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# The Borsuk-Ulam Theorem for Double Coverings of Seifert Manifolds

*We study a Borsuk-Ulam type theorem for pairs $(M, \tau)$ with $\tau$ a fixed point free involution of $M$, and such that both $M$ and $N\colon = M/\tau$ are Seifert manifolds. In this note our point of view will be to start with a Seifert manifold $N$. Any non-trivial element $\xi \in H^1(N; Z_2)$ then gives rise to a pair $(M_\xi, \tau_\xi) = (M, \tau)$ with $M$ (necessarily) also a Seifert manifold, and a double covering $p\colon M \twoheadrightarrow N$, with $\tau$ being the fixed point free involution on $M$ associated to this double covering as the non-trivial deck transformation. We then seek the largest value of $n$, called the $\mathbb{Z}_2$-index of $(M, \tau)$, such that the Borsuk-Ulam property holds for maps into $\mathbb{R}^n$, i.e. such that for every continuous map $f\colon M \to \mathbb{R}^n$, there is an $x \in M$ such that $f(x) = f(\tau(x))$. In case $M$ is a 3-manifold (such as a Seifert manifold), the $\mathbb{Z}_2$-index can take only the values 1, 2, 3.*

## 1 Introduction

The study of involutions on manifolds has been of great interest and importance within topology, as illustrated by the books of J. Matoušek [12] and S. L. de Medrano [11] (and in particular, for involutions on Seifert manifolds, cf. the book of Montesinos [13]). The most famous theorem in the subject is undoubtedly the classical Borsuk-Ulam theorem, which applies to the antipodal involution of a sphere. This theorem together with various generalizations and applications continues to be of great interest. For example, a generalization of the Borsuk-Ulam theorem that applies to a fixed point free involution on any manifold has recently been studied by Gonçalves, Hayat, and Zvengrowski [7]. The case of manifolds of dimension 2 and the corresponding Borsuk-Ulam theorem has also been recently studied by Gonçalves and Guaschi [6]. The above mentioned book of Matoušek gives an extensive set of references related to the Borsuk-Ulam theorem; in addition to these further interesting aspects and generalizations of the classical Borsuk- Ulam theorem appear (among others) in work by K. D. Joshi [10], J. Jaworowski [9], A. Dold [5], and more recently in work of P. L. Q. Pergher, D. de Mattos, E. L. dos Santos [16], P. L. Q. Pergher, H. K. Singh, T. B. Singh [17], as well as survey papers among which we mention H. Steinlein [22], and I. Nagasaki [14].

In this paper we attempt to initiate this study for the Seifert manifolds, a large and important class of 3-manifolds introduced by Seifert [19] in 1933. This is possible, using the aforementioned paper [7] and the knowledge of the $\mathbb{Z}_2$-cohomology rings of these manifolds, cf. [2], [3], [4] for the orientable case and more recently [1] for all Seifert manifolds. We will suppose throughout that all manifolds under consideration are closed and connected.

Given a (closed, connected) $m$-manifold $N$, any non-trivial element $\xi \in H^1(N;\mathbb{Z}_2)$ gives rise to an epimorphism $\phi\colon \pi_1 N \twoheadrightarrow \mathbb{Z}_2$ and a pair $(M_\xi, \tau_\xi) = (M, \tau)$, where $p\colon M \twoheadrightarrow N$ is a double covering, $M$ is a (closed, connected) $m$-manifold, and $\tau$ is the fixed point free involution on $M$ associated to this double covering as the non-trivial deck transformation. This correspondence is via the sequence of isomorphisms

$$hom(\pi_1(N), \mathbb{Z}_2) \approx hom((\pi_1(N))_{ab}, \mathbb{Z}_2) \approx hom(H_1(N), \mathbb{Z}_2) \approx H^1(N;\mathbb{Z}_2). \tag{1}$$

**Definition 1.1.** *(i) We say that the Borsuk-Ulam property BU(n) holds for $(M,\tau)$ if for every continuous map $f\colon M \to \mathbb{R}^n$, there is an $x \in M$ such that $f(x) = f(\tau(x))$.*

*(ii) The $\mathbb{Z}_2$-index $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)$ is then defined as the largest $n \le \infty$ such that $BU(n)$ holds.*

>From [7] it is known that $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \ge 1$ always holds, and $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ if and only if $\xi \in \mathrm{Im}(\rho\colon H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2)$, where $\rho$ is the coefficient homomorphism induced by the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$. Furthermore, it is shown there that $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \le m = \dim(M)$ and $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = m$ if and only if $\xi^m \ne 0 \in H^m(N;\mathbb{Z}_2)$. It follows that the inequality $1 \le \mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \le m$ is always satisfied. In particular, for $m = 3$, the $\mathbb{Z}_2$-index can only equal $1, 2$, or $3$. These facts are formally stated in Section 2 as Theorem 2.1.

In the present work, we suppose that $N$ is a Seifert manifold (of dimension $m = 3$), presented in the usual way by its Seifert invariants (cf. [15], [19]). The presentation of $\pi_1(N)$, associated to these invariants, is the standard presentation found in [15], and allows one to list the (non-trivial) homomorphisms $\phi\colon \pi_1 N \twoheadrightarrow \mathbb{Z}_2$. We classify the $\phi$'s for which the $\mathbb{Z}_2$-index equals 1, equals 2, or equals 3. The main results are expressed in terms of the Seifert invariants of $N$ and the homomorphism $\phi$.

This work contains five sections. In Section 2, we recall some basic facts about Seifert manifolds. In Section 3 we consider the situation of maps into $\mathbb{R}^2$; the main results are Proposition 3.4 and Theorem 3.5. The former gives necessary and sufficient conditions for $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$, and the latter (which is essentially the negation of the former) for $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \geq 2$. In Section 4 we consider the situation of maps into $\mathbb{R}^3$; the main result is Theorem 4.3 which gives necessary and sufficient conditions for $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 3$. In Section 5 we make some general comments about the relation between the $\mathbb{Z}_2$-index $= 2$ and the $\mathbb{Z}_2$-index $=$ 3 cases. In this section we also study several specific examples that effectively illustrate the techniques, for a variety of Seifert manifolds, and also show that the distinction between the various cases can be surprisingly delicate.

Another (and probably more natural) approach to these questions is to start with the manifold $M$ and fixed point free involution $\tau$, then construct $N$ as the orbit space $M/\tau$. For Seifert manifolds $M$ this can lead to cases that are not covered in the present paper, indeed cases where $N$ is not a Seifert manifold in the classical sense, depending on the geometry (in the sense of Thurston) of $M$. The authors hope to complete the study, from this point of view, in subsequent research, with [7] being the first step in this direction and the present note the second step. We also note that the condition $\xi^m \neq 0$ mentioned above becomes $\xi^3 \neq 0$ for a 3-manifold, and for orientable 3-manifolds this condition also arises in the study of general relativity (where one says such 3-manfolds have "type 1"), cf. [21]. The condition $\xi^3 \neq 0$ is equivalent to the existence of a degree 1 (or odd degree) map of the 3-manifold onto $\mathbb{R}P^3$.

