Збірник праць Ін-ту математики НАН України 2013, том 10, N 4–5, 92–114

UDC 517.54

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Zero-free polynomial approximation on the union of two sets amply intersecting

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Dedicated to the memory of Professor Promarz M. Tamrazov

The problem of approximating a function by polynomials having no zeros on the domain of the function is related to the Riemann Hypothesis. If the restriction of the function to each of two sets can be so approximated, we give conditions under which it can be so approximated on the union of these two sets. This helps to localize the problem.

1. Introduction. The most important value of a polynomial is the value zero. It is natural to enquire about the possibility of approximating a given function on a compact set $K \subset \mathbb{C}$ by polynomials having no zeros on K. Recently, Johan Andersson [1] has shown the remarkable fact that this question is equivalent to a question regarding the Riemann zeta-function. A positive answer to the latter question would generalize the spectacular universality property of the zeta-function due to Sergei Mikhailovich Voronin, which is related to the Riemann Hypothesis (see [2]).

If f can be (uniformly) approximated on K by polynomials, then f necessarily belongs to the class $A(K)$ of continuous functions on K, which are holomorphic on the interior K^o. Let us say that K is a set of polynomial approximation, if every $f \in A(K)$ can be approximated by polynomials. The celebrated theorem of Sergei Nikitovich Mergelyan states that K is a set of polynomial approximation if and only if $\mathbb{C} \setminus K$ is connected.

Let us denote by $A_o(K)$ the set of $f \in A(K)$ having no isolated zeros in K^o. If $f \in A(K)$ can be approximated by polynomials which are zero-free on K, then necessarily $f \in A_o(K)$. We shall say that a set of polynomial approximation K is a set of *zero-free* polynomial approximation if every $f \in A_o(K)$ can be approximated by polynomials which are zero-free on K.

Suppose K_1 and K_2 are compact sets of zero-free polynomial approximation. It is easy to see that their union $K_1 \cup K_2$ is also a set of zero-free polynomial approximation if the two sets are disjoint and, in [2], it is shown that this is also true if their intersection is at most a singleton. The main result of the present paper is rather at the opposite extreme. Namely, we consider the case that the intersection is “ample"in a sense to be now specified.

For two sets $A, B \subset \mathbb{C}$, we denote by $d(A, B)$ the distance between A and B :

$$d(A, B) = \inf\{|a - b| : a \in A, b \in B\}.$$

Note that we follow the convention that $\inf \mathbb{R} = +\infty$, so that $d(A, \emptyset) = = +\infty$. Let us say that two intersecting sets A and B in $\mathbb{C}$ intersect *amply* if $d(A \setminus B, B \setminus A) > 0$.

Our principal result is the following.

Theorem 1. *Suppose K_1 and K_2 are compact sets of zero-free polynomial approximation whose intersection is ample and path-connected. Then, every function $f \in A_o(K_1 \cup K_2)$ which is zero-free on $K_1 \cap K_2$ can be approximated by polynomials zero-free on $K_1 \cup K_2$.*

For earlier results on zero-free polynomial approximation, see [1 — 5]. Since our main result concerns zero-free approximation on the union of two sets, in Section 2, we recall some fundamental theorems regarding rational approximation on unions and intersections. In Section 3, we state some basic facts regarding exponentials and logarithms in $\mathbb{C}$. In Section 4, we recall some basic results on exponentials in Banach algebras, which have as a consequence that approximation by polynomials zero-free on a compact set K is equivalent to approximation by exponentials e^g, with $g \in A(K)$. This allows us to prove Theorem 1 in Section 5. In Section 6, we give a slight improvement on an interesting recent theorem of Sergey Khrushchev [5] regarding zero-free approximation. In Section 7, we give some examples and open problems. Finally, in Section 8, we show that there is no topological obstruction to zero-free approximation.

It is easy to approximate a function f on a compact set K by polynomials zero-free on K, if f itself is zero-free on K. But our assumption is minimal, $f \in A_o(K)$. Thus, we merely assume that f has no isolated zeros on K^o. In particular, f is allowed to have zeros on the boundary ∂K. We formulated (but did not publish) this problem in the 70's, while considering the related problem of approximating a function mapping K to $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by rational functions having no poles on K (see [6]).

Suppose f has no zeros on K^o. If we can uniformly approximate the identity function, $I(z) = z$, on K by a function $\varphi \in A(K)$, such that $\varphi(K) \subset K^o$, then $f \circ \varphi$ approximates f, so our problem is equivalent to that of approximating $f \circ \varphi$ by polynomials zero-free on K. But $f \circ \varphi$ has no zeros on all of K, so we have reduced the problem to one we can solve. In considering the analogous problem in [6], this led us to consider the following more interesting variant of the problem of whether there is such a φ.

For a compact set K, we denote by $Aut(K)$ the family of homeomorphisms of K onto itself which are holomorphic on K^o. We say that K is *conformally rigid* if the identity $I(z) = z$ is isolated in $Aut(K)$. Thus, K is conformally rigid if and only if, there is an $\epsilon > 0$, such that $\varphi \in Aut(K)$ and $|I - \varphi| < \epsilon$ implies $\varphi = I$. Conformally rigid compacta might provide counterexamples to our zero-free polynomial approximation problem. Dieter Gaier told me that he devoted two years of his life (see [7]) attempting to characterize conformally rigid domains. Gaier's work relied on fundamental results of Promarz M. Tamrazov, whom I had the pleasure of knowing since 1972; who, with his charming and beautiful wife kindly received my family in his home in the Soviet era; and to who's memory this paper is dedicated with fondness and admiration.

2. Rational approximation. Before undertaking our study of approximation on a compact set K by rational functions having neither poles nor zeros on K, we recall the classical theory of rational approximation, which is that of approximation by rational functions having no poles on K (with no restrictions regarding the zeros of these functions). For the purposes of the present paper, we are particularly interested in approximation on unions and intersections of sets on which we can approximate.

For a compact set $K \subset \overline{\mathbb{C}}$, we denote by $R(K)$ the uniform limits on K of rational functions having no poles on K. Also, we denote by $A(K)$ the family of continuous complex-valued functions on K which are holomorphic on the interior K^o. Obviously, $R(K) \subset A(K)$. Let us say that a compact set K is a set of *rational approximation* if $R(K) = A(K)$, that is, if every function plausibly approximable by rational functions is indeed approximable. One may assume that $K \subset \mathbb{C}$. A complete characterization of sets of rational approximation was given by Anatoliy Georgievich Vitushkin (see [8]) in terms of continuous analytic capacity. For a Borel set $E \subset \mathbb{C}$, we denote by $\alpha(E)$ the continuous analytic capacity of E.

Theorem 2 (Vitushkin). *For an arbitrary compact set $K \subset \mathbb{C}$, the following are equivalent:*
(i) K is a set of rational approximation;
(ii) For each $z \in \partial K$, there exists $r \geq 1$, such that

$$\limsup_{\delta \searrow 0} \frac{\alpha(D(z;\delta) \setminus K^o)}{\alpha(D(z;r\delta) \setminus K)} < +\infty.$$

A fundamental problem for many years was to prove the semiadditivity of continuous analytic capacity. This was finally established by Xavier Tolsa [9].

