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## Algebroid functions with essential singularities

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*Dedicated to memory of Professor Promarz M. Tamrazov*

In this article, it is considered the question of analyzing the behaviour of an algebroid function near a singularity. This is an old question, going back to Puiseux, Cramer and others. Here it is considered the problem of estimating the length of the algebraic cycles of the branches of an algebroid function at an algebraic singularity in terms of the data relative to the coefficients  $A_k(z)$  of the equation defining the algebroid function. A further question considered is the value distribution of an algebroid function near an essential singularity, we prove in this direction the corresponding result to the Casoratti–Weierstrass Theorem. The final aim should be the Great Picard Theorem for algebroid functions.

**1. Introduction.** An algebroid function  $w = f(z)$  of order  $n$  is a multivalued function, which we shall assume to be defined in the entire plane  $\mathbb{C}$ , given by an equation of the form

$$F(w, z) = A_n(z)w^n + A_{n-1}(z)w^{n-1} + \cdots + A_0(z) = 0, \quad (1)$$

where  $A_0(z), A_1(z), \dots, A_n(z)$  are meromorphic functions with no common zeros or poles and  $F(w, z)$  is irreducible, that is, it cannot be decomposed as a product

$$F(w, z) = F_1(w, z) F_2(w, z),$$

where  $F_1(w, z), F_2(w, z)$  are two non-constant functions of the same kind.

In this way we obtain a  $n$ -valued meromorphic function  $w(z)$  outside the critical points where  $A_n(z)$  has a zero or one of the  $A_i(z)$ ,  $i = 1, 2, \dots, n$  has a pole and also outside those points where the so-called discriminant of  $F(w, z)$  vanishes where two or more of the  $n$  roots  $w_i(z)$  of the equation (1) are equal.

The algebroid functions are a natural extension of meromorphic functions and have been thoroughly studied, see K.Hensel and Landsberg [1]. Here we shall allow the coefficients  $A_i(z)$  to have isolated essential singularities, this case has not been so much considered, we shall mention here the works of G. Remoundos [2, 3]. We shall be interested in the behaviour of these functions near the singularities.

We shall call algebroid functions with essential singularities those functions obtained in this way, that is, those given by an irreducible equation of the form (1), where the coefficients  $A_i(z)$  might have isolated essential singularities. In this article, we shall be concerned with the behaviour of algebroid functions, allowing essential singularities, near a singularity, in particular we shall prove the corresponding Casoratti–Weierstrass Theorem.

**2. The discriminant of an algebroid function.** Let  $z = \alpha$  be a regular point for the coefficients  $A_i(z)$ ,  $i = 1, 2, \dots, n$ , of the equation (1) and assume that  $A_n(z)$  does not vanish at this point, then we get in a small neighbourhood of this particular point  $n$  roots  $w_1(z), w_2(z), \dots, w_n(z)$  of the equation. We shall call the symmetric function  $D(z)$  of the  $w_i$ 's defined by

$$D(z) = \prod_{i \neq j} (w_i(z) - w_j(z)),$$

the discriminant of the algebroid function. It turns out that at a regular point,  $D(z)$  vanishes if and only if two or more roots  $w_1(z), w_2(z), \dots, w_n(z)$  are equal.

In Hensel und Landsberg [1], it is shown that if in addition to the above condition about the regularity of the coefficients, we also assume that  $D(z) \neq 0$ , then the  $n$  roots  $w_i(z)$ ,  $i = 1, 2, \dots, n$  are analytic near  $\alpha$ , that is, then they can be expanded as a power series of  $z - \alpha$

$$w_i(z) = \sum_{n=0}^{\infty} a_{ni} (z - \alpha)^n.$$

If the coefficients  $A_i(z)$ ,  $i = 1, 2, \dots, n$ , are all of them meromorphic, the function  $D(z)$  is itself also meromorphic, the possible zeros or poles of



with only a finite number of non-integral terms and common denominator  $a \in \mathbb{N}$ , that is,

$$\frac{p_{0i}}{a_i} < \frac{p_{1i}}{a_i} < \frac{p_{2i}}{a_i} < \dots$$

where  $a_i, p_{ji} \in \mathbb{Z}$ .

