

*Sebastian Ruzkowski*

*(Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, Toruń, Poland)*

## Time scale version of the Ważewski retract method

sebrus@mat.umk.pl

In the paper there are discussed approaches to the Ważewski retract method on time scales. In particular there is presented planar case without a restrictive assumption that the whole boundary of a set of constraints, where we look for solutions, is a set of egress points. One example illustrating the main theorem is presented.

### Introduction

In 1947 Tadeusz Ważewski (see [1]) gave a simple but excellent topological principle, now called the Ważewski retract method, which has been used by many authors to prove the existence of solutions of a given differential equation which remain in a prescribed set of constraints. In particular, the method helps to find bounded solutions in several differential problems. It generalizes the direct method of Lyapunov and is based on examining so-called ‘egress’ and ‘strict egress’ points on a boundary of the set of constraints. It is worth noting that the set does not need to be an attractor or repeller. It is sufficient to check that the set of egress points, which is usually assumed to be equal to the set of strict egress points, is not a retract or, more generally, strong deformation retract of the whole set. This topological principle became a base and a motivation for a construction of a very well known and useful topological invariant, the Conley index (see, e.g., [2] for a comparison of these two topological tools).

The Ważewski retract method was generalized and adopted to: differential inclusions (see, e.g., [2] or [3] and references therein), difference equations (e.g. [4, 5]) or, recently, dynamic equations on time scales ([6, 7, 8]). This last area of research has been intensively developed since 90's as a unification and generalization of the theory of difference equations and differential equations, and has found applications in many mathematical models in biology and physics, where discrete and continuous dynamics have to be studied simultaneously. Moreover, various impulsive differential problems can be transformed to dynamic equations on time scales.

While several results on dynamic equations on time scales are just simple transformations of continuous or discrete analogs, the ones concerning qualitative theory are not. The results on the Ważewski topological principle for dynamic equations on time scales are still not satisfactory. In fact, the only cases explored enough are the ones where the set of constraints is negatively invariant (see [6, 7]).

When we drop the above simplification, we meet several essential problems. The main of them is to construct a retraction, which has to be a continuous map, from an initial section  $\Omega_{t_0}$  of the tube of constraints onto the  $t_0$ -section  $E_{t_0}$  of the set of egress points. We need a deep geometrical study to overcome this problems. The Shöenflies theorem, a convexity and strict convexity play an important role in proofs of non-whole boundary case (see [8]).

The paper is organized as follows. In section 2 we recall some information on the calculus on time scales that will be useful in the sequel. Section 3 shows topological ideas contained in [6] and [7]. Section 4 presents results from [8], where the positive or negative invariance as well as a repulsivity of the set is not assumed anymore. One transparent example is given to illustrate the results.

## 2 Preliminaries

### 2.1 Basics of a calculus on time scales

The interested reader can consult [9, 10] to get a complete introduction or to find proofs of statements of this section.

A time scale is any closed subset of the set  $\mathbb{R}$  of real numbers and we denote it by  $\mathbb{T}$ .

Basic functions describing  $\mathbb{T}$  are jump operators  $\sigma, \rho: \mathbb{T} \rightarrow \mathbb{T}$  i  $\mu: \mathbb{T} \rightarrow \mathbb{R}$ , defined as follows:

- $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$  (forward jump operator)
- $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$  (backward jump operator)
- $\mu(t) = \sigma(t) - t$  (graininess function)

where we assume:  $\inf \emptyset = \sup \mathbb{T}$  and  $\sup \emptyset = \inf \mathbb{T}$ .

**Proposition 2.1** (Induction Principle). *Let  $t_0 \in \mathbb{T}$  and assume that  $\{S(t) : t \in [t_0, \infty) \cap \mathbb{T}\}$  is a family of statements satisfying:*

- *The statemnt  $S(t_0)$  is true.*
- *If  $t \in [t_0, \infty) \cap \mathbb{T}$  is right-scattered and  $S(t)$  is true for all  $s \in [t_0, t) \cap \mathbb{T}$ , then  $S(\sigma(t))$  is also true.*
- *If  $t \in [t_0, \infty) \cap \mathbb{T}$  is right-dense and  $S(t)$  is true, then there is a neighborhood  $U$  of  $t$  such that  $S(s)$  is true for all  $s \in U \cap (t, \infty) \cap \mathbb{T}$ .*
- *If  $t \in (t_0, \infty) \cap \mathbb{T}$  is left-dense and  $S(s)$  is true for all  $s \in [t_0, t) \cap \mathbb{T}$ , then  $S(t)$  is true.*

*Then  $S(t)$  is true for all  $t \in [t_0, \infty) \cap \mathbb{T}$ .*

**Definition 2.2.**  $\Delta$ -derivative of a function  $f : \mathbb{T} \rightarrow \mathbb{X}$  in a point  $t$ , where  $\mathbb{X}$  is a linear normed space, is the point  $f^\Delta(t) \in \mathbb{X}$  (if it exists) such that:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall s \in B(t, \delta) \cap \mathbb{T} \quad \|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leq \varepsilon |\sigma(t) - s|$$

**Proposition 2.3.** *If a function  $f$  is continuous in  $t$  and:*

- *$t = \sigma(t)$ , then:  $f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}$*
- *$t \neq \sigma(t)$ , then:  $f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$*
- *in general, for  $t \in \mathbb{T}^\kappa$  we have*

$$f^\Delta(t) = \lim_{s \rightarrow t} \mathbb{T} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

*where  $\mathbb{T}^\kappa$  is the set  $\mathbb{T}$  without the point  $\max \mathbb{T}$  if this point exists and is isolated.*

## 2.2 $\Delta$ -differential equations

**Definition 2.4.** By a local solution of a system of equations:

$$\begin{cases} x^\Delta(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \quad (1)$$

we will mean a continuous function  $x: \mathbb{T} \cap (a, b) \rightarrow \mathbb{X}$  such that  $a < \rho(t_0)$ ,  $b > \sigma(t_0)$ ,  $x(t_0) = x_0$  and for all  $t \in \mathbb{T}^\kappa \cap (a, b)$  equation  $x^\Delta(t) = f(t, x(t))$  is fulfilled.

**Definition 2.5.** A solution  $x_2$  is an extension of a solution  $x_1$ , if  $x_1$  and  $x_2$  are local solutions of the same system of equations,  $\text{Dom}(x_2) \subsetneq \text{Dom}(x_1)$  and  $x_2|_{\text{Dom}(x_1)} = x_1$ .

If we cannot extend a local solution, then we call it a global solution.