Lemma 2. *There is an absolute constant C, such that for arbitrary Borel sets $E_i, i \geq 1$ in $\mathbb{C}$,*

$$\alpha\left(\bigcup_{i=1}^{\infty} E_i\right) \leq C \sum_{i=1}^{\infty} \alpha(E_i).$$

As an application of Lemma 1, we have the following.

Lemma 2. *If K_1 and K_2 are sets of rational approximation, then $K_1 \cap K_2$ is a set of rational approximation.*

Proof. We may assume that $K \subset \mathbb{C}$. Fix $z \in \partial(K_1 \cup K_2)$ and let r_1 and r_2 be associated to K_1 and K_2 respectively according to Theorem 2. Put $r = \min r_1, r_2$ and let C be the constant from Lemma 1. Then, by Theorem 2 and Lemma 1,

$$\limsup_{\delta \searrow 0} \frac{\alpha(D(z;\delta) \setminus (K_1 \cap K_2)^o)}{\alpha(D(z;r\delta) \setminus (K_1 \cap K_2)} \leq$$

$$\leq \limsup_{\delta \searrow 0} C \frac{\alpha(D(z;\delta) \setminus K_1^o)}{\alpha(D(z;r\delta) \setminus K_1)} + \limsup_{\delta \searrow 0} C \frac{\alpha(D(z;\delta) \setminus K_2^o)}{\alpha(D(z;r\delta) \setminus K_2)} < +\infty.$$

Thus, by Theorem 2, $K_1 \cap K_2$ is a set of rational approximation.

A compact set $K \subset \overline{\mathbb{C}}$ is *analytically negligible* if every continuous function on $\overline{\mathbb{C}}$ which is holomorphic on an open set V can be approximated uniformly on $\overline{\mathbb{C}}$ by functions continuous on $\overline{\mathbb{C}}$ and holomorphic on $V \cup K$. The union of two sets of approximation need not be a set of approximation, but, in the positive direction, the following result of Alexander Munro Davie and Bernt Karsten Øksendal [10] is fundamental.

Lemma 3 (Davie–Øksendal). *If K_1 and K_2 are sets of rational approximation and $\partial K_1 \cap \partial K_2$ is analytically negligible, then $K_1 \cup K_2$ is also a set of rational approximation.*

Since a piecewise analytic curve is analytically negligible (see [8]) and a countable union of analytically negligible sets is analytically negligible, and since a closed subset of an analytically negligible set is again analytically negligible, we have the following.

Lemma 4. *If K_1 and K_2 are sets of rational approximation and the boundary of one of the two sets is piecewise analytic, then $K_1 \cup K_2$ is also a set of rational approximation.*

A more general problem than that of characterizing compact sets K, on which all plausibly approximable functions are approximable (that is, $R(K) = A(K)$), is that of characterizing those functions f on K which can be approximated by rational functions, for an *arbitrary* compact set K. Vitushkin also solved that problem, but we shall not make use of that result. Alice Roth proved the following Fusion Lemma (see [11]), which allows one to approximate two functions simultaneously.

Lemma 5 (Fusion Lemma). *Let K_1, K_2 be disjoint compacts sets of the Riemann sphere $\overline{\mathbb{C}}$. There is a constant $A > 0$ such that if $k \subset \overline{\mathbb{C}}$ is a compact set and r_1, r_2 are rational functions with $\max_{z\in k} |r_1(z) - r_2(z)| <$ $< \epsilon$, then there is a rational function r, such that*

$$\max_{z\in K_j \cup k} |r(z) - r_j(z)| < A\epsilon, \quad j = 1, 2.$$

Roth pointed out that a consequence of her Fusion Lemma is the beautiful localization theorem of Errett A. Bishop (see [11]).

Theorem 3 (Localization Theorem). *Let $K \subset \overline{\mathbb{C}}$ and $f : K \to \mathbb{C}$. Then, $f \in R(K)$ if and only if, for each $z \in K$, there is disc D_z containing z such that $f|(K \cap \overline{D_z}) \in R(K \cap \overline{D_z})$.*

From the Localization Theorem (also directly from the Vitushkin Theorem 2), we have the following.

Theorem 4. *If Q_1 and Q_2 are compact sets of rational approximation having the form $Q_j = K_j \cup k, j = 1, 2,$ with K_j compact and disjoint, then $Q = Q_1 \cup Q_2$ is a set of rational approximation.*

Proof. Choose $z \in Q$. There are three cases: $z \in K_1, z \in k \backslash K_1, z \in K_2$.

Suppose $z \in K_1$. There is a disc D_z containing z, such that $\overline{D_z} \cap K_2 = \emptyset$. Thus,

$$Q \cap \overline{D_z} = (K_1 \cup k \cup K_2) \cap \overline{D_z} = (K_1 \cup k) \cap \overline{D_z} = Q_1 \cap \overline{D_z} .$$

Similarly, if $z \in K_2$, there is a disc D_z containing z, such that

$$Q \cap \overline{D_z} = Q_2 \cap \overline{D_z}.$$

If $z \in k \setminus K_1$, then there is a disc D_z containing z, such that $\overline{D_z} \cap K_1 = \emptyset$, so

$$Q \cap \overline{D_z} = (K_1 \cup k \cup K_2) \cap \overline{D_z} = (k \cup K_2) \cap \overline{D_z} = Q_2 \cap \overline{D_z} .$$

Thus, for each $z \in Q$, there is a disc D_z containing z , such that, for $j = 1$ or $j = 2$, $Q \cap \overline{D_z} = Q_j \cap \overline{D_z}$. The theorem then follows from the Localization Theorem and also from the Vitushkin Theorem 2.

3. Exponentials and logarithms in $\mathbb{C}$. The object of this paper is to approximate a function f on a compact set K by polynomials zero-free on K. It is natural to assume that K is a set of polynomial approximation, and in this case, we have noted that it is sufficient to approximate f by functions in $A(K)$ which are zero-free on K, for the latter functions can be approximated by polynomials which, for sufficiently good approximations, are also zero-free. If $g \in A(K)$, then $e^g \in A(K)$ and e^g is zero-free, so it is sufficient to approximate by exponentials e^g. In this section we gather some basic facts regarding exponentials and logarithms on subsets of $\mathbb{C}$.

Lemma 6. *Suppose E is connected, $f \in C(E)$ is zero-free and $\log_1 f$ and $\log_2 f$ are two branches of $\log f$. Then, $\log_1 f$ and $\log_2 f$ differ by an integral multiple of $2\pi i$.*

Proof. We note that $\log_1 f - \log_2 f$ is a branch of $\log 1$ on E. It is sufficient to show that every branch of $\log 1$ on E is of the form $n2\pi i$ for some $n \in \mathbb{Z}$. Let $\log 1$ be a branch. Then, for each $z \in E$, $\log 1(z) = n_z 2\pi i$, for some $n_z \in \mathbb{Z}$. It is sufficient to show that n_z takes the same value for all $z \in E$. If E is a singleton, this is trivial. If E is not a singleton, then no point of E is isolated, since E is connected. Since n_z is continuous, it must be locally constant. Thus, for each n the set $E_n = \{z \in E : n_z = n\}$

is clopen in E. Since E is connected, each E_n is empty or all of E. Since E is not empty, it is equal to E_n for some n. This completes the proof.