The sequences of exponents  $\epsilon_{0i}, \epsilon_{1i}, \epsilon_{2i}, \dots$  are obtained in the following way.

Let us consider the quotients

$$\epsilon = -\frac{\rho_l - \rho_g}{l - g}, \quad l, g = 1, 2, \dots, n,$$

so that

$$\rho_g + g\epsilon = \rho_l + l\epsilon = \gamma,$$

and

$$\rho_k + k\epsilon \geq \gamma, \quad k = 1, 2, \dots, n.$$

In this way, one obtains a number  $\nu$  of  $\epsilon$ 's which is less or equal than  $n$ , say,  $\epsilon_0, \epsilon'_0, \dots, \epsilon_0^{(\nu)}$ . These are all the possible exponents  $\epsilon_{0i}$  and for each of these exponents  $\epsilon_0^{(j)}$  there are  $n_j$  possible initial coefficients  $e_{0i}$  which will be the solutions of a polynomial equation

$$\varphi_j(e) = a_{n_j} e^{n_j} + \dots + a_0 = 0,$$

in such a way that

$$\sum_j n_j = n,$$

so that for the  $\nu$  exponents  $\epsilon_0, \epsilon'_0, \dots, \epsilon_0^{(\nu)}$ , there are  $n = \sum_j n_j$  possible initial terms and, in fact, it is shown that for each of these possible initial terms there is one and only one power series which is a solution of (1), that is, the  $n$  branches at  $\alpha$  of the algebroid function are obtained in this way.

Let us call  $u(\zeta)$  one of this branches and set

$$u(\zeta) = e_0 \zeta^\epsilon + e_1 \zeta^\epsilon + \dots$$

the corresponding power series expansion in the variable  $\zeta$ . A necessary and sufficient condition in order that among the exponents  $\epsilon_0, \epsilon_1, \dots$  some of them are proper fractions, is that the corresponding



we shall arrive at a new equation of the same type as (1)

$$\begin{aligned} {}_1F(u', \zeta) &= F(e_0 \zeta^{\epsilon_0} + u', \zeta) \\ &= A_0^1(\zeta) + A_1^1(\zeta) u' + \cdots + A_n^1(\zeta) (u')^n, \end{aligned}$$

and apply to  ${}_1F(u', \zeta)$  the same procedure as we did with  $F(u, \zeta)$  and obtain the exponent

$$\epsilon_1 = -\frac{e'_t - e'_r}{t - r} = \frac{q}{s_1},$$

and the coefficient  $e_1$  as a solution of a polynomial equation

$$\varphi_1(e) = 0,$$

of a certain multiplicity  $\lambda_1$ , which will satisfy the relation

$$\lambda_1 \leq s_1 \leq \lambda_0 \leq s_0,$$

as proved in Hensel and Landsberg [1], where  $\lambda_0, s_0$  are the  $\lambda$  and  $s$  obtained for the original  $F$  and  $\lambda_1, s_1$  are the corresponding numbers for  ${}_1F$ .

Proceeding in this way we shall obtain the sequences of exponents and coefficients  $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ ,  $e_0, e_1, e_2, \dots$  and simultaneously the sequences  $\lambda_0, \lambda_1, \lambda_2, \dots$ ,  $s_0, s_1, s_2, \dots$  satisfying

$$s_0 \geq \lambda_0 \geq s_1 \geq \lambda_1 \geq s_2 \geq \lambda_2 \geq \dots,$$

in particular, we conclude  $s_{k+1} \leq s_k$  so that from a certain term onwards the sequence  $\{s_k\}$  must be constant and, in fact, it is shown in [1] that  $s_k = 1$  from a certain point onwards, say for  $k \geq \tau$ .