## 2 Introductory Remarks and Notation for 3-manifolds

Let $N$ be a 3-manifold. In Section 1 the isomorphism (1) between $H^1(N;\mathbb{Z}_2)$ and $hom(\pi_1(N),\mathbb{Z}_2)$ was introduced. Under this correspondence, the image in $H^1(N;\mathbb{Z}_2)$ of a homomorphism $\phi\colon \pi_1(N) \to \mathbb{Z}_2$ will be denoted by $\xi_\phi = \xi$. Any non-zero element $\xi \in H^1(N;\mathbb{Z}_2)$ corresponds to an epimorphism $\phi\colon \pi_1 N \twoheadrightarrow \mathbb{Z}_2$ which induces a short exact sequence:

$$1 \to \operatorname{Ker}\phi \rightarrowtail \pi_1\mathrm{N} \twoheadrightarrow \mathbb{Z}_2 \to 0.$$

>From the theory of covering spaces, we know that there exists a connected 3-manifold $M = M_\phi$ such that $\operatorname{Ker}\phi = \pi_1(\mathrm{M})$ is a normal, index 2, subgroup of $\pi_1(N)$, and $M \twoheadrightarrow N$ is the regular double covering of $N$ corresponding to $\operatorname{Ker}\phi$. We also know that the non-trivial deck transformation is a fixed point free involution $\tau_\phi = \tau$ on $M$ such that the quotient $M/\tau$ is homeomorphic to $N$. We will use this correspondence freely whenever necessary.

From [7] Theorems (3.1) and (3.2) we have:

**Theorem 2.1.** *Let $N$ be a 3-manifold and $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ an epimorphism. Let $(M,\tau)$ and $\xi \in H^1(N;\mathbb{Z}_2)$ be determined as above.*

*(i) One has $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ if and only if the homomorphism $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ factors through the projection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ (equivalently $\xi \in \operatorname{Im}(\rho\colon H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2))$), otherwise $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) \in \{2,3\}$,*

*(ii) One has $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 3$ if and only if $\xi^3 \neq 0$.*

We now focus on the situation where $N$ is *any* Seifert manifold (orientable or not), as introduced in [19]. We shall answer the following question: given a presentation of $N$ in terms of Seifert invariants, for which $\phi$ is $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 1, 2, \text{or } 3$ ?

Following the notation of Orlik [15], from now on, $N$ will be a Seifert manifold described by a list of Seifert invariants

$$\{e; (\in, g); (a_1, b_1), \ldots, (a_n, b_n)\}$$

(note that Orlik uses $b$ for the Euler number $e$). We do not need them to be "normalized" as in [15] and [19]: we only assume that $e$ is an integer, the type $\in$ will described below, $g$ is the genus of the base surface (the orbit space obtained by identifying each $S^1$ fibre of $N$ to a point), and for each $k$, the integers $a_k, b_k$ are coprime with $a_k \neq 0$ (in case $b_k = 0$ then $a_k = \pm 1$).

As in [15], p.74 (and elsewhere), it is convenient to add an additional (non-exceptional) fibre $a_0 = 1, b_0 = e$. We shall then use the following presentation of the fundamental group of $N$:

$$\pi_1(N) = \left\langle \begin{array}{c} s_0, \ldots, s_n \\ v_1, \ldots, v_{g'} \\ h \end{array} \middle| \begin{array}{cc} [s_k, h] \quad \text{and} \quad s_k^{a_k} h^{b_k}, & 0 \le k \le n \\ v_j h v_j^{-1} h^{-\varepsilon_j}, & 1 \le j \le g' \\ s_0 \ldots s_n V & \end{array} \right\rangle, \tag{2}$$

where the generators and $g', V$ are described below. Also note that if $e = 0$ then the relation $s_0^{a_0} h^{b_0}$ reduces to $s_0 = 1$, so in this case $s_0$ is usually omitted.

- The type $\in$ of $N$ equals:

  $o_1$ if both the base surface and the total space are orientable (which forces all $\varepsilon_j$'s to equal 1);

  $o_2$ if the base surface is orientable and the total space is non-orientable, hence $g \ge 1$ (which forces all $\varepsilon_j$'s to equal $-1$);

  $n_1$ if both the base surface and the total space are non-orientable (hence $g \ge 1$) and moreover, all $\varepsilon_j$'s equal 1;

  $n_2$ if the base surface is non-orientable (hence $g \ge 1$) and the total space is orientable (which forces all $\varepsilon_j$'s to equal $-1$);

  $n_3$ if both the base surface and the total space are non-orientable and moreover, all $\varepsilon_j$'s equal $-1$ except $\varepsilon_1 = 1$, and $g \ge 2$;

  $n_4$ if both the base surface and the total space are non-orientable and moreover, all $\varepsilon_j$'s equal $-1$ except $\varepsilon_1 = \varepsilon_2 = 1$, and $g \ge 3$.

  We note that these six types, in Seifert's original notation, are respectively denoted $Oo$, $No$, $Nn$, $On$, $NnI$, $NnII$, where the first (capital) letter refers to the orientability or non-orientability of the total space $N$, while the second (lower case) letter refers to the same for the base surface.

- The orientability of the base surface and its genus $g$ determine the number $g'$ of the generators $v_j$'s and the word $V$ in the last relator of $\pi_1(N)$ as follows:

  – when the base surface is orientable, $g' = 2g$ and $V = [v_1, v_2] \ldots [v_{2g-1}, v_{2g}]$;

  – when the base surface is non-orientable, $g' = g$ and $V = v_1^2 \ldots v_g^2$.

- The generator $h$ corresponds to the generic regular fibre.
- The generators $s_k$ for $0 \leq k \leq n$ correspond to (possibly) exceptional fibres.

Throughout this paper, we shall use the following notations (the last one $S_\phi$ depends on $\phi$, all the previous ones only on $N$).

**Notation 2.2.** *Let $N$ be a Seifert manifold described by a list of Seifert invariants*

$$\{e; (\in, g); (a_1, b_1), \ldots, (a_n, b_n)\}.$$

- *Denoting by $a$ the least common multiple of the $a_k$'s,*

$$c = \sum_{k=0}^{n} b_k (a/a_k).$$

- *The number of even $a_k$'s will be denoted by $d$.*
- *We distinguish three cases:*
  - *Case 1,* $d = 0$ *and $c$ is even;*
  - *Case 2,* $d = 0$ *and $c$ is odd;*
  - *Case 3,* $d > 0$.
- *In Case 3, the indices $k$ are reordered by decreasing 2-valuation $\nu_2(a_k)$. Hence the set of even $a_k$'s will be $\{a_0, \ldots, a_{d-1}\}$ and the set of $k$'s for which $a_k$ has maximal 2-valuation, denoted by $S_N$, will be $\{0, \ldots, J-1\}$ for some $0 < J \leq d$. Note that after this reordering, in Case 3,* $a_0 \neq 1$.
- *$S_\phi$ will denote the set of $k$'s for which $\phi(s_k) = 1$.*

Note that these cases are not related to the type $\in$, each of the three cases can occur with any of the six types. The next lemma will be useful in Section 3.

**Lemma 2.3.** *In Case 3 ($d > 0$), $c$ has the same parity as $J$. Furthermore, one also has* $S_\phi \subseteq \{0, \ldots, d-1\}$, $|S_\phi|$ *is even, and* $\phi(h) = 0$.