**Proposition 2.6.** *If for all  $t_0 \in \mathbb{T}^\kappa$  and  $x_0 \in \mathbb{X}$  there exists a unique local solution of system (1), then for all  $t_0 \in \mathbb{T}^\kappa$  and  $x_0 \in \mathbb{X}$  there exists a unique global solution of the same equation  $x: \mathbb{T} \cap (a, b) \rightarrow \mathbb{X}$ , where  $\mu(a) = 0$  or  $a = -\infty$ , and  $b - \rho(b) = 0$  or  $b = \infty$ .*

In analogy to standard local processes on  $\mathbb{R}$  we can define a local  $\Delta$ -process. We have then a formal definition:

**Definition 2.7.** A continuous function  $\Pi: M \rightarrow \mathbb{X}$  (where  $M \subset \mathbb{X} \times \mathbb{T}^2$ ) is a local  $\Delta$ -process if:

$$\text{P1 } \forall_{x \in \mathbb{X}, t \in \mathbb{T}} \exists_{\alpha < t < \beta} (\mu(\alpha) = 0 \vee \alpha = -\infty) \wedge (\beta - \rho(\beta) = 0 \vee \beta = \infty) \wedge \\ \wedge \{s \in \mathbb{T}; (x, t, s) \in M\} = (\alpha, \beta) \cap \mathbb{T},$$

$$\text{P2 } \forall_{x \in \mathbb{X}, t \in \mathbb{T}} \Pi(x, t, t) = x,$$

$$\text{P3 } \forall_{(x, t, s), (x, t, r) \in M} (\Pi(x, t, s), s, r) \in M \wedge \Pi(\Pi(x, t, s), s, r) = \Pi(x, t, r).$$

**Definition 2.8.** We say that an equation  $x^\Delta(t) = f(t, x(t))$  generates a local  $\Delta$ -process  $\Pi$ , if for all  $x_0 \in \mathbb{X}$  and  $t_0 \in \mathbb{T}$  a function  $\Pi(x_0, t_0, \cdot)$  is a global and unique solution of (1) and  $\Pi$  is a local  $\Delta$ -process.

Analogously as processes on  $\mathbb{R}$ , a  $\Delta$ -process induce homeomorphisms along trajectories:

---

There is also possibility of understanding a solution as a function that fulfills the equation  $x^\Delta(t) = f(t, x(t))$   $\mathbb{T}$ -almost everywhere (that concept had been introduced in [11]). If we accept this definition the next part of this paper needs only nonsignificant changes.

**Proposition 2.9.** *If an equation  $x^\Delta(t) = f(t, x(t))$  generates a local  $\Delta$ -process  $\Pi$ , and if all solutions of the problem*

$$\begin{cases} x^\Delta(t) = f(t, x(t)) \\ x(t_0) \in A \end{cases}$$

*exist in time  $t_1$ , then  $\Pi(\cdot, t_0, t_1)|_A$  is a homeomorphism between  $A$  and its image.*

*Proof.*  $\Pi$  is continuous so  $\Pi(\cdot, t_0, t_1)$  and  $\Pi(\cdot, t_1, t_0)$  are continuous on their domains which implies what was to prove.  $\square$

We will need the preservation of orientation by  $\Pi$ . Below we show a simple theorem which gives an example of a class of functions implying that property.

**Definition 2.10.** A function  $f : \mathbb{T} \times \mathbb{X} \rightarrow \mathbb{X}$  is rd-continuous if it is continuous in all  $t \in \mathbb{T}$  such that  $\mu(t) = 0$ , that is in so-called right dense points (this justifies "rd" in the name).

**Proposition 2.11.** *Let  $f : \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be rd-continuous and Lipschitz continuous (with Lipschitz constant  $L(t)$ ) with respect to the second variable. If for all  $t \in \mathbb{T}$  inequality  $L(t)\mu(t) < 1$  is fulfilled, then an equation  $x^\Delta(t) = f(t, x(t))$  generates a local  $\Delta$ -process  $\Pi$  and for all  $t \in \mathbb{T}$  we have that function  $\Pi(\cdot, t, \sigma(t))$  preserves an orientation of  $\mathbb{R}^n$ .*

*Proof.* An equation  $x^\Delta(t) = f(t, x(t))$  has a global and unique solution (see [9, p.322, 324]) with a continuous dependence on the initial conditions, so this equation generates a local  $\Delta$ -process  $\Pi$ . Moreover:

$$\begin{aligned} \Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t)) &= x + \mu(t)f(t, x) - (0 + \mu(t)f(t, 0)) = \\ &= x + \mu(t)(f(t, x) - f(t, 0)) \end{aligned}$$

so, by  $L(t)\mu(t) < 1$ , for  $x \neq 0$  we have:

$$\langle \Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t)), x \rangle > 0$$

which means that vectors  $\Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t))$  and  $x$  are in the same halfspace, so  $\Pi(\cdot, t, \sigma(t))$  preserves an orientation of  $\mathbb{R}^n$ .  $\square$

### 3 Ważewski method for the whole boundary egress set

There are shown two approaches to basic Ważewski Theorem in this chapter, which means: the case of set  $\Omega$ , for which all trajectories starting from the boundary immediately leaves that set. In this section we assume, that  $\mathbb{X} = \mathbb{R}^n$ .

#### 3.1 Approach 1.

If local  $\Delta$ -process  $\Pi$  generated by  $\Delta$ -equation is not well defined in  $(x, t_0, t_1)$ , it means, that solution starting in  $(x, t_0)$  reaches to boundary of  $\mathbb{R}^n$  - infinity (one point compactification of  $\mathbb{R}^n$ ). This observation leads to convenient notation for points  $(x, t_0, t_1)$  outside of the domain of local  $\Delta$ -process  $\Pi$ :  $\Pi(x, t_0, t_1) := \infty$ .

We will use a function of positively closest point of change of interval character of  $\mathbb{T}$ .

