Збірник праць Ін-ту математики НАН України 2013, том 10,N6, 224–235

*V. Sharko, D. Gol'cov*

# Semi-free $R^1$ action and Bott map

## 1 Introduction

Let $M^n$ be a compact closed manifold of dimension at least 3. We study the $R^1$-Bott functions on $M^n$. Separately investigated $R^1$-invariant Bott functions on $M^{2n}$ with a semi-free circle action which has finitely many fixed points. The aim of this paper is to find exact values of minimal numbers of singular circles of some indices of $R^1$-invariant Bott functions on $M^{2n}$.

Closely related to $R^1$-Bott function on a manifold $M^n$ is a more flexible object, the decomposition of round handle of $M^n$. In its turn, to study the round handles decomposition of $M^n$ we use a diagram, i.e. a graph which carries the information about the handles.

## 2 $R^1$-Bott maps

Let $M^n$ be a smooth manifold and $f : M^n \to \mathbf{R^1}$ smooth function or $f : M^n \to \mathbb{R}^1$ non-homotopy to zero a smooth map. Suppose that $x \in M^n$ one of its critical points of $f$. In neighborhood $U$ of critical point $x$ in both cases the map $f$ can be viewed as a function with values in $\mathbf{R}$. Consider the Hessian $\Gamma_x(f) : T_x \times T_x \to \mathbf{R}$ at this point. Recall that the index of the Hessian is called the maximum dimension of $T_x$, where $\Gamma_x(f)$ is negative definite. The index of $\Gamma_x(f)$ is called the index of the critical point $x$, and the corank of $\Gamma_x(f)$ is called the corank of $x$. Suppose that the set of critical points of $f$ forms a disjoint union of smooth submanifolds $K_j^i$ whose their dimensions do not exceed $n-1$. A connected critical submanifold $K_{j_0}^{i_0}$ is called **non-degenerate** if the

Hessian is non-degenerate on subspaces orthogonal to $K_{j_0}^{i_0}$ (i.e. has corank equal to $n - i_0$) at each point $x \in K_{j_0}^{i_0}$.

**Definition 2.1.** *A mapping $f : M^n \to \mathbb{R}^1$ is called a Bott map if all of its critical points form nondegenerate critical submanifolds which do not intersect the boundary of $M^n$.*

Consider the following important example of Bott map:

**Definition 2.2.** *A mapping $f : M^n \to \mathbb{R}^1$ is called an $R^1$-Bott map if all of its critical points form nondegenerate critical circles.*

Note that an $R^1$-Bott map do not exist on any smooth manifold (see Theorem 2.3).

**Theorem 2.1.** *Let $M^n$ be a smooth closed manifold and suppose that on $M^n$ there is $R^1$- Bott map $f : M^n \to \mathbb{R}^1$. Denote by $\gamma \subset M^n$ its critical circle and let $f(\gamma) = a$. Then there is interval $(a-\varepsilon, a+\varepsilon) \subset \mathbb{R}^1$ and a system of coordinates in a neighborhood of $\gamma$ of one of the following types:*

1) *Trivial $\nu : S^1 \times D^{n-1}(\varepsilon) \to M^n$; where $D^{n-1}(\varepsilon)$, a disc of radius $\varepsilon$, $\nu(R^1 \times 0) = \gamma$, and $f(\nu(\theta, x)) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_{n-1}^2$, for $(\theta, x) \in S^1 \times D^{n-1}(\varepsilon)$.*

2) *Twisted $\tau : ([0,1] \times D^{n-1}(\varepsilon)/\sim) \to M^n$, where $\tau$ is a smooth embedding such that $(\tau([0,1]) \times 0/\sim) = \gamma$ and $f(\tau(t,x)) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_{n-1}^2$, for $(t,x) \in (\tau : [0,1] \times D^{n-1}(\varepsilon)/\sim)$. Here $([0,1] \times D^{n-1}(\varepsilon)/\sim)$ is diffeomorphic to $S^1 \times D^{n-1}(\varepsilon)$ by identifying $0 \times D^{n-1}(\varepsilon)$ and*
*$1 \times D^{n-1}(\varepsilon)$ by the mapping:*
*$(0, x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_{n-1}) \leftrightarrow (1, -x_1, ..., x_\lambda, -x_{\lambda+1}, ..., x_{n-1})$.*