For an open set $U \subset \mathbb{C}$, denote by $\mathcal{O}(U)$ the family of holomorphic functions in U. Let us say that an open (not necessarily connected) set is simply connected if every component is simply connected.

Lemma 7. *Suppose U is a simply-connected open set and $f \in C(U)$ is zero-free. Then, f has a logarithm. Equivalently, $f \in \exp C(U)$. If, moreover $f \in \mathcal{O}(U)$, then $f \in \exp \mathcal{O}(U)$.*

Proof. This follows immediately from the universal covering.

Lemma 8. *If $f \in A(K) \cap \exp C(K)$, then $f \in \exp A(K)$.*

Proof. Suppose $f \in A(K)$ and $f = e^g, g \in C(K)$. Then, there is a branch of $\log f$ such that $g = \log f$. Thus, $g \in A(K)$ which completes the proof.

4. Reduction to exponential approximation. The set of zero-free elements in the algebra $C(K)$ is precisely the group $C(K)^{-1}$ of invertible elements of the algebra $C(K)$. For a general unital algebra A, we denote by A^{-1} the group of invertible elements of A. Our problem is a particular case of trying to approximate an element of a unital Banach algebra A by elements of the group A^{-1}. In a unital Banach algebra A, we can define the class $e^A = \exp A$ (see [12]). If we wish to approximate by elements of A^{-1}, it is sufficient to approximate by elements of e^A, since the latter is a subgroup of the former.

Lemma 9 (see [12]). *Let A be a unital Banach algebra. Then e^A is the component of the identity in A^{-1} and consequently e^A is closed in A^{-1}.*

Setting $A = C(K)$, we see that $\exp C(K)$ is the component of the identity element of the multiplicative topological group $C(K)^{-1}$, that is, the component of the function 1. More information concerning the relation between e^A and A^{-1} is given by the following.

Theorem 5 (Arens–Royden [13]). *Let $\mathcal{M}$ be the maximal ideal space of a commutative unital Banach algebra A. Then*

$$A^{-1}/e^A \cong H^1(\mathcal{M}, \mathbb{Z}).$$

In this theorem, the symbol $\cong$ denotes group homomorphism. For a compact set $K \subset \mathbb{C}$ the maximal ideal space of the algebra $C(K)$ is K itself so we have the following.

Corollary 1. *For a compact set K,*

$$C(K)^{-1}/\exp C(K) \cong H^1(K, \mathbb{Z}).$$

Hence the Arens–Royden gives a necessary and sufficient condition on a compact set K in order for every continuous zero-free function to have a logarithm. Namely, this is the case if and only if $H^1(K, \mathbb{Z}) = \{0\}$.

Theorem 6. $C(K)^{-1} = \exp C(K)$ *if and only if* $\mathbb{C} \setminus K$ *is connected.*

Proof. This is an immediate consequence of the previous corollary, but let us give a simple proof not explicitly involving cohomology. Suppose $\mathbb{C}\setminus K$ is connected and f is a continuous zero-free function on K. Firstly, we extend f to a continuous function $\widetilde{f}$ on all of $\mathbb{C}$. By considering a regular exhaustion of $\overline{\mathbb{C}} \setminus K$, we may surround K by finitely many disjoint Jordan curves $J_1, J_2, \cdots, J_n$ such that $\widetilde{f}$ remains zero-free inside and on each J_j. Denote by $\overline{D}_1, \overline{D}_2, \cdots, \overline{D}_n$ the corresponding disjoint Jordan domains. By the monodromy theorem, we may define a branch of $\log \widetilde{f}$ inside each D_j and clearly it extends to $\overline{D}_j$. The restriction of $\log \widetilde{f}$ to K is a branch of $\log f$ defined on K.

Suppose, conversely, that $\mathbb{C} \setminus K$ is not connected and let z_o be in a bounded complementary component. Then, $z - z_o$ is continuous and zero-free on K, but it is impossible to define a branch of $\log(z - z_o)$ on K. This completes the proof.

Let $P(K)$ denote the closure in $C(K)$ of the polynomials. The maximal ideal space of $P(K)$ is the polynomial hull $\widehat{K}$ of K. Consequently, we have the following corollary of the Arens–Royden Theoem.

Corollary 2. *For a compact set K,*

$$P(K)^{-1}/\exp P(K) \cong H^1(\widehat{K}, \mathbb{Z}).$$

Since $\widehat{K}$ is the union of K with its bounded complementary components, it follows from Mergelyan's theorem that $P(K) = A(K)$ if and only if $\mathbb{C} \setminus K$ is connected and so we deduce the following.

Corollary 3. *For a compact set K, we have* $A(K)^{-1} = \exp A(K)$ *if and only if* $\mathbb{C} \setminus K$ *is connected.*

The following theorem summarizes much of the previous discussion (see [14]).

Theorem 7. *For a compact set K, the following are equivalent:*

$\mathbb{C} \setminus K$ *is connected;*
$H^1(K, \mathbb{Z}) = \{0\}$;
$C(K)^{-1} = \exp C(K)$;
$A(K)^{-1} = \exp A(K)$;
$A(K) = P(K)$.

Recall that our main object is approximation by polynomials zero-free on K, which is equivalent to approximation by functions in $A(K)^{-1}$. Let us say that a compact set K is a set of *exponential approximation* if $\exp A(K)$ is dense $A_o(K)$. We have now seen that, if $\mathbb{C} \setminus K$ is connected, then approximation by functions in $A(K)^{-1}$ is also equivalent to approximation by functions in $\exp A(K)$. We emphasize this as follows.

Theorem 8. *If $\mathbb{C} \setminus K$ is not connected, then K is not a set of zero-free polynomial approximation. If $\mathbb{C} \setminus K$ is connected, then K is a set of zero-free polynomial approximation if and only if K is a set of exponential approximation.*

Proof. The second part follows directly from the previous corollary and Mergelyan's Theorem. Suppose $\mathbb{C}\setminus K$ is not connected and choose a point z_o in some bounded complementary component. The usual proof of Runge's polynomial approximation theorem [11] shows that $f(z) = (z - z_o)^{-1}$ cannot be approximated on K by polynomials, in particular by polynomials zero-free on K. Since f is zero-free on K, the proof is complete.

5. Proof of Theorem 1. Having reduced the problem of zero-free polynomial approximation to that of exponential approximation, we shall show how exponential approximation yields a proof of our main theorem. We need the inequalities in the following two simple lemmas.

