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Formal Aspects of Topological Complexity

We study the concept of topological complexity from the viewpoint of fibrewise Lusternik-Schnirelmann category and discuss certain formal aspects which include the equivalence of various descriptions, the axiomatic characterization, and the possibility to obtain a decomposition into Δ*-sets of different dimensions.*

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1. Introduction

The concept of topological complexity was introduced by M. Farber in [4, 5] in his study of the navigation problem in robotics. Broadly speaking, the navigation problem refers to the problem of finding a continuous motion that transforms a mechanical system from some given initial position to a desired final position. To give a mathematical formulation of this problem one introduces the so-called *configuration space*, i.e. a topological space that describes all possible states of the mechanical system. For such a configuration space X one then considers the space X^I of all continuous paths $\alpha\colon I \to X$, and the evaluation map $\mathrm{ev}\colon X^I \to X \times X$ that to a path α assigns its end-points, $\mathrm{ev}(\alpha) := (\alpha(0), \alpha(1))$. A *navigation plan* for X is a rule that takes as input a pair of points $x, y \in X$, and returns as output a path α in X starting at x and ending at y. In other words, a navigation plan is a section of the evaluation map, i.e. a function $s\colon X \times X \to X^I$ such that $\mathrm{ev} \circ s = 1_{X\times X}$. Observe that while the movement through the configuration space is always assumed to be continuous with respect to the topology of the configuration space, this is not necessarily the case for the navigation plan. In fact, one can easily show that a continuous navigation plan exists if and only if X is contractible. Thus, for non-contractible spaces one is naturally led to consider navigation plans that are continuous only when restricted to subsets of $X \times X$.

Farber [4] exploited the fact that $\mathrm{ev}\colon X^I \to X \times X$ is a fibration, and defined the topological complexity of path-connected space X to be the Schwarz genus [19] of the fibration ev, i.e. the minimal n for which $X \times X$ can be covered by open subsets $U_1, \ldots, U_n$ such that each of them admits a continuous section $s_i\colon U_i \to X^I$ of ev. A very similar approach was previously used by S. Smale [20] and A. Vassiliev [21] in their investigation of the topological complexity of algorithms for finding roots of polynomial equations. Observe that strictly speaking, the sections $s_i\colon U_i \to X^I$ do not determine a navigation plan for X because the elements of the open cover of X must overlap, so over their intersections one has a multiple choice of navigation plans. To avoid this difficulty, one may decompose $X \times X$ into disjoint subsets such that the restriction of some global navigation plan to each of them is continuous. Clearly for a non-contractible configuration space, every such global navigation plan must be discontinuous, and that fact is sometimes described as the instability of the navigation planning algorithm. Farber [5] tackled this problem and proved that the topological complexity provides a suitable measure for the level of this instability.

It is clear from the definition that the topological complexity $\mathrm{TC}(X)$ is a homotopy invariant of X, and so it has recently attracted a lot of interest among homotopy theorists. This resulted in a series of interesting developments, variations and reformulations of the original idea. In particular, methods from the classical Lusternik-Schnirellman (LS) category, in particular the Whitehead-Ganea approach was developed in a series of papers [11], [12] and [13] by G. Calcines and L. Vandembroucq.

The alternative fibrewise LS category viewpoint was introduced by N. Iwase and M. Sakai in [15], and further applied and developed in [8], [9] and [10]. The fibrewise formulation avoids the use of function spaces, so the resulting theory has more geometric flavour and opens the possibility of extensive application of the methods of LS category to problems in topological complexity. In the first two section of this paper we use the Iwase-Sakai approach to give a uniform overview of known facts about the absolute and relative topological complexity together with slick and efficient proofs. The remaining sections exploit the alternative approach to obtain a couple of new results on the axiomatic approach to the topological complexity and on some useful dimension-wise decompositions.

2. Topological complexity as fibrewise category

In this section we show that the topological complexity of X can be described in terms of decompositions of the product $X \times X$ into subsets that can be deformed into the diagonal. and investigate the relations between different kinds of such decompositions.