*Proof.* With the above notational conventions, $a/a_k$ is odd if and only if $k < J$, and for such $k$'s, $b_k$ is also odd since it is coprime to $a_k$. Hence, modulo 2, $c = \sum b_k(a/a_k) \equiv \sum_{0\le k<J} 1 = J$. The fact that $S_\phi \subseteq \{0,\dots,d-1\}$ follows directly from the definition of $d$ and the reordering convention in Case 3. If we take any $k \in \{0,\dots,d-1\}$ we have $a_k$ even and $b_k$ odd, hence $0 = \phi(s_k^{a_k}h^{b_k}) = a_k\phi(s_k) + b_k\phi(h) = \phi(h)$. Finally, note that $\phi(V) = 0$ in both the case of orientable or non-orientable base surface, since $\phi$ is a homomorphism and $\mathrm{Im}(\phi) \subseteq \mathbb{Z}_2$. Then $0 = \phi(s_0 \cdots s_n V) = \phi(s_0) + \dots \phi(s_n)$ implies $|S_\phi|$ is even. □

We close this section with an abelianized version of (2), which gives a presentation of $H_1(N) = H_1(N;\mathbb{Z})$. This will also be useful for the work in Section 3.

$$H_1(N) = \left\langle \begin{matrix} s_0,\dots,s_n \\ v_1,\dots,v_{g'} \\ h \end{matrix} \;\middle|\; \begin{matrix} a_k s_k + b_k h, & 0 \le k \le n \\ (1-\varepsilon_j)h, & 1 \le j \le g' \\ s_0 + \dots + s_n + V & \end{matrix} \right\rangle, \tag{3}$$

where $V = 0$ for types $o_1$ and $o_2$, and $V = 2(v_1 + \dots + v_g)$ for the four remaining types.

## 3 Study of $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \ge 2$

As before, let $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ be an epimorphism and $\xi \in H^1(N;\mathbb{Z}_2)$ the corresponding cohomology class as given in (1). By Theorem 2.1, the set of $\xi$'s for which $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ is the image of the coefficient homomorphism $\rho\colon H^1(N;\mathbb{Z}) \to H^1(N;\mathbb{Z}_2)$, so our initial goal in this section is to compute $\mathrm{Im}(\rho)$ (a less direct method, leading to the same results, would be to compute the kernel of the Bockstein homomorphism $H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z})$). This is done in Propositions 3.1, 3.3, and 3.4. Then, in 3.5, we determine when $\xi \notin \mathrm{Im}(\rho)$, and this is equivalent to $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) \ge 2$.

>From the presentation (3) of $H_1(N)$ we shall compute $H^1(N;\mathbb{Z}_2)$ (Proposition 3.1, which will be repeated later as a small part of Theorem 4.1), and similarly compute $H^1(N;\mathbb{Z})$ (Proposition 3.3). We use the fact that $H^1(N;\mathbb{Z}_2)$ naturally identifies to the subspace of cocycles contained in $C^1(N,\mathbb{Z}_2) := hom(C_1(N),\mathbb{Z}_2)$, where $C_1(N)$ is the free abelian group with generators $v_j, s_k, h$. Furthermore, using the isomorphism (1), we see that the 1-cocycles are simply the 1-cochains that

vanish on the abelianized relations for $\pi_1(N)$, as given in (3). We denote by $\hat{v}_j$ $(1 \leq j \leq g')$, $\hat{s}_k$ $(0 \leq k \leq n)$, $\hat{h}$, the elements of the dual basis of $C^1(N, \mathbb{Z}_2)$ corresponding respectively to $v_j, s_k$, and $h$. In Proposition 3.3, the same notations and identifications will be used, replacing $\mathbb{Z}_2$ by $\mathbb{Z}$, recalling also that $H^1(X;\mathbb{Z})$ is a free abelian group for any finite $CW$-complex $X$.

**Proposition 3.1.** *Let* $\alpha = \hat{h} + \sum_{k=0}^{n} b_k \hat{s}_k$ *and* $\alpha_k = \hat{s}_k + \hat{s}_0,\ 1 \leq k \leq n$. *A basis of the* $\mathbb{Z}_2$*-vector space* $H^1(N;\mathbb{Z}_2) \subseteq C^1(N, \mathbb{Z}_2)$ *is (with Notation 2.2):*

- *Case 1 :* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \alpha\}$,
- *Case 2 :* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
- *Case 3 :* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \alpha_1, \ldots, \alpha_{d-1}\}$.

*Proof.* Consider an arbitrary element

$$u = x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k + \sum_{j=1}^{g'} y_j \hat{v}_j \in C^1(N;\mathbb{Z}_2)$$

(with $z_k, y_j, x \in \mathbb{Z}_2$). Due to the presentation (2) of $\pi_1(N)$, $u \in H^1(N;\mathbb{Z}_2)$ if and only if the following $n+2$ equations, coming from the relations in (3), are satisfied:

$$a_k z_k + b_k x = 0, \quad k = 0, \ldots, n, \qquad \text{and} \qquad z_0 + \ldots + z_n = 0.$$

When $d = 0$ all $a_k$ and $a$ are odd, so $c = \sum b_k$ and this system is thus equivalent to:

$$z_k = b_k x \ (k = 0, \ldots, n) \qquad \text{and} \qquad cx = 0.$$

The elements of $H^1(N;\mathbb{Z}_2)$ are therefore the $u$'s of the form:

$$u = x\left(\hat{h} + \sum_{k=0}^{n} b_k \hat{s}_k\right) + \sum_{j=1}^{g'} y_j \hat{v}_j,$$

with no restriction on $x$ in Case 1 ($d = 0$ and $c$ even), but with $x = 0$ in Case 2 ($d = 0$ and $c$ odd).

When $d > 0$, the system is equivalent to:

$$x = 0, \qquad z_k = b_k x \ (k = d, \dots, n) \qquad \text{and} \qquad z_0 + \dots + z_n = 0,$$

which simplifies to:

$$x = z_d = \dots = z_n = 0 \qquad \text{and} \qquad z_0 = z_1 + \dots + z_{d-1}.$$

So, in Case 3, the elements of $H^1(N; \mathbb{Z}_2)$ are the $u$'s of the form:

$$\sum_{1 \le k \le d-1} z_k(\hat{s}_k + \hat{s}_0) + \sum_{j=1}^{g'} y_j \hat{v}_j,$$

which completes the proof. □

**Remark 3.2.** *In future use of this proposition and the following ones it will be important to note that if the cohomology class $u \in H^1(N; \mathbb{Z}_2)$ corresponds to the epimorphism $\phi\colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ via the isomorphism (1), then $z_k = \phi(s_k)$, $y_j = \phi(v_j)$, and $x = \phi(h)$. In this case we also write $u = \xi_\phi = \xi$, as in Section 1.*

**Proposition 3.3.** *The abelian group $H^1(N; \mathbb{Z})$ is free and generated by the following elements of $C^1(N, \mathbb{Z})$:*

- *if $\in = o_2$: $\{\hat{v}_1, \dots, \hat{v}_{g'}\}$,*
- *if $\in = n_2, n_3, n_4$: $\{\hat{v}_2 - \hat{v}_1, \dots, \hat{v}_{g'} - \hat{v}_1\}$,*
- *if $\in = o_1$:*
  - *if $c = 0$: $\{\hat{v}_1, \dots, \hat{v}_{g'},\ a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k\}$,*
  - *if $c \neq 0$: $\{\hat{v}_1, \dots, \hat{v}_{g'}\}$,*
- *if $\in = n_1$:*
  - *if $c$ is even: $\{(c/2)\hat{v}_1 + a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k,\ \hat{v}_2 - \hat{v}_1, \dots, \hat{v}_{g'} - \hat{v}_1\}$,*
  - *if $c$ is odd: $\{c\hat{v}_1 + 2a\hat{h} - 2\sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k,\ \hat{v}_2 - \hat{v}_1, \dots, \hat{v}_{g'} - \hat{v}_1\}$.*

*Proof.* It was noted earlier in this section that $H^1(N;\mathbb{Z})$ is free. As in the proof of Proposition 3.1, consider an arbitrary element

$$u = x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k + \sum_{j=1}^{g'} y_j \hat{v}_j \in C^1(N;\mathbb{Z}),$$

now with $z_k, y_j, x \in \mathbb{Z}$. We obtain that $u \in H^1(N;\mathbb{Z})$ if and only if the following equations are satisfied:

$$a_k z_k + b_k x = 0, \quad k = 0, \ldots, n;$$

$$(1 - \varepsilon_j)x = 0, \quad j = 1, \ldots, g';$$

$$\sum_{k=0}^{n} z_k = 0 \text{ if } \in = o_1, o_2; \ \sum_{j=0}^{n} z_k + 2\sum_{j=1}^{g'} y_j = 0 \text{ if } \in = n_1, n_2, n_3, n_4.$$