Lemma 10. *For $w_1 = \rho_1 e^{i\varphi_1}$ and $w_2 = \rho_2 e^{\varphi_2}$, with $|\varphi_1 - \varphi_2| \leq \pi/4$,*

$$|\varphi_1 - \varphi_2| \leq \frac{\pi}{2\min\{\rho_1, \rho_2\}} |w_1 - w_2| \,.$$

Proof. We may assume $0 < \rho_1 \leq \rho_2$ and $\varphi_1 = 0$. Then, $|w_2 - w_1| \geq$ $\geq |\rho_1 e^{i\varphi_2} - \rho_1| \geq \Im \rho_1 e^{i\varphi_2} = \rho_1 \sin\varphi_2 \geq \rho_1(2/\pi)\varphi_2$. Forgetting our assumption that $\rho_1 \leq \rho_2$ and $\varphi_1 = 0$, this yields $|w_1 - w_2| \geq$ $\geq \min\{\rho_1, \rho_2\}(2/\pi)|\varphi_1 - \varphi_2|$.

Lemma 11. *For $|w_1|, |w_2| \geq m > 0$ and $|w_1 - w_2| < \delta < m$, we may* *choose* $\arg w_1, \arg w_2$ *so that*

$$|\arg w_1 - \arg w_2| \leq \arcsin \frac{\delta}{m} < \frac{\pi}{2} \frac{\delta}{m}.$$

Proof. We may assume that $0 = \arg w_1 \leq \arg w_2$, so we only need to estimate $\arg w_2$. The point w_2 lies in the band $0 \leq \Im w < \delta$ and outside the circle $|w| < m$. Thus, w_2 lies in the first quarter plane and we only need to estimate the value of $\arg w_2$ lying between 0 and $\pi/2$. The maximum is attained at the intersection of $\Im w = \delta$ with $|w| = m$. Thus,

$$0 \leq \arg w_2 \leq \arcsin \frac{\delta}{m} < \frac{\pi}{2} \frac{\delta}{m}.$$

It is easy to see (by considering a circle) that the possibility of approximating a function by exponentials on a set is not a local property. However, following lemma allows us to approximate simultaneously two exponentials given on overlapping sets by a single exponential on the union of the two sets.

Lemma 12 (Exponential Fusion). *Let K_1, K_2 be disjoint compacts sets of the Riemann sphere $\overline{\mathbb{C}}$. There is a constant $A > 0$ such that if $k \subset \overline{\mathbb{C}}$ is a compact set and, for $j = 1, 2$, r_j are rational functions with $|r_j| \leq M$ on $K_j \cup k$ and $\max_{z \in k} |r_1(z) - r_2(z)| < \epsilon$, where $A\epsilon \leq 1$, then, there exists a rational function r such that*

$$\max_{z \in K_j \cup k} |e^{r(z)} - e^{r_j(z)}| < e^M \cdot A\epsilon \cdot e, \quad j = 1, 2.$$

Proof. By the Fusion Lemma, there is a rational function r such that

$$\max_{z \in K_j \cup k} |r(z) - r_j(z)| < A\epsilon, \quad j = 1, 2.$$

Thus, for $z \in K_j \cup k$,

$$|e^{r(z)} - e^{r_j(z)}| = |e^{r_j(z)}| \cdot |e^{r(z) - r_j(z)} - 1| \leq$$

$$e^M \cdot \left| \sum_{n=1}^{\infty} \frac{(r(z) - r_j(z))^n}{n!} \right| \leq e^M \cdot |r(z) - r_j(z)| \cdot \sum_{n=1}^{\infty} \frac{1}{n!} \leq$$

$$\leq e^M \cdot A\epsilon \cdot e.$$

Lemma 13. *Let K_1, K_2, k be compact sets, with $K_1 \cap K_2 = \emptyset$ and k path connected. Suppose f is a function defined on $K = K_1 \cup k \cup K_2$ and for $j = 1, 2$, we have $f|(K_j \cup k) \in \overline{\exp A(K_j \cup k)}$. Suppose also that $f(z) \neq 0, z \in k$. Then, $f \in \overline{\exp A(K)}$.*

Proof. Let $\epsilon > 0$ and let A be the constant associated with the couple K_1, K_2 by the Fusion Lemma. Let f_j be the restriction of f to $K_j \cup k$. By hypothesis, for each $\delta >$ there are functions $g_j \in A(K_j \cup k)$, such that

$$\max_{z \in K_j \cup k} |f_j(z) - e^{g_j(z)}| < \delta, \quad j = 1, 2.$$

Thus,

$$\max_{z \in k} |e^{g_1(z)} - e^{g_2(z)}| < 2\delta. \tag{1}$$

Set $\max_{z \in k} |f(z)| = M$ and $\min_{z \in k} |f(z)| = m > 0$. For $\delta < m/2$ sufficiently small, we may assume that

$$m/2 < |e^{g_j(z)}| < 2M, \quad z \in k, \quad j = 1, 2. \tag{2}$$

We wish to show that we can choose branches of $\arg e^{g_1}$ and $\arg e^{g_2}$ which are close to each other on k. Namely, we fix a branch of $\arg e^{g_1}$ and we shall choose an appropriate branch of $\arg e^{g_2}$. To this end, fix a point $a \in k$ and let $\sigma : [0, 1] \to E$ be a path in E, with initial point $\sigma(0) = a$. Let $T \subset [0, 1]$ be the set of $t \in [0, 1]$ such that there is a (continuous) branch of $\arg e^{g_2(\sigma(s))}$, for $0 \leq s \leq t$, such that $|\arg e^{g_1(\sigma(s))} - - \arg e^{g_2(\sigma(s))}| < \pi/8$. The set T is open by definition. By equation (1) and Lemma 11, we may assume that δ is so small that $a \in T$, so $T \neq \emptyset$. Also, if $|\arg e^{g_1(\sigma(s))} - \arg e^{g_2(\sigma(s))}| < \pi/8$, and δ is small, then by Lemma 11, $|\arg e^{g_1(\sigma(s))} - \arg e^{g_2(\sigma(s))}|$ is actually much smaller than $\pi/8$, say $\leq \pi/9$. Thus, T is closed. By connectivity, it follows that $T = [0, 1]$. We have shown that, along an arbitrary path σ in E, with initial point a, there is a branch of $\arg e^{g_2(\sigma(s))}$, with $|\arg e^{g_1(\sigma(s))} - \arg e^{g_2(\sigma(s))}| < \pi/8$. If $\sigma(s) = \sigma(t)$, for some $s < t$, then, we claim that $\arg e^{g_2(\sigma(s))} = \arg^{g_2(\sigma(t))}$. Indeed, a choice of $\arg e^{g_2}$ at each point of σ corresponds to a choice of $n_z \in \mathbb{Z}$ at each point of σ, with $\arg e^{g_2} = \Im g_2 + i n_z 2\pi$. Since, we have made a continuous choice of $\arg e^{g_2}$ along σ, we may choose n_z constant along σ. Thus, if $\sigma(s) = \sigma(t)$, we have

$$\arg e^{g_2(\sigma(s))} = \Im g_2(\sigma(s)) + n 2\pi i = \Im g_2(\sigma(t)) + n 2\pi i = \arg e^{g_2(\sigma(t))}.$$

For every such path, the choice of n_a is unique. Thus, we have the same n, for all paths in E starting from a. Set $h_1 = g_1$ and $h_2 = g_2 + n2\pi i$. We have shown that $|\arg e^{h_1(z)} - \arg e^{h_2(z)}| < \pi/8$, for $z \in E$. From (1), (2) and Lemma 10, we have

$$|\arg e^{h_1(z)} - \arg e^{h_2(z)}| < \frac{2}{m/2} 2\delta, \quad z \in E\,.$$