*The number $\lambda$ is called the index of the critical circle $\gamma$.*

Let $M^n$ be a smooth manifold, and $f : M^n \to R^1$ an $R^1$-Bott map. Each nice $R^1$-Bott map defines a filtration on manifold $M^n : \mathbb{M}_0(f) \subset \mathbb{M}_1(f) \subset ... \subset \mathbb{M}_{n-1}(f) \subset M^n$. The existence of a nice $R^1$-Bott map from manifold $M^n$ into the circle is equivalent to existance of a $R^1$-round handle decomposition on the manifold $M^n$. We recall some necessary definitions.

**Definition 2.3.** *We define an n-dimensional round handle $R_\lambda$ of index $\lambda$ by $M_\lambda = M^1 \times D^\lambda \times D^{n-\lambda-1}$, where $D^i$ is a disc of dimension i.*
*Define twisted n-dimensional round handle $TM_\lambda$ of index $\lambda$ $(0 < \lambda < n-1)$ by $TM_\lambda = [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim$, where identification is given by the map: $(0, x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_{n-1}) \leftrightarrow (1, -x_1, ..., x_\lambda, -x_{\lambda+1}, ..., x_{n-1})$.*

**Definition 2.4.** *We say that the manifold $M^n_\lambda$ is obtained from a smooth manifold $M^n$ by attaching a round handle of index $\lambda$ if*
*$M^n_\lambda = M^n \bigcup_\varphi S^1 \times D^\lambda \times D^{n-\lambda-1}$, where $\varphi : R^1 \times \partial D^\lambda \times D^{n-\lambda-1} \longrightarrow \partial N^n$ is a smooth embedding.*

*Manifold $M^n_\lambda$ is obtained from a smooth manifold $M^n$ by gluing a twisted round handles of index $\lambda$, if $M^n_\lambda = N^n \bigcup_\varphi [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim$, where $\varphi : ([0,1] \times \partial D^\lambda \times D^{n-\lambda-1}/\sim) \to M^n$ is a smooth embedding.*

**Definition 2.5.** *The $M^1$- round handle decomposition on the closed manifold $M^n$ is called a filtration*

$$M^{n-1} \times [0, \varepsilon] \bigcup M^n_0(R) \subset M^n_1(R) \subset ... \subset M^n_{n-1}(R) = M^n,$$

*where $M^{n-1}$ is a closed submanifold of $M^n$, the manifold $M^n_i(R)$ obtained from the manifold $M^n_{i-1}(R)$ by gluing round and twisted round handles of index i .*

In what follows we recall the relationship between $S^1$ and the decomposition by round handles ([11]).

**Theorem 2.2.** *Let $M^n$ be a smooth closed manifold. The following two conditions are equivalent:*

1) *On the manifold $M^n$ there is a nice $R^1$-Bott map with the critical circles $\gamma_1, ..., \gamma_k$ of index $\lambda_1, ..., \lambda_k$ with trivial coordinate systems and critical circles $\tilde{\gamma}_1, ..., \tilde{\gamma}_l$ of indices $\mu_1, ..., \mu_l$ with twisted coordinate systems.*

2) *Manifold $M^n$ admits a decomposition by round handles consisting of round handles $R_{\lambda_1}, ..., R_{\lambda_k}$ of index $\lambda_1, ..., \lambda_k$ and of twisted round handles*
*$TR_{\mu_1}, ..., TR_{\mu_l}$ of indices $\mu_1, ..., \mu_l$ so that the critical circle $\gamma_i$ corresponds to a round handle $R_{\lambda_i}$ $(1 \le i \le k)$, and the critical circle $\tilde{\gamma}_j$ corresponds to a twisted round handle $TR_{\mu_j}$ $(1 \le j \le l)$.*

Thus each nice $R^1$-Bott map from manifold $M^n$ into the $\mathbb{R}^1$ generates a round handle decomposition of $M^n$ and vice versa.

We are interested in conditions when an $R^1$-Bott map on $M^n$ has the property that all of its critical circles have trivial coordinate system. We recall the necessary facts from an [4].