Let us first treat the four easiest cases. As soon as some $\varepsilon_j$ equals $-1$ (i.e. $\in = o_2, n_2, n_3, n_4$), the equation involving such a $\varepsilon_j$ implies $x = 0$, which, by the first $n+1$ equations, forces all $z_k$'s to be also zero. The remaining last equation thus reduces to $0 = 0$ if $\in = o_2$ and to $y_1 = -\sum_{j>1} y_j$ if $\in = n_2, n_3, n_4$. This already enables us to assert that the elements of $H^1(N;\mathbb{Z})$ are the $u$'s of the form:

- if $\in = o_2$: $\sum y_j \hat{v}_j$
- if $\in = n_2, n_3, n_4$: $\sum_{j>1} y_j(\hat{v}_j - \hat{v}_1)$.

In the two remaining cases $\in = o_1, n_1$ (where the conditions $(1-\varepsilon_j)x = 0$ are vacuous since $\varepsilon_j = 1$), first note that the first $n+1$ equations imply that each $a_k$ divides $x$, hence so does $a$ (their l.c.m.). Letting $x = ax'$, these equations may be rewritten:

$$z_k = -b_k(a/a_k)x', \quad k = 0, \ldots, n.$$

The remaining last equation thus becomes:

$$cx' = 0 \quad \text{if} \quad \in = o_1; \qquad cx' = 2\sum y_j \quad \text{if} \quad \in = n_1.$$

When $\in = o_1$, this last equation forces $x'$ (hence also the $z_k$'s) to be 0 if and only if $c \neq 0$. Hence the elements of $H^1(N;\mathbb{Z})$ are the $u$'s of the form:

- if $\in = o_1$ and $c \neq 0$: $\sum y_j \hat{v}_j$
- if $\in = o_1$ and $c = 0$: $x'\left(a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k\right) + \sum y_j \hat{v}_j$.

In the last remaining case ($\in = n_1$), the last equation forces $x'$ to be even whenever $c$ is odd, which naturally leads us to consider two subcases:

- if $c$ is even, this equation amounts to $y_1 = (c/2)x' - \sum_{j>1} y_j$;
- if $c$ is odd, letting $x' = 2x''$ allows rewriting the equation as $y_1 = cx'' - \sum_{j>1} y_j$.

Hence the elements of $H^1(N;\mathbb{Z})$ are the $u$'s of the form:

- if $\in = n_1$ and $c$ is even:

$$x'\left(a\hat{h} - \sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k + (c/2)\hat{v}_1\right) + \sum_{j>1} y_j(\hat{v}_j - \hat{v}_1)$$

- if $\in = n_1$ and $c$ is odd:

$$x''\left(2a\hat{h} - 2\sum_{k=0}^{n} b_k(a/a_k)\hat{s}_k + c\hat{v}_1\right) + \sum_{j>1} y_j(\hat{v}_j - \hat{v}_1),$$

  which completes the proof.

□

>From Theorem 2.1 and Propositions 3.1 and 3.3 we deduce:

**Proposition 3.4.** *With the notations of Proposition 3.1 and Notation 2.2, the subspace* $\mathrm{Im}(\rho) \subseteq H^1(N;\mathbb{Z}_2)$ *has basis:*

- *if* $\in = o_2$*:* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
- *if* $\in = n_2, n_3, n_4$*:* $\{\hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1\}$,
- *if* $\in = o_1$*: if* $c \neq 0$ *then* $\{\hat{v}_1, \ldots, \hat{v}_{g'}\}$,
  - *if* $c = 0$ *and* $d = 0$ *then* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \alpha\}$,
  - *if* $c = 0$ *and* $d > 0$ *then* $\{\hat{v}_1, \ldots, \hat{v}_{g'}, \sum_{1 \leq k \leq J-1} \alpha_k\}$,
- *if* $\in = n_1$*:*

– *if $c$ is odd:* $\{\hat{v}_1, \dots, \hat{v}_{g'}\}$,
– *when $c$ is even and $d = 0$:* $\{\hat{v}_2 + \hat{v}_1, \dots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \alpha\}$,
– *if $c$ is even and $d > 0$:* $\{\hat{v}_2 + \hat{v}_1, \dots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \sum_{1 \le k \le J-1} \alpha_k\}$.

*Proof.* Most of this statement follows immediately from Propositions 3.1 and 3.3; we shall address the only non-obvious parts which are the two possibilities ($\in = o_1, c = 0$) and ($\in = n_1, c$ even). Note that in these two cases, Case 2 ($d = 0, c$ odd) does not occur.

If $\in = o_1$ and $c = 0$, we must compute the image in $H^1(N; \mathbb{Z}_2)$ of $u := a\hat{h} - \sum_{k=0}^{n} b_k (a/a_k)\hat{s}_k \in H^1(N; \mathbb{Z})$, in terms of the generators of $H^1(N; \mathbb{Z}_2)$.

- If $d = 0$ (Case 1, $a, a_k$ are all odd): $\rho(u) = \hat{h} + \sum_{k=0}^{n} b_k \hat{s}_k = \alpha$.
- If $d > 0$ (Case 3, $a/a_k$ is odd only for $0 \le k \le J-1$, $a$ is even, and $b_0, \dots, b_{d-1}$ are odd ):

$$\rho(u) = 0 \cdot \hat{h} + \sum_{k=0}^{n} b_k (a/a_k)\hat{s}_k = \sum_{k=0}^{J-1} b_k \hat{s}_k = \sum_{k=0}^{J-1} \hat{s}_k.$$

  By Lemma 2.3 $J$ is even, so we may rewrite this sum as $\sum_{k=1}^{J-1}(\hat{s}_k + \hat{s}_0) = \sum_{k=0}^{J-1} \alpha_k$, as desired.
- If $\in = n_1$ and $c$ is even, the proofs in the two cases (Case 1 and Case 3) are identical to the corresponding previous two cases for $\in = o_1$, except that $(c/2)\hat{v}_1$ is added to $u$ and hence also to $\rho(u)$.

□

We now take into account Remark 3.2, Notation 2.2, and Proposition 3.4 (in its negated form), to prove the main theorem of this section.

**Theorem 3.5.** *One has* $\operatorname{ind}_{\mathbb{Z}_2}(M_\phi, \tau_\phi) \in \{2, 3\}$ *in exactly the following cases:*

- *Either $\in = o_1$ and $c \neq 0$, or $\in = n_1$ and $c$ is odd, or $\in = o_2$, and in addition* $\{\phi(h), \phi(s_0), \dots, \phi(s_n)\} \neq \{0\}$,
- *Either $\in = n_2$ or $\in = n_3$ or $\in = n_4$, and in addition* $\{\phi(\sum v_j), \phi(h), \phi(s_0),$
  $\dots, \phi(s_n\} \neq \{0\}$,

- $\in = o_1, c = 0, d > 0$ *and* $S_\phi \neq \emptyset, S_N$
- $\in = n_1$, *c is even and:*
  - *if* $d = 0$*:* $\sum_{j=1}^{g'} \phi(v_j) \neq (c/2)\phi(h)$
  - *if* $d > 0$*: either* $S_\phi \neq \emptyset, S_N$*, or* $S_\phi = \emptyset$ *and* $\sum_{j=1}^{g'} \phi(v_j) \neq 0$, *or* $S_\phi = S_N$ *and* $\sum_{j=1}^{g'} \phi(v_j) \neq (c/2)$.