**Definition 3.1.** Essential forward jump operator is a function  $\text{ess}\sigma : \mathbb{T} \rightarrow \mathbb{T} \cup \text{sup } \mathbb{T}$  with formula:

$$\text{ess}\sigma(t) := \inf\{s \in \mathbb{T} ; s > t \wedge (\mu(s) > 0 \vee \mu(t) > 0)\}.$$

Let  $\tilde{\Omega}$  be closed subset of  $\mathbb{R} \times \mathbb{R}^n$ , such that for each  $r \in \mathbb{R}$  the set  $\tilde{\Omega}_r := \{x \in \mathbb{R}^n ; (r, x) \in \tilde{\Omega}\}$  is nonempty and bounded,  $\partial(\tilde{\Omega}_r)$  is not a re-trakt of  $\tilde{\Omega}_r$  and  $\{r\} \times \partial(\tilde{\Omega}_r)$  is a re-trakt of  $\partial(\tilde{\Omega})$ . We will use curtailment of  $\tilde{\Omega}$  to the time scale  $\mathbb{T}$ :

$$\Omega := \bigcup_{t \in \mathbb{T}} \{t\} \times \tilde{\Omega}_t.$$

**Theorem 3.2.** For above set  $\Omega$  and equation  $x^\Delta(t) = f(t, x(t))$ , which generates local  $\Delta$ -process  $\Pi$ , if for all  $t \in \mathbb{T}$  and for all  $s \in (t, \text{ess}\sigma(t)]$  we have  $\Pi(\text{cl}(\Omega^c)_t, t, s) \subset (\Omega_s)^c$  then for each  $t_0 \in \mathbb{T}$  there exists point  $x_0 \in \Omega_{t_0}$ , such that the solution starting from  $(t_0, x_0)$  remain in  $\Omega$  for every  $t \in \mathbb{T}$  bigger than  $t_0$ .

Similar theorem is proved in [7], but they are focused on simple time scales (with values of grainies function equal 0 or bigger than  $\epsilon$ ), and then using inverse systems and analitic means, they obtain general case.

This condition means that starting from the outside of set  $\Omega$  there is no trajektory such that enters set  $\Omega$  up to time of essential forward jump of starting time, which means that the whole boundary of  $\Omega$  is a set of egress points in a specyfic sens.

*Proof.* We will prove by induction principle for time scales (Proposition 2.1), that  $\Pi(\Omega_t, t, s) \subset \Omega_s$  for  $s, t \in \mathbb{T}$  where  $s \leq t$ .

Obviously  $\Pi(\Omega_t, t, t) = \Omega_t$ .

We have  $\text{ess}\sigma(t) = \sigma(t)$  for right-scattered points, therefore:  $\Pi(\text{cl}(\Omega^c)_t, t, \sigma(t)) \in (\Omega_{\sigma(t)})^c$ , so  $\Pi(\Omega_{\sigma(t)}, \sigma(t), t) \subset \Omega_t$ .

For points  $t$  in compact interval in time scale, that are not right boundary of that interval we have that  $\text{ess}\sigma(t)$  is right boundary of that interval. In particular we have  $\text{ess}\sigma(t) - t > \varepsilon > 0$ , so for  $s \in (t, t + \varepsilon)$  we have  $\Pi(\Omega_t, t, s) \subset \Omega_s$ .

For other right-dense points  $t$  we know that there exists sequence  $(t_n) \subset \mathbb{T}$  diminishing to  $t$  such that  $\mu(t_n) > 0$  for all  $n$ , for which we have  $\Pi(\Omega_{\sigma(t_n)}, \sigma(t_n), t_n) \subset \Omega_{t_n}$ , so by continuity of  $\Pi$  we find  $\varepsilon > 0$  such that for  $s \in (t, t + \varepsilon) \cap \mathbb{T}$  we have  $\Pi(\Omega_t, t, s) \subset \Omega_s$ .

For left-dense points we obtain needed property also by continuity.

By induction we have  $\Pi(\Omega_t, t, s) \subset \Omega_s$  for  $s, t \in \mathbb{T}$  where  $s \leq t$ .

Let us fix  $t_0 \in \mathbb{T}$ . We can choose sequence  $(t_n)_{n=0..∞} \subset \mathbb{T}$  increasing to  $\sup \mathbb{T}$  and define  $(\Omega^n)_{n=1..∞}$  by:

$$\Omega^n := \Pi(\Omega_{t_n}, t_n, t_0).$$

We know, that:

$$\Omega^{n+1} = \Pi(\Pi(\Omega_{t_{n+1}}, t_{n+1}, t_n), t_n, t_0) \subset \Omega^n,$$

therefore  $(\Omega^n)_{n=1..∞}$  is descending family of compact sets. Intersection of descending family of nonempty compact sets is nonempty and we choose  $x_0$  in that intersection. We now know, that trajectory starting in  $(t_0, x_0)$  is in  $\Omega$  up to any time  $t_n$ , and  $t_n \rightarrow \sup \mathbb{T}$ , so this is the searched trajectory. □

### 3.2 Approach 2.

Let  $b_i, c_i : \mathbb{T} \rightarrow \mathbb{R}$  (for  $i = 1..n$ ) be  $\Delta$ -differentiable functions where  $b_i < c_i$ . We define  $\Omega$  using that functions:

$$\Omega := \{(t, x) \in \mathbb{T} \times \mathbb{R}^n ; b_i(t) \leq x_i \leq c_i(t) \text{ for all } i\}$$

and we will use notation:

$\partial_{\mathbb{T}}\Omega := \{(t, x) \in \mathbb{T} \times \mathbb{R}^n ; b_i(t) \leq x_i \leq c_i(t) \text{ for all } i$   
 wherein at least one inequality is equality\}.

All points  $p \in \partial_{\mathbb{T}}\Omega$  can be presented in one of the following ways:

$$p = (t, x_1, \dots, x_{i-1}, b_i(t), x_{i+1}, \dots, x_n) \in \Omega_b^i$$

or

$$p = (t, x_1, \dots, x_{i-1}, c_i(t), x_{i+1}, \dots, x_n) \in \Omega_c^i.$$

**Theorem 3.3.** *Let  $b_i, c_i: \mathbb{T} \rightarrow \mathbb{R}$  be  $\Delta$ -differentiable and  $f: \mathbb{T} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  generates local  $\Delta$ -process  $\Pi$ . If for all  $(t, x) \in \Omega_b^i$  we have  $f(t, x) < b_i^\Delta(t)$  and for all  $(t, x) \in \Omega_c^i$  we have  $f(t, x) > c_i^\Delta(t)$  then for each  $t_0 \in \mathbb{T}$  there exists  $x_0 \in \Omega_{t_0}$ , such that solution starting in  $(t_0, x_0)$  remains in  $\Omega$  for every  $t \in \mathbb{T}$  bigger than  $t_0$ .*

*Proof.* (ad absurdum)

Let us notice, that for  $(t, x) \in \Omega_b^i$ , where  $\mu(t) > 0$ , we have  $\Pi(x, t, \sigma(t)) = x + \int_t^{\sigma(t)} f(\tau, x) \Delta\tau < x + \int_t^{\sigma(t)} b_i^\Delta(\tau) \Delta\tau = b_i(\sigma(t))$ , and similarly for  $\mu(t) = 0$  we have  $\Pi(x, t, t + \epsilon) = x + \int_t^{t+\epsilon} f(\tau, x) \Delta\tau < x + \int_t^{t+\epsilon} b_i^\Delta(\tau) \Delta\tau = b_i(t + \epsilon)$ , which means that trajectories starting in  $\Omega_b^i$  immediately leaves set  $\Omega$ . By analogy, trajectories starting in  $\Omega_c^i$  also immediately leaves set  $\Omega$ .