**Lemma 2.1.** *Let $M^n$ be a smooth closed manifold, $f : M^n \to \mathbb{R}^1$ an $R^1$-Bott map, and $c$ its critical value. Suppose $\varepsilon > 0$, and that on the interval $[c-\varepsilon, c+\varepsilon]$ there are no other critical values. Assume that on the surface level $f^{-1}(c)$ there are critical circles $\gamma_1, ..., \gamma_k$ of indices $\lambda_1, ..., \lambda_k$ with trivial coordinate systems and there are critical circles $\tilde{\gamma}_1, ..., \tilde{\gamma}_l$ of indices $\mu_1, ..., \mu_l$ with twisted coordinate systems, then the homology groups $H_*(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$ is generated exactly by the handles which correspond to the critical circles*
*$\gamma_1, ..., \gamma_k, \tilde{\gamma}_1, ..., \tilde{\gamma}_l$. Each circle $\gamma_i$ generates two subgroups that are isomorphic to $\mathbf{Z}$, a direct product of the homology group $H_{\lambda_i}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$, and the other in the homology group $H_{\lambda_{i+1}}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$. Each circle $\tilde{\gamma}_j$ generates a subgroup $\mathbf{Z}_2$ which is direct product in a group $H_{\mu_j}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$.*

**Corolary 2.1.** *Let $M^n$ be a smooth closed manifold, $f : M^n \to \mathbb{R}^1$ an $S^1$-Bott map, and $c_1, ..., c_k$ its critical values. Suppose $\varepsilon_i > 0 (1 \le i \le k)$ such that the interval $[c_i - \varepsilon_i, c_i + \varepsilon_i]$ has no other critical values. Then on a level surface $f^{-1}(c_i)$ there are only critical circles with trivial coordinate systems if and only if the nonzero homology groups $H_*(f^{-1}[c_i - \varepsilon_i, c_i + \varepsilon_i], f^{-1}(c_i - \varepsilon_i), \mathbf{Z})$ are free Abelian groups.*

Thus we have a homological criterion when $R^1$-Bott map do not have critical circle with twisted coordinate systems.

In the next section, we give another class of $R^1$-Bott map which do not possess the critical circle with twisted coordinate systems.

**Definition 2.6.** *Let $M^n$ be a smooth closed manifold. The number*
*$\chi_i(M^n) = \mu(H_i(M^n, \mathbf{Z})) - \mu(H_{i-1}(M^n, \mathbf{Z})) + \ldots + (-1)^{i+1}\mu(H_0(M^n, \mathbf{Z}))$ is called the i-th Euler characteristic of $M^n$,where $\mu(H)$ is a minimal number of generators $H$.*

**Definition 2.7.** *A dimension $\lambda$ of closed manifold $M^n$ is called singular if $H_\lambda(M^n, \mathbf{Z})$ is a nonzero finite group distinct from $\mathbf{Z}_2 \oplus ... \oplus \mathbf{Z}_2$ and $\chi_{\lambda-1}(M^n) = \chi_{\lambda+1}(M^n) = 0$.*

**Definition 2.8.** *Let $M^n$ be a smooth closed manifold. A round handle decomposition is called quasiminimal, if one of the following holds:*

1) *the number of round handles of index $i$ equals to $\rho(\chi_i(M^n)) + \varepsilon_i$, where $\varepsilon_i = 0$,if dimension $i+1$ is nonsingular and $\varepsilon_i = 1$,if dimension $i+1$ is singular,*

2) *the number of round handles of index $i$ equals to $\rho(\chi_i(M^n))$, if dimension $i+1$ is singular, then there is only one handle of index $i+2$.*

*In both cases, the number of round handles of index $i+1$ equals to $\rho(\chi_{i+1}(M^n))$. A round handle decomposition is called minimal, if number of round handles of index $i$ equals to $\rho(\chi_i(M^n))$ for all $i$.*

Using the decomposition of manifold on handles and the diagram technique, we can easily prove the following fact [4].