*Proof.* Writing as usual $\xi = \xi_\phi \in H^1(N;\mathbb{Z}_2)$, the condition given in Theorem 2.1(1) tells us that $\mathrm{ind}_{\mathbb{Z}_2}(M_\phi, \tau_\phi) \in \{2,3\}$ if and only if $\xi \notin \mathrm{Im}(\rho)$. Now Proposition 3.4 identifies $\mathrm{Im}(\rho)$, so in each case we simply have to negate the conditions given in Proposition 3.4.

- When either $\in = o_1$ and $c \neq 0$, or $\in = n_1$ and $c$ is odd, or $\in = o_2$, $\mathrm{Im}(\rho) = \langle \hat{v}_1, \ldots, \hat{v}_{g'} \rangle$. Therefore $\xi = x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k + \sum_{j=1}^{g'} y_j \hat{v}_j \notin \mathrm{Im}(\rho)$ if and only if some $z_k$ or $x$ is non-zero, which is identical to the given condition (see Remark 3.2).
- When either $\in = n_2$ or $\in = n_3$ or $\in = n_4$, $\mathrm{Im}(\rho) = \langle \hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1 \rangle$. Therefore $\xi \in \mathrm{Im}(\rho)$ if and only if $\xi = \sum_{j=2}^{g'} y_j(\hat{v}_j + \hat{v}_1)$, or equivalently $\xi = \sum_{j=1}^{g'} y_j \hat{v}_j$ with $\sum_{j=1}^{g'} y_j = 0$. So $\xi \notin \mathrm{Im}(\rho)$ if and only if some $x_k \neq 0$ or $x \neq 0$ or $\sum_{j=1}^{g'} y_j \neq 0$, which is identical to the given condition.
- When $\in = o_1, c = 0, d = 0$, we see from Propositions 3.4 and 3.1 (Case 1) that $\rho$ is surjective. So $\xi \in \mathrm{Im}(\rho)$, i.e. $\mathrm{ind}_{\mathbb{Z}_2}(M, \phi) = 1$, and hence this case does not appear on the list in Theorem 3.5.
- When $\in = o_1, c = 0, d > 0$, we have Case 3 so Lemma 2.3 applies, and we shall use it several times here. In particular we will use $0 = c \equiv J \pmod 2$ and $x = \phi(h) = 0$ without further mention. Here $\mathrm{Im}(\rho) = \langle \hat{v}_1, \ldots, \hat{v}_{g'}, \sum_{k=1}^{J-1} \alpha_k \rangle$, hence $\xi \in \mathrm{Im}(\rho)$ if and only if, for some $y_j, z \in \mathbb{Z}_2$,

$$\xi = \sum_{j=1}^{g'} y_j \hat{v}_j + z \sum_{k=1}^{J-1} (\hat{s}_k + \hat{s}_0) = \sum_{j=1}^{g'} \hat{v}_j + \sum_{k=0}^{J-1} z\hat{s}_k.$$

  Since, as already noted, $x = 0$, we deduce $\xi = \sum_{j=1}^{g'} y_j \hat{v}_j + \sum_{k=0}^{J-1} z\hat{s}_k \notin \mathrm{Im}(\rho)$ if and only if either $\phi(\hat{s}_k) = z_k \neq 0$ for some $k \geq J$, or $z_J = \ldots = z_n = 0$ and $\{\phi(\hat{s}_0), \ldots, \varphi(\hat{s}_{J-1})\} = \{z_0, \ldots, z_{J-1}\} = \{0, 1\}$ (i.e. $\varphi(\hat{s}_0), \ldots, \phi(\hat{s}_{J-1})$ are not all equal). These conditions are easily seen, recalling Notation 2.2, to be equivalent to $S_\phi \neq \emptyset$, $S_N$, as stated.

- When $\in = n_1$, $c$ even, and $d = 0$, we have Case 1 so $z_k = b_k x$, $0 \leq k \leq n$, as seen in the proof of Proposition 3.1. Here $\mathrm{Im}(\rho) = \langle \hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \alpha \rangle$. Noting that $\alpha(h) = x$ and $\hat{v}_1(h) = 0$, this gives that $\xi \in \mathrm{Im}(\rho)$ if and only if

$$\xi = \sum_{j=2}^{g'} y_j(\hat{v}_j + \hat{v}_1) + x[(c/2)\hat{v}_1 + \alpha] = \sum_{j=1}^{g'} y_j \hat{v}_j + x\alpha$$

$$= \sum_{j=1}^{g'} y_j \hat{v}_j + x\hat{h} + \sum_{k=0}^{n} z_k \hat{s}_k,$$

where $y_1 = x(c/2) + y_2 + \ldots y_{g'}$, or equivalently $\sum_{j=1}^{g'} y_j = x(c/2)$. It follows that $\xi \notin \mathrm{Im}(\rho)$ if and only if $\sum_{j=1}^{g'} y_j \neq (c/2)x$, and this is the same as the stated condition.

- When $\in = n_1, c$ even, and $d > 0$ we again have Case 3 so as in the previous Case 3, $J$ is even and $x = 0$. Now

$$\mathrm{Im}(\rho) = \langle \hat{v}_2 + \hat{v}_1, \ldots, \hat{v}_{g'} + \hat{v}_1, (c/2)\hat{v}_1 + \sum_{k=1}^{J-1} \hat{s}_k + \hat{s}_0 \rangle.$$

Then $\xi \in \mathrm{Im}(\rho)$ if and only if $\xi = \sum_{j=2}^{g'} y_j(\hat{v}_j + \hat{v}_1) + tc_1\hat{v}_1 + t\sum_{k=0}^{n} \hat{s}_k = \sum_{j=1}^{g'} y_j \hat{v}_j + t\sum_{k=0}^{n} \hat{s}_k$, where $y_1 = y_2 + \ldots + y_{g'} + t(c/2)$, or equivalently $\sum_{j=1}^{g'} y_j = t(c/2)$. It follows that $\xi \notin \mathrm{Im}(\rho)$ if and only if either $S_\phi \nsubseteq S_N$, or $S_\phi \subseteq S_N$ and $\sum_{j=1}^{g'} \phi(v_j) \neq (c/2)\phi(s_k)$ for at least one $k$, $1 \leq k \leq J-1$. Again, these conditions are easily seen to be equivalent to the stated conditions in this case.

□

# 4 Study of $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)=3$

According to 2.1(ii), one has $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)=3$ if and only if $\xi^3\neq 0$. We therefore begin this section by stating known results (cf. [1], [2], [3], [4]) for the $\mathbb{Z}_2$-cohomology ring of a Seifert manifold $N$. For $H^1(N;\mathbb{Z}_2)$ this necessarily overlaps with some of the computations done in Section 3, and the notations used in Section 3 are consistent with those in the references (where we now will write $\hat{v}_j=\theta_j$). For types $\in = o_1, o_2$ what we now call $\theta_1,\theta_2,\theta_3,\theta_4,\dots$ correspond respectively to $\theta_1,\theta_1',\theta_2,\theta_2',\dots$ in [4], while the notation is identical for the remaining four types. As far as the cup products it suffices to list just the non-zero products in positive dimensions, on the generators, also taking account that $xy=yx$ in $H^*(\_\,;\,\mathbb{Z}_2)$.