From the above considerations we obtain that the order  $a$  of the cycle is given by

$$a = m.c.m \{s_0, s_1, \dots, s_{\tau-1}\}, \quad (4)$$

where  $s_\tau = 1$  for  $k \geq \tau$ .

Next we describe some particular cases,

I) Let us assume that  $e_0$  is a simple root of the coefficient equation

$$\varphi_0(e) = 0,$$

that is,  $\lambda_0 = 1$ , in this case we must have

$$1 = s_1 = s_2 = \dots$$

and as a consequence of (4) we deduce that  $a = s_0$ , that is,  $s_0$  is the order of the corresponding cycle.

We also consider the following case,

II) Let now  $\lambda_0 = 2$ , that is,  $e_0$  is a root of order two of the coefficient equation  $\varphi_0(e) = 0$ , then we obtain from (4)

$$s_1 = 1 \text{ or } 2.$$

In the case  $s_1 = 1$ , we deduce again from (4) that  $a = s_0$ , so that we have again a cycle of order  $s_0$ .

In the case  $s_1 = 2$ , then we can consider two subcases, namely that  $s_0$  be an even or an odd number.

II<sub>1</sub>) If  $s_0$  is even then  $a$  is again equal  $s_0$ , that is, the order of the cycle is again  $s_0$ .

II<sub>2</sub>) If  $s_0$  is an odd number, we can consider in its turn two possibilities, namely, either  $s_1 = 1$  where again  $a = s_0$  and a cycle of order  $s_0$  occurs at  $z = \alpha$  or  $s_1 = 2$ , where a cycle of order  $a = 2s_0$  occurs at  $z = \alpha$ .

In general, it follows from the above considerations that if  $\lambda_k$  is the multiplicity of  $e_k$ , for some  $k = 0, 1, \dots$  the order of the cycle will be less or equal than

$$s_0 \cdot s_1 \cdot \dots \cdot s_k \cdot \lambda_k! , \quad (5)$$

in particular for  $k = 0$ , we obtain

**Theorem 1.** *For  $k = 0$ , the order of the corresponding cycle should be less or equal than  $(s_0 \cdot \lambda_0)!$ .*

A natural question arises,

**Problem 1.** *Is the estimate (5) optimal or we can improve the upper bound for the order of a cycle given the numbers  $s_0, \lambda_0$ ?*

A related and more general question is the following,

**Problem 2.** *Given  $n \in \mathbb{N}$  and  $n_1, n_2, \dots, n_k \in \mathbb{N}$  such that  $n_1 + n_2 + \dots + n_k = n$  and let  $\alpha \in \mathbb{C}$ , can we construct an algebroid function  $w = w(z)$  of order  $n$ , such that at  $z = \alpha$  presents  $k$  cycles of orders  $n_1, n_2, \dots, n_k$ .*

**4. Essential singularities. Casoratti–Weierstrass theorem.** Now we shall allow the equation

$$F(w, z) = A_n(z)w^n + A_{n-1}(z)w^{n-1} + \dots + A_0(z) = 0,$$

to have coefficients with isolated singularities. Exactly as we did in the previous case, we can assume the equation to be normalized, that is,  $A_n(z) \equiv 1$ .

Let  $z = \alpha$  be an essential singularity for some coefficient  $A_k(z)$  and let  $w_1(z), \dots, w_n(z)$  be the branches of the algebroid function in a punctured disc  $D^*(\alpha, r)$ , where there is no other essential singularity of the coefficients.