**Proposition 2.1.** *Let $M^n$ be a smooth closed simply connected manifold $(n > 5)$. Then $M^n$ admits a* **quasiminimal** *decomposition into round handles. If manifold $M^n$ have not singular dimensions, then $M^n$ admits a* **minimal** *decomposition into round handles.*

**Definition 2.9.** *Let the manifold $M^n$ admits $R^1$-Bott function, then* **$R^1$-Morse number** $M_i^{R^1}(M^n)$ **of index** *$i$ is the minimum number of singular circles of index $i$ taken over all $R^1$-Bott functions on $M^n$.*

**Lemma 2.2.** *Let on a closed manifold $M^n$ exist a smoth function $f : M^n \to \mathbb{R}$ such that each connected component of the singular set $\Sigma_f$ of $f$ is either a nondegenerate critical point $p_i (i = 1, ..., k)$ or a nondegenerate critical circle $S^1_j (j = 1, ..., l)$. Then the Euler characteristic of the manifold $M^n$ is equal to $\chi(M^n) = \sum_{i=1}^{k} (-1)^{index(p_i)}$.*

**Proof.** It is known that for any Morse function on the manifold $M^n$ $g : M^n \to \mathbb{R}$ with critical points $p_i (i = 1, ..., q)$ there is the formula $\chi(M^n) = \sum_{i=1}^{q} (-1)^{index(p_i)}$. By small perturbation of the function $f$ any non-degenerate critical circle $S^1_j$ of index $\lambda$ can be replaced by non-degenerate critical points of idexes $\lambda$ and $\lambda + 1$ [1]. Therefore the contribution in the formula of Euler characteristic this critical points will not give and we obtain the desired formula. □

# 3 Manifolds with free $R^1$-action

Let on smooth manifold $M^n$ there is smooth free circle action. Then of course the set $M^n/S^1$ is a manifold and natural projection $p : M^n \to$

$M^n/S^1$ is fibre bundle. Any smooth $R^1$-invariant map $f : M^n \to \mathbb{R}^1$ from the manifold $M^n$ into the circle $\mathbb{R}^1$ is called an $R^1$**-invariant round Bott map** if each connected component of the singular set $\Sigma_f$ is non-degenerate critical circle.

It is clear that if $f$ be a $R^1$-invariant round Bott map from the manifold $M^n$ then it projection $\pi_*(f) : M^n/S^1 \to \mathbb{R}^1$, is a Morse map. And conversaly, if $g : M^n/S^1 \to \mathbb{R}^1$ be a Morse map from the manifold $M^n/S^1$ then $\pi_*^{-1}(g) = g \circ \pi : M^n \to \mathbb{S}^1$ is $R^1$-invariant round Bott map from the manifold $M^n$. The critical point of the index $\lambda$ of the map $g$ correspond to critical circle of the index $\lambda$ of the map $\pi_*^{-1}(g)$.

**Definition 3.1.** *Let on smooth manifold $M^n$ there are smooth free circle action $\theta : M^n \times S^1 \to M^n$ and $R^1$-invariant round Bott map $f : M^n \to \mathbb{S}^1$. For the triple $(M^n, \theta, f)$* $R^1$**-equivariant round Morse-Bott number of index** *$i$, $\mathfrak{M}_i^{eqS^1}(M^n, \theta, f)$ is the minimum number of singular circles of index $i$ taken over all homotopic to $f$ $R^1$-invariant round Bott map from $M^n$ into $\mathbb{R}^1$.*

**Definition 3.2.** *Let on smooth manifold $M^n$ there is Morse maps $f : M^n \to \mathbb{R}^1$. For the couple $(M^n, f)$* **Morse-Novikov number of index** *$i$, $\mathfrak{M}_i(M^n, f)$ is the minimum number of critical points of index $i$ taken over all homotopic to $f$ Morse maps from $M^n$ into $\mathbb{R}^1$.*

It is clear that there is following fact.