**Theorem 4.1.** *Let $N$ be any Seifert manifold described by a list of Seifert invariants*

$$\{e;(\in,g);(a_1,b_1),\dots,(a_n,b_n)\},$$

*the type $\in$ being $o_1,o_2,n_1,n_2,n_3,n_4$.*

*Using Notation 2.2, the cohomology groups $H^*(N;\mathbb{Z}_2)$ are: $H^0=\mathbb{Z}_2\{1\}$ (the unit for the cup-product), $H^3=\mathbb{Z}_2\{\gamma\}$, and (with $1\le j\le g'=2g$ for the types $o_1$ and $o_2$, and $1\le j\le g'=g$ for the other types):*

- *Case 1 (if $d=0$ and $c$ is even): $H^1=\mathbb{Z}_2\{\theta_1,\dots\theta_{g'},\ \alpha=\hat{h}+\sum_{k=0}^{n}b_k\hat{s}_k\}$, $H^2=\mathbb{Z}_2\{\varphi_1,\dots,\varphi_{g'},\beta\}$.*

- *Case 2 (if $d=0$ and $c$ is odd): $H^1=\mathbb{Z}_2\{\theta_1,\dots,\theta_{g'}\}$, $H^2=\mathbb{Z}_2\{\varphi_1,\dots,\varphi_{g'}\}$,*

- *Case 3 (if $d>0$): $H^1=\mathbb{Z}_2\{\theta_1,\dots\theta_{g'},\alpha_1,\dots\alpha_{d-1}\}$, $H^2=\mathbb{Z}_2\{\varphi_1,\dots,\varphi_{g'},\beta_1,\dots,\beta_{d-1}\}$.*

*The non-trivial cup-products, on the generators of $H^1\otimes H^1$ and $H^1\otimes H^2$, are:*

- *In all three Cases, for the types $o_1$ and $o_2$, $\theta_{2i-1}\varphi_{2i}=\theta_{2i}\varphi_{2i-1}=\gamma$, while for the other types $\theta_j\varphi_j=\gamma$.*

- *in Case 1, for the types $o_1$ and $o_2$, $\theta_{2i-1}\theta_{2i}=\beta$, while for the other types $\theta_j^2=\beta$.*

– *in Case 1,* $\theta_j\alpha = \varphi_j$, $\alpha\beta = \gamma$, $\alpha\varphi_j = \gamma$ *when* $\varepsilon_j = -1$ *(as specified in Section 2 for each of the types), and*

$$\alpha^2 = (c/2)\beta + \sum_{\varepsilon_j=-1} \varphi_j.$$

– *in Case 3 (i.e.* $d > 0$*),* $\alpha_k\beta_k = \gamma$, $k > 0$, *and, for* $k, l > 0$,

$$\alpha_k\alpha_\ell = \frac{a_0}{2}\beta_0 + \delta_{k,\ell}\frac{a_k}{2}\beta_k,$$

*where* $\beta_0$ *denotes* $\sum_{1\le k\le d-1}\beta_k$.

>From this theorem, we deduce:

**Proposition 4.2.** *With the same notations, let* $\xi = \xi_\phi \in H^1(N;\mathbb{Z}_2)$.

- *In Case 1,*

$$\xi^3 = \begin{cases} \phi(h)(c/2)\cdot\gamma \text{ when } \in = o_1, \\ \phi(h)((c/2) + \sum\phi(v_i))\cdot\gamma \text{ when } \in = o_2, n_1, \\ \phi(h)((c/2) + g)\cdot\gamma \text{ when } \in = n_2, \\ \phi(h)((c/2) + \phi(v_1) + g - 1)\cdot\gamma \text{ when } \in = n_3, \\ \phi(h)((c/2) + \phi(v_1) + \phi(v_2) + g)\cdot\gamma \text{ when } \in = n_4. \end{cases}$$

- *In Case 2,* $\xi^3 = 0$.
- *In Case 3,* $\xi^3 = (\sum\phi(s_k)(a_k/2))\cdot\gamma$.

*Proof.*

Case 1. Let $\xi = x\cdot\alpha + \sum y_j\cdot\theta_j$ with $x = \phi(h)$ and $y_j = \phi(v_j)$, then

$$\xi^2 = x\cdot\alpha^2 + \sum y_j\cdot\theta_j^2 = x((c/2)\cdot\beta + \sum_{\varepsilon_j=-1}\varphi_j) + y\cdot\beta,$$

with $y = 0$ when $\in = o_1, o_2$, $y = \sum y_j$ when $\in = n_1, n_2, n_3, n_4$, and $\sum_{\varepsilon_j=-1}\varphi_j = 0$ for types $o_1, n_1$. For the various types, this now gives :

– when $\in = o_1$, $\xi^3 = (x\alpha + \sum_{j=1}^{2g} y_j\theta_j)\cup x(c/2)\beta = x(c/2)\cdot\gamma$.

– when $\in = o_2$,

$$\begin{aligned}\xi^3 &= (x\alpha + \sum_{j=1}^{2g} y_j\theta_j) \cup x((c/2)\beta + \sum_{j=1}^{2g} \varphi_j) \\ &= x((c/2) + 2g + \sum_{j=1}^{2g} y_j) \cdot \gamma = x((c/2) + \sum_{j=1}^{2g} y_j) \cdot \gamma.\end{aligned}$$

– when $\in = n_1, n_2, n_3, n_4$,

$$\begin{aligned}\xi^3 &= (x\alpha + \sum_{j=1}^{g} y_j\theta_j) \cup \left( (x(c/2) + y)\beta + x \sum_{\varepsilon_j=-1} \varphi_j \right) \\ &= x\left( (c/2) + y + \#\{j \mid \varepsilon_j = -1\} + \sum_{\varepsilon_j=-1} y_j \right) \cdot \gamma \\ &= x\left( (c/2) + \sum_{\varepsilon_j=1} y_j + \#\{j \mid \varepsilon_j = -1\} \right) \cdot \gamma.\end{aligned}$$

Case 2. $\xi = \sum y_j\theta_j$, hence $\xi^2 = \sum y_j\theta_j^2 = 0$ and $\xi^3 = 0$.

Case 3. Letting $z_k = \phi(s_k)$, recall from the proof of Proposition 3.1 that $z_k = 0$ for $k \geq d$, $z_0 = \sum_{k>0} z_k$, and $\xi = \sum_{1\leq k\leq d-1} z_k\alpha_k + \sum y_j\theta_j$, hence

$$\begin{aligned}\xi^2 &= \sum_{1\leq k\leq d-1} z_k\alpha_k^2 + \sum y_j\theta_j^2 \\ &= \sum_{1\leq k\leq d-1} z_k(\frac{a_0}{2}\beta_0 + \frac{a_k}{2}\beta_k) + 0 \\ &= \frac{a_0}{2}(\sum_{1\leq k\leq d-1} z_k)\beta_0 + \sum_{1\leq k\leq d-1} z_k\frac{a_k}{2}\beta_k \\ &= \frac{a_0}{2}z_0\beta_0 + \sum_{1\leq k\leq d-1} z_k\frac{a_k}{2}\beta_k \\ &= \sum_{0\leq k\leq d-1} z_k\frac{a_k}{2}\beta_k\end{aligned}$$

and

$$\begin{aligned}
\xi^3 &= \Big(\sum_{1\le k\le d-1} z_k\alpha_k + \sum y_j\theta_j\Big) \cup \sum_{0\le k\le d-1} z_k\frac{a_k}{2}\beta_k \\
&= z_0\frac{a_0}{2}\Big(\sum_{1\le k\le d-1} z_k\alpha_k\Big)\cup\beta_0 + \sum_{1\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma \\
&= z_0\frac{a_0}{2}\Big(\sum_{1\le k\le d-1} z_k\Big)\gamma + \sum_{1\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma \\
&= z_0\frac{a_0}{2}\gamma + \sum_{1\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma \\
&= \sum_{0\le k\le d-1} z_k\frac{a_k}{2}\cdot\gamma.
\end{aligned}$$