Thus, for sufficiently small δ,

$$|\Im h_1(z) - \Im h_2(z)| = |\arg e^{h_1(z)} - \arg e^{h_2(z)}| < \epsilon/2, \quad z \in E. \tag{3}$$

Since $\log t$ is uniformly continuous on $[m/2, 2M]$, for all sufficiently small $\delta > 0$, we have that, if $|t_1 - t_2| < 2\delta$, with $t_1, t_2 \in [m/2, 2M]$, then $|\log t_1 - \log t_2| < \epsilon/2$. Set $t_1 = e^{\Re h_1(z)}$ and $t_2 = e^{\Re h_2(z)}$. Then, by (1) we have $t_j \in [m/2, 2M]$ and

$$|t_1 - t_2| = ||e^{h_1(z)}| - |e^{h_2(z)}|| \leq |e^{h_1(z)} - e^{h_2(z)}| < 2\delta,$$

and so

$$|\Re h_1(z) - \Re h_2(z)| = |\log t_1 - \log t_2| < \epsilon/2, \quad z \in k. \tag{4}$$

Combining (3) and (4), we have

$$|h_1(z) - h_2(z)| < \epsilon\,, \quad z \in k\,.$$

Since $K_j \cup k$ are sets of rational approximation, there are rational functions r_j such that

$$\max_{z \in K_j \cup k} |h_j(z) - r_j(z)| < \epsilon, \quad j = 1, 2.$$

By the Fusion Lemma 5, there is a rational function r such that

$$\max_{z \in K_j \cup k} |r(z) - r_j(z)| < A\epsilon, \quad j = 1, 2.$$

Thus,

$$\max_{z \in K_j \cup k} |r(z) - h_j(z)| < (1 + A)\epsilon, \quad j = 1, 2.$$

Since the h_1 and h_2 are bounded, r has no poles on K and, in particular, $r \in A(K)$. By choosing ϵ sufficiently small, we may assume $(1 + A)\epsilon < 1$. Then, for $z \in K_j \cup k$, noting that $e^{g_j} = e^{h_j}$, we have

$$|f_j(z)-e^{r(z)}| \le |f_j(z)-e^{h_j(z)}|+|e^{h_j(z)}-e^{r(z)}| \le$$

$$\le \delta + |e^{h_j(z)}|\,|e^{r(z)-h_j(z)}-1| \le \frac{m}{4\pi}\epsilon + 2M\left|\sum_{n=1}^{\infty}\frac{(r(z)-h_j(z))^n}{n!}\right| \le$$

$$\le \frac{m}{4\pi}\,\epsilon + 2M\,|r(z)-h_j(z)|\sum_{n=1}^{\infty}\frac{1}{n!} \le \frac{m}{4\pi}\epsilon + 2M\,(1+A)\epsilon\, e\,.$$

Thus,

$$\max_{z\in K}|f(z)-e^{r(z)}| < \Big(\frac{m}{4\pi}+2M(1+A)e\Big)\,\epsilon\,.$$

Since ϵ can be taken arbitrarily small, we have shown that $f \in \overline{\exp A(K)}$. This completes the proof of the lemma.

Proof of Theorem 1. If $f \in A_o(Q_1 \cup Q_2)$, then, $f \in A_o(Q_j), j = 1,2$. Since each Q_j is a set of zero-free polynomial approximation, it is a set of polynomial approximation and hence $\mathbb{C} \setminus Q_j$ is connected. Consequently, $A(Q_j)^{-1} = \exp A(Q_j)$. Since f can be approximated by polynomials zero-free on Q_j and such polynomials are in $A(Q_j)^{-1}$, they are in $\exp A(Q_j)$. Thus $f|Q_j \in \overline{\exp A(Q_j)}, j = 1,2$. From Lemma 13, setting $k = Q_1 \cap Q_2$ and $K_j = Q_j \setminus (Q_1 \cap Q_2), j = 1,2$, we have that, if f is zero-free on $Q_1 \cap Q_2$, then $f \in \overline{\exp(Q_1 \cup Q_2)}$. Since the union of two compact sets having connected complements is again of connected complement, provided their intersection is connected (see [2]), $\mathbb{C} \setminus (Q_1 \cup Q_2)$ is connected and so $\exp A(Q_1 \cup Q_2) = A(Q_1 \cup Q_2)^{-1}$. Thus, f is the uniform limit of functions in $A(Q_1 \cup Q_2)^{-1}$, but by Mergelyan's theorem, such functions are the uniform limit of polynomials, which, for sufficiently good approximations are zero-free on $Q_1 \cup Q_2$. This completes the proof of Theorem 1.

6. A theorem of Khrushchev. Recall that $A_o(K)$ denotes the functions in $A(K)$ which have no isolated zeros in the interior. Clearly, $\overline{A(K)^{-1}} \subset A_o(K)$ and our problem is whether they are equal. Moreover, we have seen that $A(K)^{-1} = \exp A(K)$ if and only if $\mathbb{C} \setminus K$ is connected. A recent result of Khrushchev [5] gives a sufficient condition for approximation by functions in $\exp A(K)$. In this section, we present Khrushchev's theorem, from which several results published earlier on zero-free polynomial approximation follow.

Lemma 14. *Suppose a set E is path connected. If $f \in C(E)$ is bounded away from zero, then $f \in \overline{\exp C(E)}$ if and only if f has a logarithm.*

We remark that, we have not assumed that E is closed or open and if E is compact, then the conclusion holds whether or not E is path connected, because, then $\exp C(E)$ is the component of the identity in $C(E)^{-1}$ and is therefore closed in $C(E)^{-1}$.

Proof. If f has a logarithm, then $f \in \exp C(E)$ and so trivially $f \in \overline{\exp C(E)}$.

Conversely, suppose $f \in \overline{\exp C(E)}$. Then, for every $\epsilon > 0$, there is a $g \in C(E)$ such that $\sup_{z\in E} |f(z)-e^{g(z)}| < \epsilon$. Since f is bounded away from zero, setting $m = \inf_{z\in E} |f(z)|$, we may assume that $|e^{g(z)}| > m/2 > 0$, for $z \in E$. Thus, both $f(z)$ and $e^{g(z)}$ lie outside $D(0, m/2)$ and $f(z) \in \in D(e^{g(z)}, \epsilon)$. For ϵ sufficiently small, the disc $D(e^{g(z)}, \epsilon)$ lies in a sector of opening less than $\pi/4$, for every $z \in E$. Thus, for every $z \in E$, we may choose $\arg f(z)$ and $\arg e^{g(z)}$ so that $|\arg f(z) - \arg e^{g(z)}| < \pi/4$.

Fix a point $a \in E$ and set

$$w_1 = f(a) = \rho_1 e^{i\varphi_1}, \quad w_2 = e^{g(a)} = \rho_2 e^{i\varphi_2},$$

where $\rho_2 = e^{\Re g(a)}$ and φ_2 is a value of $\arg e^{g(z)}$ chosen so $|\varphi_1 - \varphi_2| < \pi/4$. Let $g_a = g + n_a 2\pi i$, where $n_a \in \mathbb{Z}$ is chosen so that $\Im g_a(a) = \varphi_2$. Since $e^g = e^{g_a}$, we replace g by g_a and g_a has all of the properties, which we have established for g.