The following relations hold

$$\begin{aligned} \sum_{i=1}^n w_i(z) &= A_{n-1}(z), \\ \sum_{i,k=1}^n w_i(z) w_k(z) &= A_{n-2}(z), \\ &\dots\dots\dots \\ w_1(z) w_2(z) \dots w_n(z) &= (-1)^n A_0(z). \end{aligned} \tag{6}$$

First of all we consider the question whether in these circumstances the image of such punctured disc, that is

$$\bigcup_{i=1}^n w_i(D^*(\alpha, r)), \tag{7}$$

should be an unbounded set. This fact follows immediately from the relations (6). In fact, if the set (7) were a bounded set then the coefficients  $A_k(z)$ ,  $k = 1, \dots, n - 1$ , should be bounded in  $D^*(\alpha, r)$  and as a consequence  $\alpha$  would be a removable singularity for all of them, what is contrary to our hypothesis.

The next result is the Casoratti–Weierstrass Theorem for algebroid functions.

**Theorem 2. (Casoratti–Weierstrass Theorem for algebroid functions).** *For every punctured disc  $D^*(\alpha, r)$  of an essential singularity  $z = \alpha$  of an algebroid function  $w = w(z)$ , the set*

$$\bigcup_{i=1}^n w_i(D^*(\alpha, r))$$

*is a dense set in  $\mathbb{C}$ .*

**Proof.** We proceed in a similar way to the case of analytic functions around an isolated singularity. Let us assume that the set

$$\bigcup_{i=1}^n w_i(D^*(\alpha, r)),$$

were not a dense set in  $\mathbb{C}$  and let  $\beta \in \mathbb{C}$ ,  $\epsilon > 0$ , such that

$$\bigcup_{i=1}^n w_i(D^*(\alpha, r)) \cap D(\beta, \epsilon) = \emptyset,$$

then we consider the algebroid function  $G(w, z) = 0$ , such that its roots  $w_i^G(z)$  are related to those  $w_i(z) = w_i^F(z)$  of our original algebroid equation  $F(w, z) = 0$  by

$$w_i^G(z) = \frac{1}{w_i(z) - \beta}, \quad i = 1, 2, \dots, n,$$

or equivalently

$$w_i(z) = \frac{1}{w_i^G(z)} + \beta,$$

that is, these functions  $w_i^G(z)$  should satisfy the equation

$$\left(\frac{1}{w} + \beta\right)^n + A_{n-1}(z) \left(\frac{1}{w} + \beta\right)^{n-1} + \dots + A_0(z) = 0,$$

which can be rewritten in the form

$$\frac{1}{w^n} + B_{n-1}(z) \frac{1}{w^{n-1}} + \dots + B_0(z) = 0,$$

or equivalently

$$B_0(z) w^n + B_1(z) w^{n-1} + \dots + 1 = 0,$$

whence we obtain

$$G(w, z) = w^n + A_{n-1}^G(z) w^{n-1} + \dots + A_0^G(z) = 0,$$

where

$$A_k^G(z) = \frac{B_k(z)}{B_0(z)}, \quad k = 1, 2, \dots, n-1, \quad A_n^G(z) \equiv 1.$$

Since the coefficients  $A_k^G(z)$  can be obtained as symmetric functions of the roots  $w_i^G(z)$  and these are bounded functions in  $D^*(\alpha, r)$ , we would conclude that these coefficients should also be bounded in  $D^*(\alpha, r)$ .

In principle, the coefficients  $A_k^G(z) = \frac{B_k(z)}{B_0(z)}$  might have poles, that is, the zeros of  $B_0(z)$  might accumulate at  $\alpha$ , but from the boundedness of these coefficients, this possibility is excluded, so that  $\alpha$  should be an isolated singularity of  $A_k^G(z)$  and by the boundedness this singularity is removable. But if the coefficients  $A_k^G(z)$  are analytic at  $\alpha$ , then the solutions  $w_i^G(z)$  must be of the form

$$w_i^G(z) = \sum_{l=l_0} a_l^i (z - \alpha)^{\frac{l}{a}},$$

with  $l_0 \geq 0$ , where if  $a$  is bigger than one, these roots are grouped in cycles as described in section 3 and for  $a = 1$  we get a local uniform branch.