□

Using Proposition 4.2, we conclude:

**Theorem 4.3.** *One has* $\operatorname{ind}_{\mathbb{Z}_2}(M_\phi,\tau_\phi)=3$ *if and only if*

- *either N satisfies Case 3 (i.e.* $d>0$*) and* $\sum_{\phi(s_k)=1} a_k$ *is not a multiple of* 4*,*
- *or N satisfies Case 1 (i.e.* $d=0$ *and c is even), and* $\phi(h)=1$*, and the following element of* $\mathbb{Z}_2$ *is nonzero:*
  - *when* $\in = o_1$*:* $c/2$
  - *when* $\in = o_2, n_1$*:* $(c/2)+\sum\phi(v_j)$
  - *when* $\in = n_2$*:* $(c/2)+g$
  - *when* $\in = n_3$*:* $(c/2)+\phi(v_1)+g-1$
  - *when* $\in = n_4$*:* $(c/2)+\phi(v_1)+\phi(v_2)+g$*.*

# 5 Remarks and examples

In this section we give a brief discussion of the class $\xi^2$ and several examples. The first few examples tend to involve relatively simple Seifert manifolds for which the full machinery of the previous sections is not strictly needed. The final two examples are more involved and the full machinery will be necessary. These examples cover three of the six possible Seifert manifold types, namely $\in = o_1, n_1, n_3$, as well as various Euler numbers $e$ and genus $g'$.

**Proposition 5.1.** *(a) If* $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$*, then* $\xi^2 = 0$.
*(b) If* $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 3$*, then* $\xi^2 \neq 0$.

*Proof.* (a) Consider the Bockstein homomorphisms $B\colon H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z})$ and $\beta = Sq^1\colon H^1(N;\mathbb{Z}_2) \to H^2(N;\mathbb{Z}_2)$, and recall that under the coefficient homomorphism $\rho'\colon H^2(N;\mathbb{Z}) \to H^2(N;\mathbb{Z}_2)$ one has $\rho' \circ B = \beta$. From Theorem 2.1(i) we know $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ if and only if $\xi \in \mathrm{Im}(\rho)$. Since $\mathrm{Im}(\rho) = \mathrm{Ker}(B)$, the condition is equivalent to $B(\xi) = 0$. And this implies $0 = Sq^1(\xi) = \xi^2$.

(b) This is immediate from Theorem 2.1(ii).

□

Based on 5.1 (a), it is interesting to have examples where $\xi^2 = 0$ and where the $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau)$ could equal 1 or equal 2. In fact such examples are already considered in [7], Section 5, and we will recall them here.

**Example 5.2.**

(a) Let $N = L(4,1)$, and $\xi \in H^1(N;\mathbb{Z}_2) \approx \mathbb{Z}_2$ be the generator. Then $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$ and $\xi^2 = 0$.

(b) Let $N = S^1 \times V$, $V$ being is any closed surface, and $\xi = \pi^*(u)$, where $\pi\colon N \twoheadrightarrow S^1$ is the projection and $u$ generates $H^1(S^1;\mathbb{Z}_2)$. Then $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ and $\xi^2 = 0$.

(c) As a special case of (b) let $N = S^1 \times \mathbb{R}P^2$, then $H^1(N;\mathbb{Z}_2)$ has the generator $u$ as in (b), and the additional generator $x$ corresponding to the (pull-back) of the generator of $H^1(\mathbb{R}P^2;\mathbb{Z}_2)$. Now, in addition to $\xi = u$ as in (b), we have two further possible choices $\xi = v$ or $\xi = u + v$. For each of these latter two choices we have $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$ since $\xi^2 = v^2 \neq 0$ and $\xi^3 = 0$.

Of course, the conclusions in Example 5.2 as well as the following Example 5.3 also follow easily from our main theorems. As an illustration, in 5.2(a) we have $L(4,1) = \{4; (o_1, 0)\}$ (cf. [15] 5.4(i)). Here $a_0 = 1$, $b_0 = 4$, whence $d = 0$, $c = 4$, and this implies we are in Case 1. By Theorem 4.1 the only non-zero element in $H^1(N;\mathbb{Z}_2)$ is $\alpha$, hence $\xi = \alpha$. Again by Theorem 4.1 we have $\alpha^2 = (c/2)\beta = 0$. Now applying Theorem 3.5 (first case) and Theorem 4.3 (second case), we obtain that $\mathrm{ind}_{\mathbb{Z}_2}(M,\tau) = 2$.

We also remark that in 5.2(b) and 5.2(c) one has $e = 0$, and the type is $o_1$ if $V$ is orientable, $n_1$ if $V$ is non-orientable.

Our next example illustrates to some extent the delicacy of the Borsuk-Ulam situation. The example shows that one can have two double covers of a Seifert manifold $N$ by the same Seifert manifold $M$ but with different $\mathbb{Z}_2$-indices for $(M, \tau)$. Indeed the example already arises at the level of surface topology.

**Example 5.3.**

Let $N = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, one has $\pi_1(N) = \langle v_1, v_2, v_3 | v_1^2 v_2^2 v_3^2 \rangle$, $H^1(N; \mathbb{Z}_2) \approx \mathbb{Z}_2^3$ with generators $\theta_1, \theta_2, \theta_3$ and $H^2(N; \mathbb{Z}_2) \approx \mathbb{Z}_2$ with generator $\beta$. Furthermore $\theta_i^2 = \beta$ whereas $\theta_i \theta_j = 0$, $i \neq j$ (cf. [8] Section 3.2). The characteristic class $\xi_1 = \theta_2 + \theta_3$ corresponds to the homomorphism $\phi_1 \colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ given by $\phi_1(v_1) = 0, \phi_1(v_2) = \phi_1(v_3) = 1$. Similarly the characteristic class $\xi_2 = \theta_1$ corresponds to $\phi_2 \colon \pi_1(N) \twoheadrightarrow \mathbb{Z}_2$ with $\phi_2(v_1) = 1, \phi_2(v_2) = \phi_2(v_3) = 0$. Using Proposition 4.2 of [7] we obtain at once that $\mathrm{ind}_{\mathbb{Z}_2}$ is 1 for $\xi_1$ and 2 for $\xi_2$ (this corresponds to $\xi_1^2 = 0$, $\xi_2^2 \neq 0$). The surface $M$ that is the double cover of $N$ must have Euler characteristic $\chi(M) = 2\chi(N) = -2$. Since it is not hard to see that in both cases $M$ is non-orientable, it follows that in both cases $M = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$.

By simply taking the product of $M$ and $N$ with $S^1$, we obtain similar examples with Seifert manifolds (where we take $\phi_i(h) = 0$). Indeed, writing $N_1 = N \times S^1$, we have that $N_1$ has $\in = n_1$, $g' = 3$, and no exceptional fibres whence $d = c = 0$. From 3.5, final case, we see that $\phi_1(v_1) + \phi_1(v_2) + \phi_1(v_3) = 0$ implies the $\mathbb{Z}_2$-index for $\phi_1$ equals 1, whereas $\phi_2(v_1) + \phi_2(v_2) + \phi_2(v_3) = 1$ implies the $\mathbb{Z}_2$ index for $\phi_2$ is 2 or 3. Since $\zeta^3 = 0$ for any $\zeta \in H^1(N_1; \mathbb{Z}_2)$, the $\mathbb{Z}_2$-index of $\phi_2$ must be 2.