Let us fix  $t_0 \in \mathbb{T}$ . We extend linearly  $\Omega$  to continuous tube on  $[t_0, \sup \mathbb{T}] \cap \mathbb{R}$ :

$$\Omega^* := \Omega \cup \{(t, x) \in ([t_0, \sup \mathbb{T}] \cap \mathbb{R} \setminus \mathbb{T}) \times \mathbb{R}^n ;$$

$$b_i(t_a) + (b_i(t_b) - b_i(t_a)) \frac{t - t_a}{t_b - t_a} \leq x_i \leq c_i(t_a) + (c_i(t_b) - c_i(t_a)) \frac{t - t_a}{t_b - t_a}$$

for all  $i$ \}

where  $t_a, t_b \in \mathbb{T}$  are such that  $t_a < t < t_b$  and  $(t_a, t_b) \cap \mathbb{T} = \emptyset$ .

Naturally, for above  $t_a$  and  $t_b$  we know that set  $\Omega_{[t_a, t_b]}^*$  is convex.

Note that  $r: \partial\Omega^* \rightarrow \{t_0\} \times \partial\Omega_{t_0}$

$$r(t, x) := (t_0, (b_i(t_0) + \frac{c_i(t_0) - b_i(t_0)}{c_i(t) - b_i(t)}(x_i - b_i(t)))_{i=1..n})$$

---

This is Theorem from [6], with assumptions given in the the language of  $\Delta$ -processes.



is a retraction.

Now we will find continuous function from the set  $\Omega_{t_0}$  to the boundary of  $\Omega^*$ .

Negation of the thesis means that for all  $x \in \Omega_{t_0}$  we have finite time of exit

$$t_e(x) := \sup\{t \in \mathbb{T} ; \forall_{s \in \mathbb{T} \cap [t_0, t]} (t, \Pi(x, t_0, t)) \in \Omega\} < \sup \mathbb{T}.$$

If  $\mu(t_e(x)) = 0$ , then  $(t_e(x), \Pi(x, t_0, t_e(x)))$  is in boundary of  $\Omega^*$ .

If  $\mu(t_e(x)) \neq 0$ , then  $(t_e(x), \Pi(x, t_0, t_e(x))) \in \Omega$  and  $(\sigma(t_e(x)), \Pi(x, t_0, \sigma(t_e(x)))) \notin \Omega$  and by convexity of  $\Omega_{[t_e(x), \sigma(t_e(x))]}^*$  we obtain unique intersection of  $\Omega_{[t_e(x), \sigma(t_e(x))]}^*$  with interval connecting points  $(t_e(x), \Pi(x, t_0, t_e(x)))$  and  $(\sigma(t_e(x)), \Pi(x, t_0, \sigma(t_e(x))))$ . Denote this point by  $(t_e^*(x), x_e^*)$ .

Therefore, we can define function  $p: \Omega_{t_0} \rightarrow \partial\Omega^*$ :

$$p(x) := \begin{cases} (t_e(x), \Pi(x, t_0, t_e(x))), & \text{gdy } \mu(t_e(x)) = 0 \\ (t_e^*(x), x_e^*), & \text{gdy } \mu(t_e(x)) > 0. \end{cases}$$

By continuities of  $\Pi$  and tube  $\Omega^*$  we have continuous dependence  $(t_e^*(x), x_e^*)$  and  $(t_e(x), \Pi(x, t_0, t_e(x)))$  in respect to  $x$ . It is enough to show continuous dependence between  $(t_e^*(x), x_e^*)$  and  $(t_e(x), \Pi(x, t_0, t_e(x)))$ .

For point  $x_0$  such that  $t_e(x_0)$  is left-dense and right-scatered and  $(t_e(x_0), \Pi(x_0, t_0, t_e(x_0))) \in \Omega$  we have  $p(x) \rightarrow p(x_0)$  for  $x \rightarrow x_0$  with the time of exit  $t_e(x) < t_e(x_0)$ , and  $p(x) \rightarrow p(x_0)$  for  $x \rightarrow x_0$  with the time of exit  $t_e(x) = t_e(x_0)$ . By continuity of  $\Pi$  we can choose small enough neighborhood of  $x_0$ , which do not contains another points, so  $p$  is continuous in  $x_0$ .

By analogy, for left-scattered and righth-dense points  $t_e(x_0)$  we obtain continuity of  $p$  in such points. Therefore  $p$  is continuous.

Note that function  $R: \{t_0\} \times \Omega_{t_0} \rightarrow \{t_0\} \times \partial\Omega_{t_0}$

$$R(t_0, x) := r(p(x))$$

is a composition of continuous functions, so it is a retraction, which is in contradiction with the construction of set  $\Omega$ .  $\square$

## 4 Ważewski method for non-whole boundary egress set

In this section there are presented results from [8].

## 4.1 Notation

Let  $B(x, r)$  denote an open ball centered in  $x \in \mathbb{R}^2$  and with a radius  $r$ ,  $D(x, r) = \text{cl } B(x, r)$ ,  $S(x, r) = \partial B(x, r)$  and  $S^1 := S(0, 1)$ .

**Proposition 4.1** (Shöenflies theorem). *Any homeomorphism  $h : S^1 \rightarrow h(S^1) \subset \mathbb{R}^2$  can be extended to a homeomorphism  $\tilde{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ .*

*In particular, for any homeomorphism  $h : S^1 \rightarrow h(S^1) \subset \mathbb{R}^2$  there exists a homeomorphism  $\hat{h} : D(0, 1) \rightarrow \hat{h}(D(0, 1)) \subset \mathbb{R}^2$  such that the set  $\hat{h}(S^1)$  is a boundary of  $\hat{h}(D(0, 1))$  and the equality  $h(x) = \hat{h}(x)$  holds for all  $x \in S^1$ .*

Let  $A \subset \mathbb{T} \times \mathbb{R}^2$ . Then we define:

$$A_t := \{x \in \mathbb{R}^2 ; (t, x) \in A\}.$$

Let  $\Theta : \mathbb{T} \times S^1 \rightarrow \mathbb{R}^2$  be a continuous function such that:

- $\Theta_t : S^1 \rightarrow \Theta_t(S^1) \subset \mathbb{R}^2$ , where  $\Theta_t(x) = \Theta(t, x)$ , is a homeomorphism,
- $\Theta(t, s) = \Theta(\sigma(t), s)$ .