**Corolary 3.1.** *Let on smooth manifold $M^n$ there is smooth free circle action $\theta : M^n \times R^1 \to M^n$ and let $p : M^n \to M^n/R^1$ is natural projection. Suppose that $f : M^n/R^1 \to \mathbb{R}^1$ be a Morse map. Then $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f \cdot p) = \mathfrak{M}_i(M^n/S^1, f)$.*

**Definition 3.3.** *Let on smooth manifold $M^n$ there is smooth free circle action $\theta : M^n \times R^1 \to M^n$. Then this circle action is* **minimal** *if there exist $R^1$-invariant round Bott map $f : M^n \to \mathbb{R}^1$ such that $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f) = \mathfrak{M}_i^{S^1}(M^n, f)$ for all $i$.*

Suppose that on smooth compact manifold $M^n (n > 6)$ there is smooth free circle action $\theta : M^n \times R^1 \to M^n$ and let $p : M^n \to M^n/R^1$ is natural projection. Suppose that $\pi_1(M^n) \approx \pi_1(M^n/R^1) \approx \mathbf{Z}$. Then from from results of Novikov [2] it follows that

$$\mathfrak{M}_i(M^n/R^1, f) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$$

for any non-homotopy to zero Morse map $f : M^n/R^1 \to \mathbb{R}^1$. Therefore corollary 3.1 implies that $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f \cdot p) = \mathfrak{M}_i^{R^1}(M^n, f)$

**Theorem 3.1.** *Let on smooth compact manifold $M^n (n > 6)$ there is smooth free circle action. Suppose that $\pi_1(M^n) \approx \pi_1(M^n/S^1) \approx \mathbf{Z}$. Then this circle action is minimal if and only if*

$$\mu(H_i(M^n/S^1, Z) + \mu(TorsH_{i-1}(M^n/S^1, Z) = \rho(\chi_i(M^n))$$

*for all $i$.*

**Proof**. **Necessary.** Suppose that on $M^n$ there is minimal smooth free circle action. If $n > 6$ from results of Novikov [2] it follows that Morse number in dimension $i$ of the manifold $M^n/R^1$ is equal $\mathfrak{M}_i(M^n/S^1) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$. There is equality $\mathfrak{M}_i(M^n/S^1) = \mathfrak{M}_i^{eqR^1}(M^n)$. Because of the condition of minimal free circle action there is equality $\mathfrak{M}_i(M^n/R^1) = \mathfrak{M}_i^{eqR^1}(M^n) = \mathfrak{M}_i^{R^1}(M^n) = \rho(\chi_i(M^n))$.

**Sufficiently.** Consider on manifold $M^n/R^1$ Morse function with the number of critical points of index $i$ equal $M_i(M^n/R^1) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$. By the construction and condition of the theorem we have the equalities $M_i(M^n/S^1) = M_i^{eqR^1}(M^n) = \rho(\chi_i(M^n))$. But $M_i^{R^1}(M^n) = \rho(\chi_i(M^n))$ and therefore free action of $R^1$ is minimal. □

# 4 Manifolds with semi-free $R^1$-action

Let $M^{2n}$ be a closed smooth manifold with semi-free $R^1$-action which has only isolated fixed points. It is known that every isolated fixed point $p$ of a semi-free $R^1$-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n}$. More precisely, for every isolated fixed point $p$ there exist an open invariant neighborhood $U$ of $p$ and a diffeomorphism $h$ from $U$ to an open unit disk $D$ in $\mathbb{C}^n$ centered at origin such that $h$ is conjugate to the given $S^1$-action on $U$ to the $S^1$-action on $\mathbb{C}^n$ with weight $(1, \dots, 1)$. We will use both complex, $(z_1, \dots, z_n)$, and real coordinates $(x_1, y_1, \dots, x_n, y_n)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ with $z_j = x_j + \sqrt{-1}y_j$. The pair $(U, h)$ will be called a **standard chart** at the point $p$. Let $f : M^{2n} \to \mathbb{R}^1$ be a smooth $R^1$-invariant map from the manifold $M^{2n}$ into the circle $\mathbb{R}^1$. Denote by $\Sigma_f$

the set of singular points of the map $f$. It is clear that the set of isolated singular points $\Sigma_f(p_j) \subset \Sigma_f$ of $f$ coincides with the set of fixed points $M^{R^1}$.

For a nondegenerate critical point $p_j$ there exist a standard chart $(U_j, h_j)$ such that on $U_j$ the map $f$ is given by the following formula:

$$f = f(p) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2.$$

Notice that the index of nondegenerate critical point $p_j$ is always even.