It should be noted, as was already done in Seifert's original paper [19], that the same 3-manifold (even $S^3$) can often be fibred in different ways, i.e. the Seifert "invariants" are not always true invariants in the sense that they may not be unique. However the cohomology ring with any coefficients, and fundamental group, are of course true invariants, and the determination of the $\mathbb{Z}_2$-index is based upon these. We conclude with a couple of deeper examples for which the techniques of Sections 3 and 4 must be utilized to answer the Borsuk-Ulam question.

**Example 5.4.**

Let $N$ be the Seifert manifold given by the following Seifert invariants:

$$N = \{0, (n_3, 2); (9, 4), (5, 2), (7, 2)\}.$$

Then, a presentation of $\pi_1(N)$ is:

$$\pi_1(N) = \left\langle \begin{matrix} s_1, s_2, s_3 \\ v_1, v_2 \\ h \end{matrix} \left| \begin{matrix} [s_k, h] \quad (k = 1, 2, 3) \\ [v_1, h], \quad v_2 h v_2^{-1} h \\ s_1^9 h^4, \ s_2^5 h^2, s_3^7 h^2, \ s_1 s_2 s_3 v_1^2 v_2^2 \end{matrix} \right. \right\rangle.$$

Note that $d = 0$ (since $9, 5, 7$ are odd) and $c$ is even (since $4, 2, 2$ are even), hence we are in Case 1 of Notation 2.2. The following table shows the values of all possible non-zero $\phi$'s on the generators of $\pi_1(N)$, as well as the corresponding cohomology class $\xi \in H^1(N; \mathbb{Z}_2)$ under the isomorphism (1). Also recall that here, by Theorem 4.1 (or Proposition 3.1), $H^1(N; \mathbb{Z}_2)$ has generators $\alpha, \theta_1, \theta_2$ with $\alpha = \hat{h} + 4\hat{s}_1 + 2\hat{s}_2 + 2\hat{s}_3 = \hat{h}$, and finally that $\in = n_3$ implies all $\phi(s_j) = 0$. The final column in the table gives the $\mathbb{Z}_2$-index, in each case, of $(M_i, \tau_i) := (M_{\xi_i}, \tau_{\xi_i})$. The proofs for the data in the table are given in Proposition 5.5 below.

| $\phi_i$ | $s_1$ | $s_2$ | $s_3$ | $h$ | $v_1$ | $v_2$ | $\xi_i$ | $\text{ind}_{\mathbb{Z}_2}(M_i, \tau_i)$ |
|---|---|---|---|---|---|---|---|---|
| $\phi_1$ | 0 | 0 | 0 | 1 | 0 | 0 | $\alpha$ | 3 |
| $\phi_2$ | 0 | 0 | 0 | 1 | 1 | 0 | $\alpha + \theta_1$ | 2 |
| $\phi_3$ | 0 | 0 | 0 | 1 | 0 | 1 | $\alpha + \theta_2$ | 3 |
| $\phi_4$ | 0 | 0 | 0 | 1 | 1 | 1 | $\alpha + \theta_1 + \theta_2$ | 2 |
| $\phi_5$ | 0 | 0 | 0 | 0 | 1 | 0 | $\theta_1$ | 2 |
| $\phi_6$ | 0 | 0 | 0 | 0 | 0 | 1 | $\theta_2$ | 2 |
| $\phi_7$ | 0 | 0 | 0 | 0 | 1 | 1 | $\theta_1 + \theta_2$ | 1 |

**Proposition 5.5.**

- *For* $\xi = \xi_7$ *one has* $\text{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 1$.
- *For* $\xi = \xi_2, \xi_4, \xi_5, \xi_6$ *one has* $\text{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 2$.
- *For* $\xi = \xi_1, \xi_3$ *one has* $\text{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 3$.

*Proof.* By Theorem 3.5, $\text{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 1$ if and only if $\phi(h) = \phi(v_1 + v_2) = 0$, i.e. $\phi = \phi_7$. Moreover, $N$ is in Case 1, $\in = n_3$, $c$ is a multiple of 4 and $g = 2$ hence, by Theorem 4.3, $\text{ind}_{\mathbb{Z}_2}(M_i, \tau_i) = 3$ if and only if $\phi(h) = \phi(v_1) + 1 = 1$, i.e. $\phi = \phi_1, \phi_3$. □

Our concluding example has (in contrast to the previous examples) non-zero Euler number, arbitrary genus $g \geq 0$, and a relatively large number (seven) of singular fibres.

**Example 5.6.**

Let $N_g,\ g \geq 0$, be the Seifert manifold given by the Seifert invariants

$$\{-2; (o_1, g); (16,5), (16,1), (16,1), (16,1), (2,1), (3,2), (3,1)\}.$$

With the conventions given in Notation 2.2, a presentation of $\pi_1(N)$ is (note that according to these conventions the singular fibres are reordered so that $s_0$ corresponds to $(16,5)$, $s_1$ to $(16,1)\ldots$, $s_6$ to $(3,1)$, and $s_7$ to $(1,e) = (1,-2)$ :

$$\pi_1(N) = \left\langle \begin{matrix} s_0, s_1, s_2, s_3, s_4, s_5, s_6, s_7 \\ v_1, v_2, \ldots, v_{2g-1}, v_{2g} \\ h \end{matrix} \;\middle|\; \begin{matrix} [s_k, h] \quad \text{and} \quad s_k^{a_k} h^{b_k}, \quad 0 \leq k \leq 7 \\ [v_j, h], \quad 1 \leq j \leq 2g \\ s_0 \cdots s_7 [v_1, v_2] \cdots [v_{2g-1}, v_{2g}] \end{matrix} \right\rangle.$$

One easily checks that here $a = 48,\ c = 0,\ d = 5,\ J = 4$, whence $S_N = \{0,1,2,3\}$ and we are in Case 3 of Notation 2.2. As usual, $\phi$ denotes any surjective homomorphism $\phi\colon \pi_1(N_g) \twoheadrightarrow \mathbb{Z}_2$ and $\tau$ the corresponding involution of the double cover $M$ arising from $\phi$. It is also readily seen that $\phi(h) = \phi(s_5) = \phi(s_6) = \phi(s_7)$ and $\phi(s_0) + \phi(s_1) + \phi(s_2) + \phi(s_3) + \phi(s_4) = 0$ are necessary conditions for $\phi$ to be a homomorphism.

**Proposition 5.7.**

- $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 1$ *iff either* $S_\phi = \emptyset$ *(in which case* $\phi(s_k) = 0,\ 0 \leq k \leq 7,\ g \geq 1$, *and* $\phi(v_j) = 1$ *for at least one* $j$*), or* $S_\phi = S_N$ *(in which case* $\phi(s_0) = \phi(s_1) = \phi(s_2) = \phi(s_3) = 1$*).*
- $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 3$ *iff* $\phi(s_4) = 1$ *(whence also* $\phi(s_0) + \phi(s_1) + \phi(s_2) + \phi(s_3) = 1$*).*
- *In all remaining cases* $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 2$.

*Proof.* By Theorem 3.5 we have $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) > 1$ if and only if $d > 0$ and $S_\phi \neq \emptyset, S_N$. Since here $d = 5$, the negation of the previous sentence gives the first statement of the proposition.

By Theorem 4.3 we have $\operatorname{ind}_{\mathbb{Z}_2}(M,\tau) = 3$ if and only if $d > 0$ (which is the case) and $\sum\{a_k : k \in S_\phi\}$ is not divisible by 4. We have already observed that $S_\phi \subseteq \{0,1,2,3,4\}$ and furthermore $a_0 = a_1 = a_2 = a_3 = 16$, hence $\sum\{a_k : k \in S_\phi\} \equiv 2 \cdot \phi(s_4) \pmod 4$, and this gives the second statement of the proposition. The third and final statement follows by default.

□

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