For all  $t \in \mathbb{T}$  let  $\Omega_t$  be a closure of a bounded open set surrounded by the curve  $\Theta(t, S^1)$  and

$$\Omega := \bigcup_{t \in \mathbb{T}} \{t\} \times \Omega_t.$$

For such construction we will say that  $\Omega$  is  $\Theta$ -bounded. In particular  $\Omega$  can be a constant tube  $\Omega = \mathbb{T} \times \Omega_0$ , where  $\Omega_0$  is homeomorphic to  $D(0, 1)$ .

We consider the following parts of the set  $\Omega$ :

$$\partial_{\mathbb{T}}\Omega := \Theta(\mathbb{T} \times S^1) = \bigcup_{t \in \mathbb{T}} \{t\} \times \partial(\Omega_t)$$

$$\begin{aligned} \partial_{\mathbb{T}}\Omega^+ := \bigcup_{t \in \mathbb{T}} \{t\} \times \text{cl}\{x \in \mathbb{R}^2 ; (t, x) \in \partial_{\mathbb{T}}\Omega \wedge \exists r > 0 \forall y \in B(x, r) \cap \Omega_t \forall \lambda \in (0, 1) \lambda x + \\ + (1 - \lambda)y \in \text{int } \Omega\} \end{aligned}$$

---

It is a well known property of planes, which is presented for example in [12, pp. 68, 72].

In other words,  $(\partial_{\mathbb{T}}\Omega^+)_t$  is a closure of the set of points in  $\partial(\Omega_t)$  that have strictly convex neighborhoods in  $\Omega_t$ .

For any maps  $f: \mathbb{T} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  we define where it makes sense:

$$f_t(x) := f(t, x),$$

$$\Phi_g(\cdot, \cdot), \text{ a local flow generated by the equation } y' = g(y)$$

$$w_g(x), \text{ a duration of a solution in a local flow } \Phi_g \text{ started in } x$$

We focus our attention at the following subsets of  $\Omega$ :

- Set of *egress points*:

$$E := \{(t, x) \in \partial_{\mathbb{T}}\Omega ; y' = f_t(y) \text{ generates a local flow } \Phi_{f_t} \\ \text{and } \Phi_{f_t}(x, (0, s]) \not\subset \Omega_t \text{ for any } s \in (0, w_{f_t}(x))\}$$

- Set of *escape points*:

$$Es := \{(t, x) \in \Omega ; \mu(t) \neq 0 \text{ and } x + \mu(t)f(t, x) \notin \text{int}\Omega_{\sigma(t)}\}$$

## 4.2 Theorems

Now we will prove the main theorem of the paper.

**Theorem 4.2.** *Let  $f: \mathbb{T} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a map such that:*

- (H0) *equation  $x^\Delta(t) = f(t, x(t))$  generates a local  $\Delta$ -process  $\Pi$ ,*
- (H1) *for all  $t \in \mathbb{T}$  a function  $\Pi(\cdot, t, \sigma(t))$  preserves an orientation of  $\mathbb{R}^2$ ,*
- (H2)  *$\Omega$  is  $\Theta$ -bounded (see section 4.1),*
- (H3) *there exists a closed set  $W \subsetneq S^1$  such that  $W$  is not a retract of  $D(0, 1)$  and  $\Theta(\mathbb{T} \times W) = E$ ,*
- (H4) *if  $\mu(t) \neq 0$ , then*

$$\text{(H4a)} \quad \Pi(E_t, t, \sigma(t)) \cap \Omega_{\sigma(t)} = \emptyset,$$

$$\text{(H4b)} \quad \Pi(\text{int}\Omega_t, t, \sigma(t)) \cap \partial_{\mathbb{T}}\Omega_{\sigma(t)} \subset E_{\sigma(t)}.$$

*Then, for all  $t_0 \in \mathbb{T}$ , there exists a point  $x_0 \in \Omega_{t_0}$  such that the solution starting in  $(t_0, x_0)$  remains in  $\Omega$  for all  $t \in \mathbb{T}$  bigger than  $t_0$ .*

*Proof.* (ad absurdum)

Let us fix for a while point  $t \in \mathbb{T}$  such that  $\mu(t) \neq 0$ .

By assumption (H4a), we know that  $E_t \subset Es_t$ .

By definition of  $Es$  we also know that  $\Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t \subset \partial(Es_t)$ .

By assumptions (H2) and (H3) we know that  $\Omega_t = \Omega_{\sigma(t)}$  and  $E_t = E_{\sigma(t)}$ . This allows us to define the following continuous tube:

$$\tilde{E} = \{(s, x) \in \mathbb{R} \times \mathbb{R}^2 ; (sup\{t \in \mathbb{T} ; t < s\}, x) \in E\}.$$

In other words we fill the interstices caused by a time scale.

We will construct a continuous function  $w_t : Es_t \rightarrow [t, \sigma(t)] \times E_t \subset \tilde{E}$  such that:

**w1**  $\forall_{x \in E_t} w_t(x) = (t, x)$

**w2**  $\forall_{x \in \Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t} w_t(x) = (\sigma(t), \Pi(x, t, \sigma(t)))$

which we will use to construct a continuous function from  $\Omega_{t_0}$  to  $\tilde{E}$  and by that, a retraction from  $D(0, 1)$  to  $W$ .

Let  $I$  be the set of indices of connected components  $E_t^i$  of  $\Pi(E_t, t, \sigma(t))$ . By assumption (H4a) we know that each set  $E_t^i$  is contained in a corresponding connected component  $\gamma_i$  of  $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t)) \setminus \partial_{\mathbb{T}}\Omega_t$ . The curve  $\gamma_i$  cuts from  $cl(\Omega_t^c)$  a closed bounded connected set denoted by  $Es_t^i$ .

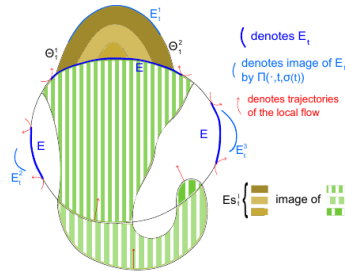


Figure 1

Figure 1

By assumption (H3) we know that  $E_t \neq \partial_{\mathbb{T}}\Omega_t$  so a boundary of  $Es_t^i$  is a closed curve and a sum of four curves:  $\theta_i^1$ ,  $E_t^i$ ,  $\theta_i^2$  and  $\partial_{\mathbb{T}}\Omega \cap Es_t^i$  (each of them homeomorphic to line segments), so it is homeomorphic to  $S^1$  (see Figure 1). By assumption (H1) the sets  $\partial_{\mathbb{T}}\Omega_t$  and  $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t))$

have the same orientation, and therefore  $E_t^i$  and  $\partial_{\mathbb{T}}\Omega \cap Es_t^i$  have an opposite orientation on the boundary of  $Es_t^i$ , so we can parameterize that boundary to obtain a homeomorphism  $h_i: \partial T_i \rightarrow \partial(Es_t^i)$  such that:

- $T_i$  is a trapezoid with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(a_i, 1)$ ,  $(b_i, 1)$  where  $[a_i, b_i] \subset [0, 1]$ ,
- $h_i([0, 1], 0) = E_t^i$ ,
- $h_i([a_i, b_i], 1) = \partial_{\mathbb{T}}\Omega_t \cap Es_t^i$ ,
- $\forall_{x \in [a_i, b_i]} h_i(x, 0) = \Pi(h_i(x, 1), t, \sigma(t))$ .

