above. We choose $h_1 = c_{11} u_1^{(o)}$ with $\int_{U_{1R}} h_1 |f'(z)|^2 dA = 1$ and $h_2 = c_{21} u_1^{(o)} + c_{22} u_2^{(o)}$ with $\int_{U_{1R}} h_2 h_1 |f'(z)|^2 dA = 0$ and $\int_{U_{1R}} h_2^2 |f'(z)|^2 dA = 1$. In this way follows the existence of a set of functions $\{h_j\}_{j=1}^n$ with the orthogonality conditions mentioned above. In order to see that $c_{ii} \neq 0$, $i = 1, 2, \dots, n$, we conclude by induction. It is evident that $c_{11} \neq 0$. A consequence of $c_{22} = 0$ would be $c_{11} = 0$ that such also $c_{22} \neq 0$. We assume that $c_{jj} \neq 0$, $j = 1, 2, \dots, k-1$ and $c_{kk} = 0$ and it would follow that h_k is a linear combination of the functions $h_1, h_2, \dots, h_{k-1}$ and from the orthogonality conditions it follows $h_k \equiv 0$. This is the contradiction.

Consider the matrix

$$C_n = (c_{ij})_{i,j=1}^n, \quad c_{ij} = 0\,,\ i < j\,.$$

C_n is regular because $c_{ii} \neq 0$ for all i. With $C_n^{-1} = D_n = (d_{ij})_{i,j=1}^n$, $d_{ij} = 0$, $i < j$, we obtain

$$u_k^{(o)} = \sum_{i=1}^{k} d_{ki} h_i\,. \tag{5}$$

In the following we need

$$b_{jk} = \int_{U_{1R}} h_j u_k^{(o)} |f'(z)|^2\, dA\,.$$

Hence

$$b_{jk} = \int_{U_{1R}} h_j u_k^{(o)} |f'(z)|^2 dA = \int_{U_{1R}} h_j (\sum_{i=1}^{k} d_{ki} h_i) |f'(z)|^2 dA = d_{kj} \tag{6}$$

caused by the orthogonality conditions.

We have using Lemma 4 with $h_i|f'(z)|$ instead of h_i

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} \geq \sum_{i=1}^{n} \int_{U_{1R}} \int_{U_{1R}} G(z,\zeta) |f'(z)|^2 h_i(z) |f'(\zeta)|^2 h_i(\zeta)\, dA_z\, dA_\zeta$$

and as a consequence of the Hilbert–Schmidt theorem

$$\int_{U_{1R}} \int_{U_{1R}} G(z,\zeta) |f'(z)|^2 h_j(z) |f'(\zeta)|^2 h_j(\zeta)\, dA_z\, dA_\zeta = \sum_{k=1}^{\infty} \frac{b_{jk}^2}{\lambda_k^{(o)}}$$

from which follows

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} \geq \sum_{j=1}^{n} \sum_{k=1}^{\infty} \frac{b_{jk}^2}{\lambda_k^{(o)}} \geq \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} \sum_{j=1}^{n} b_{jk}^2 = \sum_{k=1}^{n} \frac{1}{\lambda_k^{(o)}} \int_{U_{1R}} {u_k^{(o)}}^2 |f'(z)|^2\, dA\,,$$

because

$$\int_{U_{1R}} {u_k^{(o)}}^2 |f'(z)|^2 \, dA = \sum_{j=1}^{k} d_{kj}^2 = \sum_{j=1}^{k} b_{jk}^2.$$

The last identity is a consequence of (5) and (6). This finishes the proof. The identity given in the theorem follows from the structure of the eigenfunctions.

Remarks. 1. Theorem 3 contains the inequality

$$\sum_{j=1}^{n} \frac{1}{\lambda_j} \geq |a_1|^2 \sum_{j=1}^{n} \frac{1}{\lambda_j^{(o)}} \, ,$$

which was proved by R. Laugesen and C. Morpurgo [14]. The proof closely follows J. Hersch [15].

2. Let $\Phi(a)$ be convex and increasing for $a \geq 0$. Then a general majorization result of Hardy, Littlewood and Pólya [16] gives that

$$\sum_{j=1}^{n} \Phi\left(\frac{1}{\lambda_j}\right) \geq \sum_{j=1}^{n} \Phi\left(\frac{|a_1|^2}{\lambda_j^{(o)}}\right).$$

For $\Phi(x) = x^2$ this contains a weaker version of Theorem 2.

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