Let γ be a path in E beginning at a. As z traverses γ, the point $f(z)$ remains in the moving disc $D(e^{g_a(z)}, \epsilon)$. We may cover γ with a chain $\{D_j\}$ of such discs and in each disc choose $\arg f(z)$ so that the choices are compatible for each pair D_j, D_{j+1}. Suppose, to obtain a contradiction that for some $j < k$, the discs D_j and D_k intersect and the choices $\arg f$, which we denote by $\arg_j f$ and $\arg_k f$ do not agree on $D_j \cap D_k$. Then $|\arg_j f - \arg_k f| \geq 2\pi$. However, $|\arg_j f - \arg_k f| < |\arg_j - \arg e^{g_a}| + +|\arg e^{g_a} - \arg_k f| < \pi/4 + \pi/4$. This contradiction shows that we may define $\arg f$ continuously along each path in E, starting from a, even if the path returns to a previous point. Thus, $\arg f(z)$ has a branch on E, which is equivalent to the assertion that $\log f(z)$ has a branch on E. This completes the proof of the lemma.

The notion of local connectivity is useful to study the logarithm. The proof of the following lemma follows the presentation in [11, p. 138].

Lemma 15. *Let U be an open subset of $\overline{\mathbb{C}}$ and $z_o \in \partial U$. Then $U \cup \{z_o\}$ is locally connected at z_o if and only if it is locally path connected at z_o.*

Proof. Notice that $U \cup \{z_o\}$ is locally path connected at z_o if and only if the following holds: for every neighborhood W of z_o, there exists a neighborhood V of z_o with the property that each point $z \in U \cap V$ can be connected to z_o by an arc γ_z which is in $\subset U \cap W$, except for its end point z_o .

a) Suppose $U \cup \{z_o\}$ is path connected and W is a neighborhood of z_o. We use the neighborhood of z_o from the previous paragraph and write

$$Z = \cup\{\gamma_z : z \in U \cap V\} \cup \{z_o\};$$

hence Z is connected and $Z \subset (U \cup \{z_o\}) \cap W$. Further, $Z \supset (U \cup \{z_o\}) \cap V$ and the latter is a neighborhood of z_o in $U \cup \{z_o\}$.

b) Now suppose $U \cup \{z_o\}$ is locally connected at z_o and W is a neighborhood of z_o. It is sufficient to show the following.

Claim: There exists a neighborhood $V \subset W$ of z_o such that each point z in $U \cap V$ can be connected in $U \cap W$ with a point that is arbitrarily close to z_o.

For then we construct $\{W_n\}$ such that $W_{n+1} \subset W_n$ and $\cap W_n = \{z_o\}$ and the corresponding V_n such that $V_{n+1} \subset V_n$, and in the obvious way we connect countably many arcs to constitute γ_z, which lies in $U \cap W$ and connects z in U with z_o. In order to show the claim, choose a point $z \in U \cap V$ and let Z be the component of $U \cap W$ containing z. Suppose Z does not contain points arbitrarily close to z_o. Then Z is open and closed in $(U \cup \{z_o\}) \cap W$. Hence, it is also open and closed in $(U \cup \{z_o\}) \cap V$. But since Z does not contain z_o, it is not all of $(U \cup \{z_o\}) \cap V$. This contradicts the connectivity of $(U \cup \{z_o\}) \cap V$. Hence Z contains points arbitrarily close to z_o. Since Z is open and connected, it is path-wise connected. Thus z can be path connected in $U \cap W$ to points arbitrarily close to z_o, which establishes the claim and completes the proof.

Suppose U is a simply-connected domain in $\mathbb{C}$, $f \in C(\overline{U})$ is zero-free on U and $\log f$ is a branch of the logarithm of f on U, $z_o \in \partial U$, $f(z_o) \neq 0$, $U \cup \{z_o\}$ is locally connected at z_o. Then, $\log f$ need not extend continuously to z_o. For example, consider $U = \mathbb{C} \setminus [0, +\infty]$, $f(z) = z$ and $z_o = 1$. Then, we may define $\log z$ in U but it does not have a continuous extension to the point 1.

It does not help to suppose that U is the interior of a compact set K, for we may consider $K = \overline{D}(0,1) \setminus \overline{D}(1/2, 1/2)$, $f(z) = z$ and $z_o = 1$. Again, we may define $\log z$ in U, but we cannot extend it to the boundary point 1.

To remedy this difficulty, Khrushchev [5] introduced the notion of logarithmic continuity. Let us say that $f \in C(E)$ is logarithmically continuous if there is a (continuous) branch of $\log f$ on $E \setminus f^{-1}(0)$. If f is logarithmically continuous on E, we may extend $\log f$ to a continuous function $\log f : E \to \overline{\mathbb{C}}$, by setting $\log f(z) = \infty$ for $z \in f^{-1}(0)$. This definition of logarithmic continuity appears more general than that of Khrushchev, but it is equivalent. With the help of logarithmic continuity, Khrushchev has obtained the following fundamental result on zero-free approximation.

Theorem 9 (Khushchev [5]). *For every logarithmically continuous $f \in A(K)$ and $\epsilon > 0$, there is a function $g \in e^{A(K)}$, and hence invertible in $A(K)$, such that $|f(z) - g(z)| < \epsilon, z \in K$.*

Khrushchev claimed that logarithmically continuous functions cannot vanish on K^o. This is not correct. A correct statement is that logarithmically continuous functions in $A(K)$ can have no *isolated* zeros in K^o. That is, they must lie in $A_o(K)$.

Khrushchev's theorem gives condition under which we may approximate an individual function on a given set. We may apply it to determine classes of functions on classes of sets, for which zero-free polynomial approximation is possible.

The following is a slight generalization of Theorem 3.1 in [5].

Theorem 10. *Let K be a locally connected compact set with connected complement. Then, every function $f \in A_o(K)$ is logarithmically continuous.*

Khrushchev proved this result under the stronger assumption that f has no zeros on K^o, but his proof also yields the present formulation. As a consequence we obtain the following slight generalization of the important Corollary 3.3 in [5].

Corollary 4. *Let K be a locally connected compact set with connected complement. Then every function $f \in A_o(K)$ can be arbitrarily closely approximated by polynomials zero-free on K.*

7. Examples and open problems. Khushchev showed that if f is logarithmically continuous and K satisfies certain conditions, then $f \in \overline{\exp A(K)}$. The converse is false. Khushchev gave an example of a compact set K with connected complement and a function $f \in A(K)$ which is zero-free on K^o, such that f is in $\overline{A(K)^{-1}}$ but f is not logarithmically continuous. For compact sets with connected complement, $A(K)^{-1} = \exp A(K)$, so $f \in \overline{\exp A(K)}$, though f is not logarithmically continuous.