By the Shöenflies theorem we can extend  $h_i$  to a homeomorphism  $\hat{h}_i: T_i \rightarrow Es_t^i$ . If  $\Pi(Es_t, t, \sigma(t)) \setminus \bigcup_{i \in I} Es_t^i \neq \emptyset$ , then with other con-

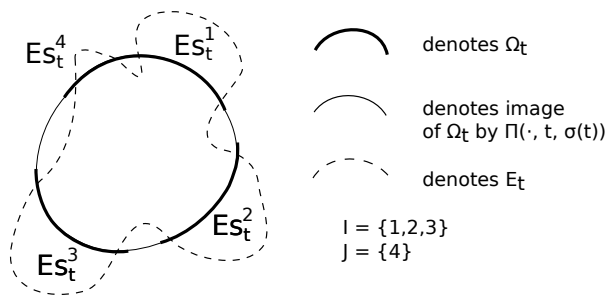


Figure 2

nected components (indexed with elements of some set  $J$ ) we make similar sets  $Es_t^j$ , each of them bounded by a part of  $\partial(\Omega_t)$  and a part of  $\Pi(\partial(\Omega_t), t, \sigma(t))$ , so bounded by a curve homeomorphic to  $S^1$  (see Figure 2). Then we have a homeomorphism  $h_j: \partial T_j \rightarrow \partial(Es_t^j)$  such that:

- $T_j$  is a triangle with vertices  $(1/2, 1/2)$ ,  $(0, 1)$ ,  $(1, 1)$ ,
- $h_j([0, 1], 1) = \partial_{\mathbb{T}}\Omega_t \cap Es_t^j$ .

and again by the Shöenflies theorem we can extend  $h_j$  to a homeomorphism  $\hat{h}_j: T_j \rightarrow Es_t^j$ .

---

With that construction  $\theta_i^1$  and  $\theta_i^2$  are the images of the side edges of  $T_i$ . In that construction point  $h_j(1/2, 1/2)$  is free to choose.

Now we know that  $\Pi(Es_t, t, \sigma(t)) \subset \bigcup_{i \in I} Es_t^i \cup \bigcup_{j \in J} Es_t^j$  which is the sum of disjoint sets, therefore we can define  $w_t: Es_t \rightarrow [t, \sigma(t)] \times E_t \subset \tilde{E}$ ,

$$w_t(x) := \begin{cases} (t + \mu(t)p_2(\cdot), \Pi(\hat{h}_i(p_1(\cdot), 0), \sigma(t), t))(\hat{h}_i^{-1}(\Pi(x, t, \sigma(t))))), & \Pi(x, t, \sigma(t)) \in Es_t^i \ (i \in I) \\ (t + \mu(t)p_2(\cdot), \Pi(\hat{h}_i(p_1(\cdot), 1), \sigma(t), t))(\hat{h}_i^{-1}(\Pi(x, t, \sigma(t))))), & \Pi(x, t, \sigma(t)) \in Es_t^j \ (j \in J) \\ (\sigma(t), \Pi(x, t, \sigma(t))) & \Pi(x, t, \sigma(t)) \in E_{\sigma(t)} \end{cases}$$

where  $p_1$  and  $p_2$  are projections respectively onto the first and second variables. By construction it is a continuous function.

Notice that for  $x \in E_t$  there exists a unique  $i \in I$  such that  $\Pi(x, t, \sigma(t)) \in E_t^i$ , and consequently  $y := \hat{h}_i^{-1}(\Pi(x, t, \sigma(t))) \in [0, 1] \times \{0\}$ . Since  $p_1(y) = y$  and  $p_2(y) = 0$ , we obtain that  $w_t(x) = (t, \Pi(\hat{h}_i(y), \sigma(t), t)) = (t, x)$ . Hence property **w1** is satisfied.

Moreover, if  $x \in \Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t$ , then  $\Pi(x, t, \sigma(t)) \in E_{\sigma(t)}$ , so also property **w2** is fulfilled.

Falsity of thesis means that for every  $x \in \Omega_{t_0}$  we have:

$$t_e(x) := \sup\{t \in \mathbb{T} ; \forall_{s \in \mathbb{T} \cup [t_0, t]} (t, \Pi(x, t_0, t)) \in \Omega\} < \sup \mathbb{T}.$$

If  $\mu(t_e(x)) = 0$ , then  $(t_e(x), \Pi(x, t_0, t_e(x)))$  is already in  $E \subset \tilde{E}$ .

If  $\mu(t_e(x)) \neq 0$ , then  $(t_e(x), \Pi(x, t_0, t_e(x))) \in \Omega$  and  $(\sigma(t_e(x)), \Pi(x, t_0, \sigma(t_e(x)))) \notin \Omega$ . So we can use the function  $w_{t_e(x)}$  to it.

Therefore, we can define  $r: \Omega_{t_0} \rightarrow \tilde{E}$ ,

$$r(x) := \begin{cases} (t_e(x), \Pi(x, t_0, t_e(x))), & \text{if } \mu(t_e(x)) = 0, \\ w_{t_e(x)}(\Pi(x, t_0, t_e(x))), & \text{if } \mu(t_e(x)) \neq 0. \end{cases}$$

Take any point  $x$  such that  $\mu(t_e(x)) = 0$ .

Then for each  $\varepsilon > 0$  there exists  $\tau \in \mathbb{T}$  such that  $0 < \tau - t_0 < \varepsilon$  and  $\Pi(x, t_0, \tau) \notin \Omega_\tau$  so, by the continuity of  $\Pi$ , for each  $\varepsilon > 0$  there exist  $\delta > 0$  and  $\tau \in \mathbb{T}$  such that  $0 < \tau - t_0 < \varepsilon$  and  $\Pi(B(x, \delta), t_0, \tau) \cap \Omega_\tau = \emptyset$ .