Let X be a path-connected space and let $\mathrm{ev}\colon X^I \to X \times X$ be the evaluation fibration $\mathrm{ev}(\alpha) = \big(\alpha(0), \alpha(1)\big)$. A subset $F \subseteq X \times X$ *admits a continuous navigation plan* if there is a continuous map $s\colon F \to X^I$ such that $\mathrm{ev} \circ s = 1_F$. Various descriptions of the topological complexity of X are related to different ways to decompose of $X \times X$ into subsets that admit continuous navigation plans. We may broadly distinguish four different approaches as follows.

1. Originally [4] the topological complexity of X was defined as the Schwarz genus of the fibraton $\mathrm{ev}\colon X^I \to X \times X$. The Schwarz genus of a fibration $p\colon E \to B$ is the minimal n for which B can be covered by n open sets $U_1, \ldots, U_n$, such that each of them admits a continuous local section $s_i\colon U_i \to E$ of p. The use of open covers is standard in homotopy theory and allows direct comparison with other invariants. For example, recall that $\mathrm{cat}(X)$, the *Lusternik-Schnirelmann category* of X, is the minimal n for which X can be covered by n open sets $U_1, \ldots, U_n$, such that each $U_i \hookrightarrow X$ is null-homotopic, (i.e. each U_i can be deformed to a point inside X). One then have the following basic estimate (cf. [7, Section 4.2])

$$\mathrm{cat}(X) \le \mathrm{TC}(X) \le \mathrm{cat}(X \times X) \le 2\,\mathrm{cat}(X) - 1\ . \tag{2.1}$$

2. For applications in robotics the unavoidable overlapping of the sets of an open cover of $X \times X$ sometimes creates problems because it introduces a level of ambiguity on which navigation plan should be used for pairs of points that lie in the intersections. It is therefore often preferable to use partitions of $X \times X$ into disjoint subsets, so that the choice of the navigation plan is uniquely determined by the input data. Furthermore, we want to avoid subspaces with bad local properties. For that reason Farber [5] considered decompositions of $X \times X$ as disjoint unions of euclidean neighbourhood retracts. Recall that X is an *euclidean neighbourhood retract* (ENR) if it is homeomorphic to a retract of an open subset of some euclidean space $\mathbb{R}^n$. More intrinsically, X is an ENR if it is locally compact, locally contractible, and embeddable in some euclidean space (see [2, Section IV,8]). The class of ENR's contains all finite-dimensional cell complexes and all manifolds. Then one can consider global navigation plans for X that are continuous when restricted to the elements of some ENR-partition of $X \times X$ (i.e. a decomposition into a disjoint union of ENR's). For example, Farber [5] proved that for a connected polyhedron X the topological complexity of X equals the minimal n for which $X \times X$ has an ENR-partition into n subsets that admit continuous navigation plans.

3. Navigation plans that come up in applications are often defined locally, on small subsets of the product $X \times X$. For example, we can describe simple-minded navigation plans on a polyhedron X as follows. We first choose a maximal tree T in the 1-skeleton of X. Then for each pair of vertices $x, y \in X$ we define a navigation plan on the product of open stars $\mathrm{st}(x) \times \mathrm{st}(y)$ by combining the unique path in T between x and y with the straight segments in the respective stars. The number of elements in such a cover of $X \times X$ by sets admitting navigation plans is in general much bigger then $\mathrm{TC}(X)$. Since most of the elements are disjoint one may aggregate them to produce covers with less elements but this is usually impractical. There us however a different way to measure the complexity of such navigation plans. Given a cover $\mathcal{U}$ of X the *weight* of $\mathcal{U}$ is the maximal number of elements of $\mathcal{U}$ that have non-empty intersection. We will see later on that the weights if such covers are bounded bellow by the topological complexity of X.

4. Finally we can combine locally defined navigation plans with the requirement that their domains of definition are disjoint ENR's. Given a global navigation plan $s\colon X \times X \to X^I$ and some cover $\{F_\lambda\}$ of $X \times X$ by mutually disjoint ENR's, such that the restrictions $s|_{F_\lambda}$ are continuous, Farber [5] defined the *order of instability* of this partition to be the weight of the cover $\{\overline{F}_\lambda\}$. Once again, the topological complexity turns out to be the precise lower bound for the orders of instability of such partitions.