In the example given by Khushchev, every boundary point belongs to only one impression. Thus, the fact that every boundary point of a compact domain belongs to only one impression does not guarantee that every $f \in A_o(K)$ is logarithmically continuous.

Suppose K satisfies the hypotheses of Corollary 4 and moreover $K = \overline{K^o}$ with K^o connected. Then, of course, K^o is simply connected. Since $K = \overline{K^o}$, it follows that $\partial K = \partial \mathbb{C} \setminus K$ and from Th. 2.1 [15] we see that K is a closed Jordan domain.

Corollary 4 (also Khrushchev's version) yields many examples where K^o is not connected. In particular, it yields the recent results for zero-free approximation on Jordan domains [1] and on *chains, forests and bouquets* of Jordan domains [4, 2].

One can mimic the proof for Jordan domains to show the following.

Theorem 11. *Let D be a C^∞-smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n > 1$, which is biholomorphic to the unit ball and f a continuous function on $\overline{D}$, which is holomorphic and zero-free on D. Then f can be uniformly approximated by zero-free functions in $\mathcal{O}(\overline{D})$.*

For the proof, we need two theorem. The first is due independently to Henkin, Kerzman and Lieb (see [16]).

Theorem 12 (Henkin–Kerzman–Lieb). *Suppose, for $n > 1$, that $\overline{D} \subset \mathbb{C}^n$ is the closure of a strictly pseudoconvex domain D with C^2-boundary. Then, every function continuous on $\overline{D}$ and holomorphic on D can be approximated uniformly by functions in $\mathcal{O}(\overline{D})$.*

Theorem 13 (Vormoor [17]). *If $\Phi : D_1 \to D_2$ is a biholomorphic mapping between two strictly pseudoconvex domains in $\mathbb{C}^n$, $n > 1$, with C^∞-boundaries, then Φ extends to a homeomorphism $\Phi : \overline{D}_1 \to \overline{D}_2$.*

Proof of Theorem 11. Let $\Phi : D \to B$ be a biholomorphic mapping of D onto the unit ball B. For $0 < r < 1$, set $f_r(z) = f(\Phi^{-1}(r\Phi(z)))$. Then, f_r is continuous on $\overline{D}$ by Vormoor's Theorem and clearly f_r is holomorphic on D. By the Henkin–Kerzman–Lieb Theorem, f_r can be uniformly approximated by functions in $\mathcal{O}(\overline{D})$. Sufficiently good such approximations are zero-free on $\overline{D}$, since f_r is zero-free. Moreover, f_r converges uniformly to f, as $r \to 1$. This completes the proof of the theorem.

Another theorem in one variable which extends to several variables is the following.

Theorem 14. *Suppose K is strictly starlike with respect to some point z_o. That is, for each $z \in K$, the segment*

$$[z_o, z) = \{z_o + r(z - z_o) : 0 \leq r < 1\}$$

is contained in K^o, *then each continuous function* f *on* K, *which is holomorphic and zero-free on* K^o, *can be uniformly approximated by zero-free function in* $\mathcal{O}(K)$.

The very simple proof is the same as in one variable. Namely, the functions $f_r(z) = f(z_o + r(z - z_o))$ may serve as the desired approximations.

Problem 1. *What more can be said regarding zero-free approximation in several variables?*

Corollary 4 also allows approximation on certain compacta whose interiors have infinitely many components.

For a sequence $E_1, E_2, \cdots$ of sets, $\limsup_{j\to\infty} E_j$ denotes the set of all z such that every neighborhood of z meets K_j, for infinitely many j.

Theorem 15. *If* $Q, Q_1, Q_2, \cdots$ *are disjoint compact sets of zero-free approximation and*

$$\limsup_{j\to\infty} K_j \subset Q,$$

then

$$K = \bigcup_{j=1}^{\infty} Q_j \cup Q$$

is a set of zero-free rational approximation.

Proof. Suppose $f \in A_o(K)$ and $\epsilon > 0$. Let g_∞ be a rational function which is zero-free on Q, such that $|g_\infty - f|Q| < \epsilon/2$. Then, there is a neighborhood U of Q such that $|g_\infty - f| < \epsilon/2$, for $z \in K \cap U$. For some n, we have $Q_j \subset U$, for each $j > n$. For each $j = 1, 2, \cdots, n$ let g_j be a rational function, which is zero-free on Q_j, such that $|g_j - f|Q_j| < \epsilon/2$. Define g on K by setting $g = g_j$ on $Q_j, j = 1, 2, \cdots, n$ and $g = g_\infty$ on $K \cap U$. Then g is holomorphic and zero-free on K and $|g - f| \leq \epsilon/2$. By Runge's theorem we may approximate g uniformly on K by rational functions and sufficiently good such approximations will also be zero-free on K. Thus, K is a set of zero-free rational approximation.

Definition (Schlangengebiet). Let G be a simply connected bounded domain with one non-degenerate prime end P, all other prime ends being simple. Moreover, let 0 be in G. Then, Gaier [7] calls G a Schlangengebiet, if for every $z \in G$, $z \neq 0$, there is a cross-cut $q(z)$ through z, which separates z from P, such that the diameter of $q(z)$ tends to 0 as $z \to P$.

Definition (Snake domain). By a snake domain we mean a strip S which gets thinner as it approaches a non-degenerate continuum. Now, we give a more formal definition. Consider a conformal mapping $\varphi : \Delta \to S$ of the unit disc onto a bounded domain S. Let P be the prime-end of S corresponding to 1 and let $[P]$ be its impression. We suppose that φ extends to a homeomorphism $\varphi : \partial\Delta \setminus \{1\} \to \partial S \setminus [P]$. For $0 < \delta < 1$, consider the lens $L_\delta = \{z \in \Delta : |z-1| < \delta$, and consider the two circular boundary arcs

$$\alpha_\delta^+ = \{z \in \partial\Delta : 0 < |z-1| < \delta, \Im z > 0\},$$

$$\alpha_\delta^- = \{z \in \partial\Delta : 0 < |z-1| < \delta, \Im z < 0\}.$$

Denote by $d(E, F)$ the distance between two sets and suppose

$$\lim_{\delta\to 0} \max \left\{ d\left(\varphi(L_\delta), \varphi(\alpha_\delta^-)\right), d\left(\varphi(L_\delta), \varphi(\alpha_\delta^+)\right) \right\} = 0.$$

This complicated condition is just the condition that $S_\delta \equiv \varphi(L_\delta)$ is getting thinner as $\delta \to 0$. We call the the impression $[P]$ of the prime end P the *end* of the snake domain S. If the continuum $[P]$ is degenerate (a singleton), then ∂S is a Jordan curve. However, we assume that $[P]$ is a non-degenerate continuum and call such a domain S a *snake domain.* A familiar example of a snake domain, is a thin strip neighborhood of the curve $y = \sin 1/x, 0 < x \leq 1$. Another familiar example is a strip which approaches the unit circle from without (or within) while winding around the circle infinitely many times. The cornucopia is the union of Δ with such an outer snake domain.

Let us define a *compact snake* K to be the closure of a snake domain S. Thus, $K = \overline{S}$. Let P be the non-degenerate prime end of S, and $[P]$ its impression. Set $bG = \partial S \setminus [P]$. Then ∂K is the disjoint union of the continuum $[P]$ and the Jordan arc $bS = \varphi(\partial\Delta \setminus \{1\})$. $\partial K = bG \cup [P]$.