Denote by $\Sigma_f(R^1)$ the set singular points of the function $f$ that are disconnected union of circles. These circles will be called singular.

A circle $s \in \Sigma f_(R^1)$ is called nondegenerate if there is an $R^1$-invariant neighborhood $U$ of $s$ on which $R^1$ acts freely and such that the point $\pi(s)$ is nondegenerate for the function $\pi_*(f) : U/R^1 \to \mathbb{R}$, induced on $U/R^1$ by the natural map $\pi : U \to U/R^1$. An invariant version of Morse lemma says that there exist an $R^1$-invariant neighborhood $U$ of the circle $s$ and coordinates $(x_1, \ldots, x_{2n-1})$ on $U/R^1$ such that the function $\pi_*(f)$ has the following presentation:

$$\pi_*(f) = \pi_*(f(\pi(s))) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{2n-1}^2.$$

By definition $\lambda$ is the **index** of singular circle $s$.

**Definition 4.1.** *A smooth $S^1$-invariant function $f : M^{2n} \to \mathbb{R}$ on a manifold $M^{2n}$ with a semi-free circle action which has isolated fixed points is called : $R^1_*$-Bott function if each connected component of the singular set $\Sigma_f$ is either a nondegenerate fixed point or a nondegenerate critical circle.*

**Theorem 4.1.** *Assume that $M^{2n}$ is the closed manifold with a smooth semi-free circle action which has isolated fixed points $p_1, \ldots, p_k$. Let for any fixed point $p_j$ consider standard chart $(U_j, h_j)$ and function*

$$f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2$$

*on $U_j$, where $\lambda_j$ is an **arbitrary integer** from $0, 1, \ldots, n$.*

*Then there exist an $R^1$-invariant $R^1_*$-Bott function $f$ on $M^{2n}$ such that $f = f_j$ on $U_j$.*

**Proof.** Consider on $U_j$ the function $f_j$. Let $\pi_*(f_j) : U_j/S^1 \to \mathbb{R}$, continuos function induced on $U_j/R^1$ by the natural map $\pi : U_j \to U_j/R^1$. It is clear that function $\pi_*(f_j)$ is smooth on manifold $(U_j \setminus$

$p_j)/R^1$. Denote by $g$ smooth extension functions $\pi_*(f_j)$ on $M^{2n}/R^1$. By small deformation of the function $g$, that is fixed on $U_j/R^1$, we shall find function $g_1$ on $M^{2n}/R^1$ such that $g_1$ equal $\pi_*(f_j)$ on $U_j/R^1$ and $g_1$ have only non-degenerate critical points on $M^{2n} \setminus \bigcup(U_j/R^1)$. Then the function $f = g_1 \circ p$ satisfy conditions of the theorem. $\square$

**Theorem 4.2.** *The number of fixed points of any smooth semi-free circle action on $M^{2n}$ with isolated fixed points is always even and equal to the Euler characteristic of the manifold $M^{2n}$.*

$f_1 = f_1(p_1) + |z_1|^2 + \ldots + |z_n|^2$ on $U_1$ and $f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_n|^2$

on $U_j$ $(2 \leq j \leq l)$ and extend such functions to $S^1$-invariant Bott function $f$ on manifold $M^{2n} \setminus U_1 \bigcup U_2 \bigcup ... \bigcup U_l$. We suppose that $U_j$ is diffeomorfic to open disk $D^{2n}$ for any $j$. Consider manifold $V^{2n} = W^{2n} \setminus \bigcup U_j$. The boudary of manifod $V^{2n}$ is disconnected union of spheres $S^{2n-1}$. By construction of manifold $V^{2n}$ there is free cirle action. The boundary of the manifold $V^{2n}/S^1$ is disconnected union of complex projective spaces $\mathbb{CP}^{n-1}$. If the number of the boundary components of the manifold $V^{2n}/S^1$ is odd then we glue pairwise boundary components and obtain compact smoth manifold with with boundary $\mathbb{CP}^{n-1}$. From the well known fact that the manifold $\mathbb{CP}^{n-1}$ is non-cobordant to zero it follows that the number of fixed points of any smooth semi-free circle action on $M^{2n}$ with isolated fixed points is even. The value of the Euler characteristic $\chi(M^{2n}) = 2k$ is follow from Lemma 3.4.$\square$