We now turn our attention from navigation plans to deformations of subsets of $X \times X$, starting from the following simple observation: every continuous navigation plan $s\colon F \to X^I$ by adjunction determines a homotopy $\widehat{s}\colon F \times I \to X \times X$, given by

$$\widehat{s}(x, y, t) := \big(x, s(x, y)(1-t)\big).$$

Since $s(x, y)(0) = x$ and $s(x, y)(1) = y$ the homotopy $\widehat{s}$ is clearly a vertical (i.e along the second factor) deformation of F to a subset of the diagonal $\Delta X = \{(x, x) \in X \times X\}$. This was already noted in [6, Section 18] and further developed by Iwase and Sakai in [15]. The main advantage of this alternative viewpoint is that a deformation of a space is much easier to visualize than a map into a path space.

Every subset of $X \times X$ that can be vertically deformed to a subset of the diagonal will be called Δ-*set*. Various characterizations of topological complexity are summarized in the following theorem.

Theorem 1. *If X is an ENR then the topological complexity of X equals the minimal n for which one (and hence all) of the following conditions is satisfied.*

(1) *There exists a cover of $X \times X$ by n open Δ-sets.*
(2) *There exists a cover of $X \times X$ by n closed Δ-sets.*
(3) *There exists an ENR-partition of $X \times X$ into n disjoint Δ-sets.*
(4) *There exists a filtration $\emptyset = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = X \times X$ by closed subsets, such that each $F_i - F_{i-1}$ is a Δ-set.*
(5) *There exists a filtration $\emptyset = U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n = X \times X$ by open subsets, such that each $U_i - U_{i-1}$ is a Δ-set.*
(6) *There exists a filtration $\emptyset = C_0 \subseteq C_1 \subseteq \ldots \subseteq C_n = X \times X$ by locally compact subsets, such that each $C_i - C_{i-1}$ is a Δ-set.*
(7) *$X \times X$ admits a cover of weight n by open Δ-sets.*
(8) *$X \times X$ admits a cover of weight n by closed Δ-sets.*
(9) *There exists an ENR-partition of $X \times X$ into disjoint Δ-sets, whose order of instability equals n.*

Proof. (1) is just a reformulation of the definition of the Schwarz genus. (2) is equivalent to (1) because as in the case of Lusternik-Schnirelmann category (cf. [17]) for spaces that are normal and neighbourhood retracts one can always work with closed instead of open coverings, and vice versa. (3) follows from [7, Proposition 4.9]. (4)-(6) correspond to the characterizations of [7, Proposition 4.12]. (7),(8) follow from [7, Corollary 4.14]. Finally (9) follows from [6, Theorem 13.1]. □

We are now going to relate the characterization (1) in the above theorem to a special case of fibrewise Lusternik-Schnirelmann category. Take a Δ-set $U \subseteq X \times X$ and consider the projection $\pi\colon X \times X \to X$ of the product to the first factor. Restrictions of the homotopy that deforms U to the diagonal to the (possibly empty) intersections $V_x := U \cap \mathrm{pr}^{-1}(x) \subseteq \{x\} \times X$ yields a family of homotopies indexed by points of X that deform sets V_x within X to the point x. This precisely corresponds to the idea of a fibrewise deformation of set to a point, on which the following definition of fibrewise Lusternik-Schnirelmann category is based (cf. [18]). A *fibrewise pointed space* is a map $p\colon E \to B$ together with a section $s\colon B \to E$: we view this structure as a continuous family of pointed spaces $p^{-1}(b)$, each of them based at the point $s(b)$. Its *fibrewise Lusternik-Schnirelmann category* is the minimal n for which E can be covered by open sets $U_1, \ldots, U_n$ such that for each i there is a fibrewise homotopy deforming U_i to a subset of the section $s(B) \subset E$.

Let us consider the fibrewise pointed space over the base X whose total space is the product $X \times X$, $\pi\colon X \times X \to X$ is projection to the first factor and the section is given by the diagonal map $\Delta\colon X \to X \times X$. We will denote this fibrewise pointed space by $X \ltimes X$ where the semi-direct sign indicates that we have a 'twisted' familly of fibres indexed by the points of X, where the base 'acts' on the fibres by sliding the base-point. We may now conclude that $\mathrm{TC}(X)$ coincides with the fibrewise Lusternik-Schnirelmann category of $X \ltimes X$.