Problem 2. *Is every compact snake or Schlangengebiet a set of zero-free approximation*?

The following two lemmas provide some preliminary information regarding this situation.

Lemma 16. *Every compact snake is a set of rational approximation.*

Proof. It is sufficient [8] to show that K has no inner boundary. First of all, every point of the Jordan arc bS is on the outer boundary. We may construct a Jordan arc α in $\mathbb{C} \setminus K$ which approaches bS as bS approaches $[P]$. Thus, every point of $[P]$ is in the closure of the complementary component of K containing α. Consequently, every point of $[P]$ lies on the outer boundary. Hence, the inner boundary of K is empty so K is a set of rational approximation.

Lemma 17. *For each continuous function f on the unit circle $\partial\Delta$, and each outer spiral σ approaching $\partial\Delta$, there is a closed spiral domain S containing σ and an extension $F \in A(S)$ with $F = f$ on $\partial\Delta$. In particular, each $f \in A(\Delta)$ extends to a cornucopia $\Delta \cup S$.*

Proof. Extend f continuously to $\mathbb{C}$. Let F, holomorphic in $\mathbb{C} \setminus \overline{\Delta}$, be a carleman approximation of f on σ. If the approximation is sufficiently rapid, then F is uniformly continuous on σ and hence is a continuous extension of f to $\partial\Delta \cup \sigma$. We may construct a spiral neighborhood S of σ, so close to σ that F is continuous on $\overline{\Delta} \cup S$. Note that these extensions are highly non-unique.

Definition. By a *cornucopia*, we mean a compact set K consisting of the closed unit disc $\overline{\Delta}$ and a (closed) strip S spiraling in towards the unit circle.

Problem 3. *Is a cornucopia a set of zero-free approximation*?

The following lemma is obvious.

Lemma 18. *Let $K = \overline{\Delta} \cup S$ be a cornucopia. If $f \in A(S)$, $g \in A(S \setminus \overline{\Delta})$, and $f(z) - g(z) \to 0$, as $z \to \partial\Delta$, then g extends continuously to S and $g = f$ on $\partial\Delta$.*

For $f \in A(K)$, where K is a cornucopia, this lemma may help to 'remove' possible zeros on $\partial S \setminus \overline{\Delta}$, by replacing f on $S \setminus \overline{\Delta}$ by such an approximation g which is zero-free on $\partial S \setminus \overline{\Delta}$.

All of the above problems are related to the following general problem which I considered in writing [6] (but did not publish) and which was published by Andersson [1].

Problem 4. *Is every compact set of polynomial approximation a set of zero-free polynomial approximation*?

One could also consider the analogous problem on closed (rather than compact) sets. The following problem might be a very simple situation in which to begin.

Problem 5. *Let E be the closed right half-plane and suppose f is an entire function whose unique zero on E is at the origin. Can f be uniformly approximated on E by entire functions having no zeros on E?*

8. Topology. Let us say that K is a compact domain if K is compact, K^o is connected and K is the closure of K^o.

The following theorem shows that there is no topological obstruction to the zero-free approximation we are attempting in this paper.

Theorem 16. *Suppose $K \subset \mathbb{C}$ is compact, $f \in C(K)$ and $f^{-1}(0) \subset \subset \partial K$. Then, for each $\epsilon > 0$, there is a $g \in C(K)^{-1}$ such that*

$$|f(z) - g(z)| < \epsilon \quad \forall z \in K .$$

Proof. Since $E = f^{-1}(0)$ is compact and f is continuous, given $\epsilon > 0$, there are finitely many disjoint Jordan domains $D_j, j = 1, 2, \ldots, n$, whose union D contains E and is contained in K and such that $|f| < \epsilon$ on D. Since $\partial D \cap E = \emptyset$, the function f is bounded away from zero on $\partial D \cap K$. Let $m = \min\{|f(z)| : z \in \partial D \cap K$. Extend f continuously to ∂D, so that $m \leq |f| \leq \epsilon$ on ∂D. Since $E \subset D \cap \partial K$, we may choose points $z_j \in D_j \setminus K, j = 1, 2, \ldots, n$. For each j, let U_j be an open disc, with $z_j \in \overline{U_j} \subset D_j \setminus K$ and let A_j be the closed annular region $\overline{D_j} \setminus U_j$. We define a continuous function h_j on $\overline{A}_j$ by first setting $h_j = f$ on ∂D_j and then extending h_j continuously to A_j while maintaining the bounds $m \leq |h_j| \leq \epsilon$. Now, we define a zero-free function $g_\epsilon \in C(K)$, by setting $g_\epsilon = f$ on $K \setminus D$ and $g_\epsilon = h_j$ on each A_j. Then, $f = g$ on $K \setminus D$ and $|f - g| \leq \epsilon + m \leq 2\epsilon$ on $K \cap D$. Thus, $|f - g| \leq 2\epsilon$ on K. Since ϵ was an arbitrary positive number, the proof is complete.

If $f \in \overline{A(K)}$, then on each component of K^o, the function f is either identically zero or zero-free. Another way of saying this is that f has no isolated zeros on K^o. This necessary condition also has a topological analog. Suppose K is a compact domain and $f \in C(K)$. Let z_o be an isolated zero of f in K^o. For sufficiently small neighborhoods G of z_o, the degree $\mu(0, f, G)$ of the mapping $f : G \to \mathbb{C}$, with respect to 0 is invariant (see [18]). We denote this value by $\mu(0, f, z_o)$ and call it the local degree of f at z_o with respect to 0. Our topological necessary condition for zero-free approximation is the following.

Theorem 17. *Let K be a compact domain. Suppose a sequence of zero-free functions $g_n \in C(K)$ converges uniformly to a function $f \in C(K)$. Then, $\mu(0, f, z_o) = 0$ at each isolated zero z_o of f in K^o.*

Proof. Let D be a closed disc centered at z_o and contained in K, such that z_o is the only zero of f in D. Then, $\mu(0, f, D) = \mu(0, f, z_o)$. By the Topological Rouché Theorem [19], for n sufficiently large, $\mu(0, f, D) = \mu(0, g_n, D)$. Since g_n omits 0, we have $\mu(0, g_n, D) = 0$ and since $\mu(0, f, D) = \mu(0, f, z_o)$, the conclusion follows.

If f holomorphic and not identically zero on a domain has a zero at a point z_o, then the local (topological) degree of f at z_o is the usual degree of the zero of f at z_o, that is, the integer n such that, in a neighborhood of z_o, the function f has a representation $f(z) = (z - z_o)^n h(z)$, with h holomorphic and $h(z_o) \neq 0$. Thus, if f is holomorphic, the condition that the local degree $\mu(0, f, z_o) = 0$ is zero at each isolated zero z_o means that f has no isolated zeros By the uniqueness property, the topological condition of the theorem, in the holomorphic case, becomes the condition that, on each component of K^o, the function f is either zero-free or identically zero. This is the condition which we considered to be natural from the start.

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