It is clear that in this case the branches

$$w_i(z) = \frac{1}{w_i^G(z)} + \beta,$$

might only present at  $\alpha$  a pole as singularity and therefore the coefficients

$$A_k(z), \quad k = 1, 2, \dots, n-1,$$

might also have at most a pole as singularity at  $\alpha$ , what is in contradiction with our hypothesis. q.e.d.

**5. The Picard theorems for algebroid functions.** Let again  $\alpha$  be an essential singularity of an algebroid function what is equivalent to the fact that  $\alpha$  is an essential singularity of at least one of the coefficients. We can assume again that the equation of the algebroid function is normalized, that is,

$$F(w, z) = w^n + A_{n-1}(z)w^{n-1} + \dots + A_0(z) = 0.$$

In the previous section, we have proved the Casoratti–Weierstrass Theorem which states that the union

$$\bigcup_{i=1}^n w_i(D^*(\alpha, \epsilon)),$$

is dense in  $\widehat{\mathbb{C}}$  for every  $\epsilon > 0$ , where the  $w'_i$ 's are the branches of the algebroid function.

A further natural question is whether the Great Picard Theorem also holds for algebroid functions. We do not have a proof of this theorem for algebroid functions. Next, we present some particular simple cases.

I) The most simple examples of a proper algebroid function is given by the equation

$$F(w, z) = w^n + A(z) = 0,$$

where  $A(z)$  vanishes at least at some point  $z = \beta$ .

In this case the branches  $w_i(z)$  are given by

$$w_i(z) = \sqrt[n]{-A(z)},$$

where for each  $i = 1, 2, \dots, n$ , we get one of the  $n$ -th roots of  $A(z)$ .

If  $z = \alpha$  is an essential singularity of  $A(z)$ , by the classical Great Picard Theorem for meromorphic functions, given a punctured disc  $D^*(\alpha, r)$ , every value  $w$  is assumed by  $A(z)$  infinitely often except for at most one exceptional value  $b$ . From this fact, it follows that the set

$$\bigcup_{i=1}^n w_i(D^*(\alpha, \epsilon)),$$

contains every value of  $\widehat{\mathbb{C}}$  except at most the  $n$ -th roots of  $-b$ , and every non-exceptional value is assumed infinitely often.

II) A more general equation which can also be dealt with following an argument of Songmin Wang [4], is

$$F(w, z) = w^n + c_{n-1}w^{n-1} + \dots + A_j(z)w^j + \dots + c_0 = 0, \quad (8)$$

that is, the equation where all the coefficients are constant except the  $j$ -th coefficient  $A_j(z)$  which, we assume to present an essential singularity at  $z = \alpha$ .

With this hypothesis, the function  $A_j(z)$  assumes every value  $b$  infinitely many times in a punctured disc  $D^*(\alpha, \epsilon)$ , with at most two exceptional values. On the other hand, for those  $z$  satisfying the equation

$$A_j(z) = b,$$

(8) is equivalent to

$$w^n + c_{n-1}w^{n-1} + \dots + bw^j + \dots + c_0 = 0, \quad (9)$$

so that every value  $w$  in  $\bigcup_{i=1}^n w_i(D^*(\alpha, \epsilon))$ , is assumed infinitely many times except at most  $2n$  exceptional points, namely the roots of the equation (9), when  $b$  is an exceptional value of  $A_j(z)$  at  $\alpha$ .

III) If we allow more than one  $A_j(z)$  to be non-constant, then we are not able, by simple considerations, to obtain similar conclusions. In fact, let us consider the simplest case, that is, a trinomial

$$F(w, z) = w^2 + A_1(z)w + A_0(z) = 0, \quad (10)$$

where one of the coefficients, say  $A_0(z)$ , presents an essential singularity at  $z = \alpha$  and where the other coefficient  $A_1(z)$  is not a constant function but it might show a regular behaviour at  $z = \alpha$ .