There are two important caveats regarding the role of the base-points (i.e. sections) that one must keep in mind when discussing the fibrewise category as related to the classical category. In the classical LS category the role of the base-points is minor, because for spaces with nice local behaviour the pointed and unpointed category coincide, and their value does not depend on the choice of the base-point. In fact, one can use the homotopy extension property and arrange that all sets of a categorical cover are deformed to the same point, and that all deformations are stationary at that point. Contrary to that, two sections of a fibrewise space may not be fibrewise homotopic, and the category with respect to one section can be completely different from the category with respect to some other section. For example the diagonal section of $\pi\colon S^2 \times S^2 \to S^2$ is clearly not homotopic to the constant section, and the fibrewise category of $\pi\colon S^2 \times S^2 \to S^2$ with respect to the diagonal section equals the topological complexity $\mathrm{TC}(S^2) = 3$, while the fibrewise category of π with respect to the constant section is the same as the ordinary category $\mathrm{cat}(S^2) = 2$.

The second point is even more delicate. First of all, we define (following [18]) the *fibrewise pointed category* of the fibrewise pointed space $p\colon E \to B$ with section $s\colon B \to E$ as the minimal n for which E can be covered by open sets $U_1, \ldots, U_n$ such that for each i $s(B) \subset U_i$ and the fibrewise homotopy deforming U_i to $s(B)$ is stationary on $s(B)$. The fibrewise pointed category is more adequate for the application of the homotopy-theoretical methods (cf. [18, Section 6], [15]), but it is not clear under what conditions the two notions coincide. In fact Iwase and Sakai [15] proposed a proof that pointed fibrewise category equals the unpointed fibrewise category for locally finite complexes but unfortunately their proof was flawed, see the Errata [16]. At the moment the best result in this direction is by A. Dranishnikov [3], who proved that the two versions of fibrewise category of X coincide when certain assumptions on the dimension of X are satisfied.

3. Subspace complexity

In this section we consider the topological complexity of subspaces of $X \times X$. We assume throughout that X is a Euclidean neighbourhood retract. Let $A \subseteq X \times X$ The *subspace topological complexity* of A, denoted $\mathrm{TC}_X(A)$ is the least integer n for which there exists a cover of A by n open Δ-subsets of $X \times X$. Of course, instead of covers by open sets we can use any of the equivalent descriptions of the topological complexity listed in Theorem 1. It is easy to see that the subspace complexity coincides with the relative complexity of A, which was defined in [7, Section 4.3] as the Schwarz genus of the restriction over A of the evaluation fibration $X^I \to X \times X$.

Let us list a few relations that follow immediately from the definition (most of them already appeared in the literature, cf. [7], Chapter 4 and in particular Section 4.3). First, we recover the topological complexity of X as

$$\mathrm{TC}(X) = \mathrm{TC}_X(X \times X). \tag{3.1}$$

If $X \subseteq Y$ and $A \subseteq B \subseteq X \times X$ then

$$\mathrm{TC}_Y(A) \leq \mathrm{TC}_X(B). \tag{3.2}$$

If $A, B \subseteq X \times X$ then

$$\mathrm{TC}_X(A \cup B) \leq \mathrm{TC}_X(A) + \mathrm{TC}_X(B). \tag{3.3}$$

Moreover, if A, B are separated open subsets of $X \times X$ (i.e. $\overline{A} \cap B = A \cap \overline{B} = \emptyset$) then

$$\mathrm{TC}_X(A \cup B) = \max\{\mathrm{TC}_X(A), \mathrm{TC}_X(B)\}. \tag{3.4}$$

The interplay between different characterizations given in Theorem 1 allows for unified and efficient proofs of the various estimates for topological complexity. To exemplify this approach we briefly summarize few most relevant results. we begin with a lemma that gives us plenty of Δ-sets.