**Definition 4.2.** *Let $f$ be an $R^1$-invariant $S^1_*$-Bott function for smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$ on a closed manifold $M^{2n}$. Denote by $\lambda_j$ the index of a critical point $p_j$ of the function $f$. The* **state** *of the function $f$ is the collection of numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, which we will be denoted by $St_f(\Lambda)$. It is clear that all numbers $\lambda_j$ are even and $(0 \leq \lambda_j \leq 2n)$.*

**Remark 4.1.** *It follows from Theorem 4.2 that for every smooth semi-free circle action on a closed manifold $M^{2n}$ with isolated fixed points $p_1, \ldots, p_{2k}$ and any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$ there exists an $R^1$-invariant $R^1_*$-Bott functions $f$ on $M^{2n}$ with state $St_f(\Lambda)$.*

**Definition 4.3.** *Let $M^{2n}$ be a closed smooth manifold with smooth semi-free circle action which has finitely many fixed points $p_1, \ldots, p_{2k}$. Fix any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$.*

*The $R^1$-Morse number $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$ of index $i$ is the minimum numbers of singular circles of index $i$ taken over all $R^1$-invariant $R^1_*$-Bott functions $f$ on $M^{2n}$ with state $St_f(\Lambda)$.*

There is an unsolved problem: for a manifold $M^{2n}$ with a semi-free circle action which has finitely many fixed points **find exact values** of numbers $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$ .

# 5 About $R^1$-equivariant Morse numbers $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$

Let $M^{2n}$ be a compact closed manifold of dimension with semi-free circle action which has finite many fixed points $p_1, , ..., p_{2k}$. Denote by $\pi : M^{2n} \to M^{2n}/R^1$ canonical map. The set $M^{2n}/R^1$ is manifold with singular points $\pi(p_1), , ..., \pi(p_{2k})$. It is clear that neighborhood of any singular point is cone over $\mathbb{CP}^{n-1}$. If $f : M^{2n} \to \mathbb{R}$ be a smooth $R^1$-invariant $R^1_*$-Bott function on the manifold $M^{2n}$, then $\pi_*(f) : M^{2n}/R^1 \to \mathbb{R}$ is continuos function such that on smooth non-compact manifold $N^{2n-1} = M^{2n}/R^1 \setminus \bigcup_{j=1}^{2k} \pi(p_j)$ it is Morse function.

Choose an invariant neighborhood $U_i$ of the point $p_j$ diffeomorphic to the open unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_{j=1}^{2k} U_j$. Consider compact manifold $V^{2n-1} = (M^{2n} \setminus U)/R^1$, its boundary is a disconnected union of complex projective spaces $\partial V^{2n-1} = \mathbb{CP}_1^{n-1} \cup \ldots \cup \mathbb{CP}_{2k}^{n-1}$. It is clear that manifold $V^{2n-1} \setminus \partial V^{2n-1}$ and manifold $N^{2n-1}$ are diffeomorphic. We use a manifold $V^{2n-1}$ for the study of $R^1$-invariant $R^1_*$-Bott functions on the manifold $M^{2n}$ with states $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$. Let $\partial_0 V^{2n-1}$ be a part of boundary of $V^{2n-1}$ consist from $r$ component $\mathbb{C}P^{2n-2}$ $(2k-1 \geq r \geq 1)$, and $\partial_1 V^{2n-1} = \partial V^{2n-1} \setminus \partial_0 V^{2n-1}$. On the manifold with boundary $V^{2n-1}$ constructed Morse function $f : V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n-1}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function $f$ we constructed on the manifold $M^{2n}$ $R^1$-equivariant $R^1_*$-Bott function $F$ with the state $St(0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction $\pi_*(F)$ on $V$ coinside with $f$. Therefore Morse number of index $i$ $M_i(V^{2n-1}, \partial_0 V^{2n-1})$ of manifold with boundary $V^{2n-1}$ is equal $\mathcal{M}_i^{S^1}(M^{2n}, St(0, \ldots, 0, 2n, \ldots, 2n)$.