The solutions of this equation are given by

$$w_{1,2}(z) = \frac{-A_1(z) \pm \sqrt{A_1^2(z) - 4A_0(z)}}{2},$$

i.e. these are the branches of the algebroid function.

In order to have a proper algebroid function given by the equation (10), the discriminant

$$\Delta(z) = A_1^2(z) - 4A_0(z),$$

should vanish at some point  $\beta$ , which should be a branch point of the algebroid function.

In this case, we do not know whether the Great Picard Theorem holds. The point  $z = \alpha$  is an essential singularity of  $\Delta(z)$  and as a consequence assumes every value of  $\widehat{\mathbb{C}}$  infinitely often with at most one exception and therefore the function

$$L(z) = \sqrt{\Delta(z)},$$

has two branches  $L_1(z), L_2(z)$  and the set

$$L_1(D^*(\alpha, r)) \cup L_2(D^*(\alpha, r)),$$

contains every value of  $\widehat{\mathbb{C}}$  and every value is assumed infinitely many times except at most one exceptional value. However we cannot say anything about the values assumed by the original branches

$$w(z) = -\frac{A_1(z)}{2} + \frac{L(z)}{2}.$$

**6. The work of Remoundos.** In this section we recall and pay attention to the work of the greek mathematician G. Remoundos, who worked extensively on the theory of algebroid functions. In particular, he proved the following extension of Picard Theorem in the plane,

**Theorem (G. Remoundos).** *An algebroid function of algebraic order  $k$  and finite order of growth assumes every value of  $\widehat{\mathbb{C}}$  infinitely many times except at most  $2k$  exceptional values.*

This theorem follows straightaway from the Second Main Theorem (SMT) of the value distribution theory of algebroid functions developed by H. L. Selberg. In fact, let us recall the SMT for algebroid functions,

**Theorem (H. L. Selberg).** *Let  $w(z)$  be an algebroid function of order  $k$  in the plane and  $a_1, \dots, a_q$ ,  $q \geq 2k$ , different values in  $\widehat{\mathbb{C}}$ , then the following inequality holds*

$$(q - 2k)T(r, w) \leq \\ \leq N(r, a_1) + \dots + N(r, a_q) - N_{Ram}(r, P) + N_{Ram}(r, w) + S(r, w),$$

*this inequality is known as the Fundamental Inequality of the value distribution of algebroid functions.*

*The terms  $N_{Ram}(r, P)$ ,  $N_{Ram}(r, w)$  are indicators of the ramifications of  $P$  and  $w$  respectively and are positive terms and  $S(r, w)$  is an error term, negligible compared with  $T(r, w)$ , that is,*

$$S(r, w) = o(T(r, w)), \quad r \rightarrow \infty.$$

The SMT together with Selberg Ramification Theorem, namely the estimate of the ramification term  $N_{Ram}(r, w)$

$$N_{Ram}(r, w) \leq 2kT(r, w) + o(T(r, w)),$$

yields the deficiency relation for algebroid functions

$$\sum_{\nu} \delta(a_{\nu}, w) \leq 2k, \tag{11}$$

where the definition of the deficiency  $\delta(a, w)$  for algebroid functions is analogous as in the plane

$$\delta(a, w) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, w)}{T(r, w)}.$$

We conclude from the deficiency relation (11) that at most  $2k$  values  $a$ , can be assumed a finite number of times and we see that the Picard Theorem of Remoundos follows from the deficiency relation for algebroid functions.

As for the Big Picard Theorem for algebroid functions near an essential singularity  $z = \alpha$ , G. Remoundos asserts in the introduction of his Mémoire [3] of 1927 in the "Memorials of the Sciences de Mathématiques" and promises that this result would appear in in the same journal further on. However this is not the case.

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