Lemma 2. *Let X be a Euclidean neighbourhood retract.*

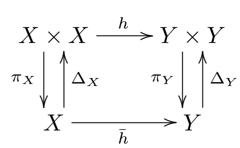
(1) *Any subspace of $X \times X$ that can be deformed within $X \times X$ into a Δ-set is itself a Δ-set. In particular, every product of two categorical subsets of X is a Δ-set (since it can be deformed to a point within $X \times X$).*
(2) *A union of a family of separated open Δ-sets is a Δ-set.*
(3) *If $h\colon X \ltimes X \to Y \ltimes Y$ is a homeomorphism of fibrewise pointed spaces then A is a Δ-set in $X \times X$ if, and only if $h(A)$ is a Δ-set in $Y \times Y$.*

Proof. (1) Let $A \subseteq X \times X$, and let $H\colon A \times I \to X \times X$ be a deformation of A, such that $A' = H_1(A)$ is a Δ-set. If we denote by $D'\colon A' \times I \to X \times X$ a vertical deformation of A' to the diagonal ΔX, than we obtain a vertical deformation D of A to the diagonal by the formula

$$\mathrm{pr}_2\overline{D}(x,y,t) := \begin{cases} \mathrm{pr}_2\,(H(x,y,3t)) & 0 \leq t \leq \frac{1}{3} \\ \mathrm{pr}_2\,(D'(H(x,y,1),3t-1)) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \mathrm{pr}_1\,(H(x,y,3-3t)) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

(2) Recall that a family of subsets of a topological space is separated if the closure of each of them does not intersect the others. Clearly, when open Δ-sets are separated, then their deformations to the diagonal combine to a continuous deformation of their union to the diagonal.

3) A homeomorphism $h\colon X \times X \to Y \times Y$ is a homeomorphism of fibrewise pointed spaces if there is a homeomorphism $\bar{h}\colon X \to Y$ such that $\bar{h} \circ \pi_X = \pi_Y \circ h$ and $h \circ \Delta_X = \Delta_Y \circ \bar{h}$, so that the following diagram commutes



Then a deformation $H\colon A \times I \to X \times X$ of A to the diagonal Δ_X yields a deformation

$$\overline{H}\colon h(A) \times I \to Y \times Y, \qquad H(y,y',t) := h(H(h^{-1}(y,y'),t)$$

of $h(A)$ to the diagonal Δ_Y. □

Part (1) of the above Lemma implies that every categorical subset of $X \times X$ is automatically a Δ-set, which immediately yields a relation between the subspace topological complexity and subspace category:

$$\mathrm{TC}_X(A) \leq \mathrm{cat}_{X\times X}(A). \tag{3.5}$$

If $B \subseteq X \times X$ can be deformed into some $A \subseteq X \times X$ (i.e., there is a deformation $H\colon B \times I \to X \times X$, such that $H_1(B) = H(B \times 1) \subseteq A$), then

$$\mathrm{TC}_X(B) \leq \mathrm{TC}_X(A). \tag{3.6}$$

In fact given a cover of A by Δ-sets $U_1, \ldots, U_n$, the pre-images $H_1^{-1}(U_1), \ldots, H_1^{-1}(U_n$ cover B and are also Δ-sets by (1) of Lemma 2. As a special case, if $B \subseteq X \times X$ can be deformed to its subset $A \subseteq B$, then by 3.2

$$\mathrm{TC}_X(A) = \mathrm{TC}_X(B). \tag{3.7}$$

Let X, Y be ENR's with $\mathrm{TC}(X) = m$ and $\mathrm{TC}(Y) = n$. Then by Theorem 1 (5) there exist a filtration $\emptyset = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_m = X \times X$ such that all $X_i - X_{i-1}$ are Δ-sets in $X \times X$ and a filtration $\emptyset = Y_0 \subseteq Y_1 \subseteq \ldots \subseteq Y_n = Y \times Y$ such that all $Y_j - Y_{j-1}$ are Δ-sets in $Y \times Y$. If we define $Z_k := \bigcup_{i+j=k+1} X_i \times Y_j$ we obtain a filtration $\emptyset = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_{m+n-1} = (X \times Y) \times (X \times Y)$. We directly verify that $Z_k - Z_{k-1} = \coprod_{i+j=k+1}(X_i - X_{i-1}) \times (Y_j - Y_{j-1})$ is a disjoint union of separated Δ-sets, and conclude that

$$\mathrm{TC}(X \times Y) < \mathrm{TC}(X) + \mathrm{TC}(Y). \tag{3.8}$$

Let G be a topological group. If $U \subseteq G$ is an open categorical set, that can be deformed to the unit $e \in G$ then $\bigcup_{g\in G}\{g\} \times gU$ is clearly a Δ-set in $G \times G$. It follows that a categorical cover of G gives rise to a cover of $G \times G$ by Δ-sets, hence