Therefore

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{y \in B(x, \delta)} t_e(y) < t_e(x) + \varepsilon.$$

Similarly we show that for each point  $x$  such that  $t_e(x) - \rho(t_e(x)) = 0$  we have

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{y \in B(x, \delta)} t_e(y) > t_e(x) - \varepsilon.$$

Using a continuity of  $\Pi$  we get a continuity of  $r$  in all  $x$  such that  $t_e(x)$  is a dense point of  $\mathbb{T}$ . Furthermore, properties **w1** and **w2** guarantee a continuity of  $r$  in every point  $x$  such that  $\mu(t_e(x)) \neq 0$  or  $t_e(x) - \rho(t_e(x)) \neq 0$ .

Hence  $r$  is continuous for all points in  $\Omega_{t_0}$ .

By the Shöenflies theorem we can extend  $\Theta_t$  to  $\hat{\Theta}_t : D(0, 1) \rightarrow \Omega_t$  for every  $t \in \mathbb{T}$ . Using this we define a map  $R : D(0, 1) \rightarrow W$ :

$$R(y) := \Theta_{t_e(\hat{\Theta}_{t_0}(y))}^{-1} \circ p_2 \circ r(\hat{\Theta}_{t_0}(y)).$$

For all  $y \in W$  we have  $R(y) = \Theta_{t_0}^{-1} p_2 r(\Theta_{t_0}(y)) = \Theta_{t_0}^{-1}(\Pi(\Theta_{t_0}(y), t_0, t_0)) = y$  and, by the continuity of  $r$ , we get that  $R$  is a retraction, what contradicts assumption (H3).  $\square$

The geometric assumption (H4') in the next theorem corresponds to the assumption (H4) and may occur to be easier to check.

**Theorem 4.3.** *Assume that:*

- (H0) equation  $x^\Delta(t) = f(t, x(t))$  generates a local  $\Delta$ -process  $\Pi$ ,
- (H1) for all  $t \in \mathbb{T}$  a function  $\Pi(\cdot, t, \sigma(t))$  preserves an orientation of  $\mathbb{R}^2$ ,
- (H2)  $\Omega$  is  $\Theta$ -bounded,
- (H3') there exists a closed set  $W \subsetneq S^1$  such that  $W$  is not a retract of  $D(0, 1)$  and  $\Theta(\mathbb{T} \times W) = \partial_{\mathbb{T}} \Omega^+ = E$ ,
- (H4') if  $\mu(t) \neq 0$ , then  $\Omega_t \subset x + T_{\Omega_t}(x)$  for all  $x \in \text{int}_{\partial_{\mathbb{T}} \Omega_t} E_t$ , where  $T_{\Omega_t}(x)$  is a Bouligand tangent cone of the set  $\Omega_t \in \mathbb{R}^2$  in a point  $x$  ( $T_K(x) = \{v \in \mathbb{X} ; \liminf_{h \rightarrow 0^+} \frac{d(x+hv, K)}{h} = 0\}$ ).

Then, for all  $t_0 \in \mathbb{T}$ , there exists a point  $x_0 \in \Omega_{t_0}$  such that a solution starting in  $(t_0, x_0)$  remains in  $\Omega$  for all  $t \in \mathbb{T}$  bigger than  $t_0$ .

*Proof.* To use Theorem 4.2 it is sufficient to show that, if  $\mu(t) \neq 0$ , then  $\Pi(E_t, t, \sigma(t)) \cap \Omega_{\sigma(t)} = \emptyset$  and  $\Pi(\text{int} \Omega_t, t, \sigma(t)) \cap \partial_{\mathbb{T}} \Omega_{\sigma(t)} \subset E_{\sigma(t)}$ .

Let us fix  $t \in \mathbb{T}$  such that  $\mu(t) \neq 0$ .

We have that  $\Pi(x, t, \sigma(t)) = x + \mu(t)f_t(x)$  and, for all  $x \in E_t$ , vectors  $f_t(x)$  are directed outside the set  $\Omega_t$  so a local strict convexity of points in  $E_t$  (assumption (H3')) guarantees that each connected component  $E_{t,i}$

of  $E_t$  has no common points with  $\Pi(E_{t,i}, t, \sigma(t))$ . Moreover, assumption (H4') ensures that the set  $\Pi(E_{t,i}, t, \sigma(t))$  is outside of the rest of  $\Omega_{\sigma(t)}$ , so the first part is fulfilled.

A connected component  $I_{t,i}$  of  $\partial_{\mathbb{T}}\Omega_t \setminus E_t$  has only points without strictly convex neighborhoods in  $\Omega_t$ . All of that points are not egress points in a local flow, so  $f_t(x)$  are directed inside the set  $\Omega_t$ . If there were  $y \in I_{t,i} \cap \Pi(I_{t,i}, \sigma(t), t)$  and  $[y, \Pi(y, t, \sigma(t))] \not\subset \partial_{\mathbb{T}}\Omega_t$ , then  $I_{t,i}$  would be a part of a spiral shaped curve which end is a beginning of a part of  $E_t$ , which would contradict assumption (H4'). Therefore there exists a small enough neighborhood  $O_{t,i}$  of  $I_{t,i}$  such that an image of  $O_{t,i} \cap \text{int}\Omega_t$  has no common points with  $I_{t,i}$ .

$\partial_{\mathbb{T}}\Omega_t$  is homeomorphic to  $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t))$  so, if  $E_{t,i}$  borders on  $I_{t,j}$ , then their images have to border as well. Therefore an image of  $I_{t,j}$  cuts out subset  $\Omega_t^j$  of  $\Omega_t$  that contains  $I_{t,j}$ . Moreover the image of  $\partial_{\mathbb{T}}\Omega_t$  does not have selfintersections so, in particular, an image of  $I_{t,j}$  is the only part of the image of  $\Omega_t$  that can have common points with  $\Omega_t^j$ . It means that

$$\Pi(\text{int}\Omega_t, t, \sigma(t)) \cap (\partial_{\mathbb{T}}\Omega_{\sigma(t)} \setminus E_{\sigma(t)}) = \emptyset$$

what was needed to prove. □

*Example 4.4.* Let  $\mathbb{T} = \bigcup_{n \in \mathbb{N}} [2n, 2n + 1]$  and  $f(t, (x, y)) = \left( \frac{e^{-t}(1-|y|\sin(t\pi))-2x}{3}, \frac{2y+\sin x}{5} \right)$ . We are interested in existence of trajectory convergent to  $(0, 0)$ .