**Theorem 5.1.** *Let $M^{2n}$ $(2n > 8)$ be a closed smooth manifold admits a smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$. Then* *for the manifold $M^{2n}$ with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$*

$$\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda) = \mathbb{D}^i(V^{2n-1}, \partial_0 V^{2n-1}) + \widehat{S}^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) +$$

$$+\widehat{S}^{i+1}_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) + dim_{N(Z[\pi])}(H^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}))$$

*for* $3 \leq i \leq 2n-4$.

**Proof.** Choose an invariant neighborhood $U_i$ of the point $p_i$ diffeomorphic to the unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_i U_i$. Let $f_i$ be a function on $U_i$ equal

$$f_i = |z_1|^2 + \ldots + |z_n|^2, \text{ and } f_j \text{ on } U_j \text{ equal } f_j = 1 - |z_1|^2 - \ldots - |z_n|^2,$$

for $i = 1, ..., r$, $j = r+1, ..., 2k-r$. Consider the manifold $V^{2n} = (M^{2n} \setminus U)/R^1$. It is clear that its boundary is a disconnected union of complex projective spaces $\partial V^{2n} = \mathbb{C}P_1^{2n-2} \cup \ldots \cup \mathbb{C}P_{2k}^{2n-2}$.

Let $\partial_0 V^{2n}$ be a part of boundary of $V^{2n}$ consist from $r$ component $\mathbb{C}P^{2n-2}$, that corespondent $U_i$ and $\partial_1 V^{2n}$ be a part of boundary consist from component $\mathbb{C}P^{2n-2}$, that corespondent $U_j$. On manifold $V^{2n} = (M^{2n} \setminus U)/R^1$ constructed Morse function $f : V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function $f$ we constructed on manifold $M^{2n}$ $S^1$-equivariant $S^1_*$-Bott function $F$ with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction F on $U_i$ coinside with $f_i$, restriction F on $U_j$ coinside with $f_j$ and restriction $\pi_*(F)$ on $V$ coinside with $f$. Therefore Morse number of cobordism $V$ equal $\mathcal{M}^{\lambda}_{R^1}(M^{2n}, St(\Lambda))$ In the paper [12] there is value of Morse number of a cobordism. □

**Literature.**

1. *Barden D.* Simply-connected five manifolds // Annals of Math., 1965, 82 n3, p. 365-385.

2. *Novikov S.P.* Multivalued functions and functionals. An analogue of the Morse theory, Soviet Math. Dokl., 1981, 24 , p. 222–226.

3. *Asimov D.* Round handle and non-singular Morse-Smale flows // Ann. Math. - 1975, 102 n.1, p. 41 - 54.

4. *Bott R.* Lecture on Morse theory, old and new// Bull. Amer.Math. Sos. - 1982, 7 n.2, p. 331 - 358.

5. *Franks J.* Morse-Smale flows and homotopy theory // Topology - 1979, 18 n2, p. 199 - 215.

6. *Franks J.* Homology and Dynamical systems // CMBS Regional Conf. Series in Math.,n.49, Amer. Math. Sos., Providence, R.I., 1982.

7. *Kogan M.* Existence of perfect Morse functions on spaces with semi- free circle action // Journal of Symplectic Geometry - 2003, 1. n3, p. 829–850.

8. *Smale S.* On the structure of Manifolds // Amer. J. Math. - 1962, 84 n.3, p. 387 - 399.

9. *Miyoshi S.* Foliated round surgery of codimension-one foliated manifolds// Topology - 1983, 21 n.3, p. 245 - 262.

10. *Morgan J.W.* Non-singular Morse-Smale flows on 3-dimensional manifolds// Topology - 1979, 18 n.1, p. 41 - 53.

11. *Newhuse S., Peixoto M.* There is a simple arc joining any two Morse-Smale Flows // Asterisque - 1976, 31 , p. 16 - 41.

12. *Sharko V.V.* New $L^2$-invariants of chain complexes and applications, $C^*$-algebra and elliptic theory// Trends Math.- Basel, Switzerland: Birkhauser, -P. 291-321.2006

13. *Thurston W.* Existence of codimension-one foliation// Ann. Math. - 1976, 104 n.2, p. 249 - 268.