We will want to use Theorem 4.2 taking  $\Omega := \bigcup_{n \in \mathbb{N}} \bigcup_{t \in [2n, 2n+1]} \{t\} \times \{(x_1, x_2) \in \mathbb{R}^2 ; -e^{(n-t)/3} \leq x_i \leq e^{(n-t)/3}\}$ .

Firstly, for  $t = 2n + 1$  and  $x_1, x_2, y_1, y_2 \in \mathbb{R}$  we have

$$\begin{aligned} & \|f(t, (x_1, y_1)) - f(t, (x_2, y_2))\| = \\ & = \|(2(x_2 - x_1)/3, (2(y_1 - y_2) + \sin(x_1) - \sin(x_2))/5)\| \leq \\ & \leq \|(2/3, 3/5)\| \|(x_1 - x_2, y_1 - y_2)\| < \|(x_1, y_1) - (x_2, y_2)\|, \end{aligned}$$

therefor we have  $L(t)\mu(t) < 1$ , so assumptions of Proposition 2.11 are met.

The set  $\Omega$  is selected so that  $\Omega_t = \Omega_{\sigma(t)}$  and is  $\Theta$ -bounded, where  $\Theta(t, (x, y)) = \frac{e^{(n-t)/3}}{\sup\{|x|, |y|\}}(x, y)$ , for  $t \in [2n, 2n + 1]$ .

We will find  $E$ .



For all  $t \in [2n, 2n + 1]$  and  $-e^{(n-t)/3} \leq x \leq e^{(n-t)/3} = |y|$  we have  $(t, (x, y)) \in \partial_{\mathbb{T}}\Omega$  and  $|\sin(x)| < |y|$ , therefore these are the egress points. Whereas for each  $t \in [2n, 2n + 1]$  i  $-e^{(n-t)/3} < y < e^{(n-t)/3} = |x|$  we have  $(t, (x, y)) \in \partial_{\mathbb{T}}\Omega$  and  $|e^{-t}(1 - |y|\sin(t\pi))| \leq e^{-t} < e^{(n-t)/3} = |x|$ , therefore vectors  $f(t, (x, y))$  are directed to the center of set  $\Omega_t$  and  $\left| \frac{e^{-t}(1 - |y|\sin(t\pi)) - 2x}{3} \right| > \left| \frac{1}{3}x \right| = \left| \frac{d}{dt}e^{(n-t)/3} \right|$ , thus these points are entry points.

For  $t = 2n + 1$  i  $(x, y) \in E_t$  we have  $\Pi((x, y), t, \sigma(t)) = (x + \frac{e^{-t} - 2x}{3}, y + \frac{2y + \sin(x)}{5}) \notin \Omega_{\sigma(t)}$ , so the assumption (H4a) is met.

For  $t = 2n + 1$  i  $(x, y) \in \Omega_t$  we have similarly:  $\Pi((x, y), t, \sigma(t)) = (x + \frac{e^{-t/3} - 2x}{3}, y + \frac{2y + \sin(x)}{5})$ , therefore the first coordinate is inside the segment  $[\frac{e^{-t/3} - e^{n/3 - t/3}}{3}, \frac{e^{-t/3} + e^{n/3 - t/3}}{3}]$ , which means that there are no common points with  $\partial_{\mathbb{T}}\Omega_t \setminus E_t$ , so assumption (H4b) is met too.

All assumptions are satisfied, therefore there exists trajectory remaining in the set  $\Omega$ , which is convergent to  $(0, 0)$  (from the selection of the set  $\Omega$ ).

### 4.3 Remarks

At first we notice that holes homeomorphic to balls in  $\Omega_t$  are available in Theorem 4.2. Indeed, for  $n$  holes we can consider:  $\Theta : \mathbb{T} \times \bigoplus_{j=0}^n S^1 \rightarrow \mathbb{R}^2$  such that  $\Theta(\{t\} \times \bigoplus_{k=0}^n S^1)$  is homeomorphic to  $S(0, 1) \cup \bigcup_{k=1}^n S((0, (k-1)/n), 1/3n)$ .

In the second remark we observe that properties of a local  $\Delta$ -process  $\Pi$  are essential, not of  $f$  itself, so in all approaches we can change our understanding of a solution of  $x^\Delta(t) = f(t, x(t))$  and treat it as a function that fulfills the equation  $\mathbb{T}$ -almost everywhere (in a Sobolev space on a time scale). Moreover, we can change assumptions to a  $\mathbb{T}$ -almost everywhere form. It is important when we look for possible generalizations to differential inclusions or multivalued  $\Delta$ -processes.

A proof technique presented in Section 4 cannot be repeated in higher dimensions because the Shönflies theorem does not raise up to them. The following open problem appears:

---

That concept had been introduced in [11].

**Open problem:**

Is it possible to use in higher dimensional spaces the geometric idea presented in the proof of Theorem 4.2 under some additional restrictions to  $\Theta$  or  $\Pi$ ?

Nevertheless, this geometric idea opens new perspectives in the Ważewski retract method on time scales and allows us to study more classes of systems (for example hyperbolic systems).

## References

- [1] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20** (1947), 279-313.
- [2] R. Srzednicki, *Ważewski method and Conley index*, Handbook of differential equations, 591-684, Elsevier/North-Holland, Amsterdam, 2004.
- [3] G. Gabor and M. Quincampoix, *On existence of solutions to differential inclusions remaining in a prescribed closed subset of a finite-dimensional space*, J. Differential Equations 185 (2002), No. 2, 483-512.
- [4] C.V. Coffman, *Asymptotic behavior of solution of ordinary difference equations*, Trans. Amer. Math. Soc., 110 (1964), 22-51.
- [5] J. Diblík, *Anti-Lyapunov method for systems of discrete equations*, Nonlinear Anal. 57 (2004), no. 7-8, 1043-1057.
- [6] J. Diblík, M. Ružičková, Z. Šmarda, *Ważewski's method for systems of dynamic equations on time scales*, Nonlinear Analysis 71 (2009), e1124-e1131.
- [7] L. Adamec, *A theorem of Ważewski and dynamic equations on time scales*, J. Difference Equ. Appl. 13 (2007), no. 1, 63-78.
- [8] G. Gabor, S. Ruszkowski *Ważewski theorem on time scales with a set of egress points that is not the whole boundary*, Nonlinear Anal., Vol. 75 no. 18, 6541-6549.

- 
- [9] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
- [10] M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.
- [11] A. Cabada, D.R. Vivero, *Expression of the Lebesgue  $\Delta$ -integral on time scales as a usual Lebesgue integral; application to the calculus of  $\Delta$ -antiderivatives*, Math. Comput. Modelling 43 (2006), no. 1-2, 194-207.
- [12] E.E. Moise, *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag, New York, Berlin, Heidelberg, 1977