$$\mathrm{TC}(G) = \mathrm{cat}(G). \tag{3.9}$$

4. Axiomatic characterization of topological complexity

Some of the properties listed in the previous section are sufficient to characterize precisely the subspace topological complexity among integer-valued functions with similar properties. In fact, we are going to show that the formulas 3.2, 3.3 and 3.6, together with a normalization requirement are sufficient to determine the topological complexity of a space. This approach is analogous to the axiomatic characterization of the Lusternik-Schnirelmann category as in [1].

Let us define the *abstract topological complexity* on a space X to be a function denoted $\mathrm{tc}(\cdot)$ that assigns a positive integer to every non-empty subset A of $X \times X$ and satisfies the following properties:

- **(tc1)** $\mathrm{tc}(\Delta X) = 1$;
- **(tc2)** If $A \subseteq B \subseteq X \times X$ then $\mathrm{tc}(A) \leq \mathrm{tc}(B)$;
- **(tc3)** If $A, B \subseteq X \times X$ then $\mathrm{tc}(A \cup B) \leq \mathrm{tc}(A) + \mathrm{tc}(B)$;
- **(tc4)** If $A, B \subseteq X \times X$, and B can be vertically deformed within $X \times X$ to a subset of A
  then $\mathrm{tc}(B) \leq \mathrm{tc}(A)$.

By the results from the previous section we know that the subspace topological complexity $\mathrm{TC}_X(\cdot)$ satisfies the conditions for the abstract topological complexity. We may now consider the set of all abstract topological complexities and order them as follows: if tc_1 and tc_2 are two abstract topological complexities, let

$$\mathrm{tc}_1(\cdot) \leq \mathrm{tc}_2(\cdot) \quad \Longleftrightarrow \quad \mathrm{tc}_1(A) \leq \mathrm{tc}_2(A) \text{ for all } A \subseteq X.$$

Let $\mathrm{tc}(\cdot)$ be an abstract topological complexity, and let U be a non-empty Δ-subset of $X \times X$. Then U can be vertically deformed to a subset of ΔX, so by (tc1) and (tc4) we have $\mathrm{tc}(U) \leq \mathrm{tc}(\Delta X) = 1$, therefore $\mathrm{tc}(U)$. Furthermore, If $A \subseteq X \times X$ can be covered by n open Δ-subsets $U_1, \ldots, U_n$ of $X \times X$ then by (tc2) and (tc3)

$$\mathrm{tc}(A) \leq \mathrm{tc}(U_1 \cup \ldots \cup U_n) \leq \mathrm{tc}(U_1) + \ldots + \mathrm{tc}(U_n) = n.$$

Since $\mathrm{TC}_X(A)$ is precisely the minimal number of open Δ-subsets of $X \times X$ that are necessary to cover A we may conclude from the above discussion that

$$\mathrm{tc}(A) \leq \mathrm{TC}_X(A).$$

We have therefore proved the following result

Theorem 3. *The subspace topological complexity* $\mathrm{TC}_X(\cdot)$ *is the maximal element among all abstract topological complexities defined on subspaces of* $X \times X$.

5. Dimension-wise Δ-sets

The standard minimal decompositions of $S^n \times S^n$ into a disjoint union of ENR Δ-sets that yield the topological complexities of the spheres are well known. For odd-dimensional spheres we can take

$$A = \{(x, y) \in S^n \times S^n \mid x + y \neq 0\}$$

and

$$B = \{(x, y) \in S^n \times S^n \mid x + y = 0\},$$

and the dimensions are $\dim(A) = 2n$ and $\dim(B) = n$. On the other side, for even-dimensional spheres we may take

$$A = \{(x, y) \in S^n \times S^n \mid x + y \neq 0\},$$

$$B = \{(x, y) \in S^n \times S^n \mid x + y = 0\} - C,$$

and

$$C = \{(N, -N), (-N, N)\}$$

(where $N \in S^n$ denotes the north pole), and the respective dimensions of the sets involved are $2n, n$ and 0. One naturally wanders whether it is possible to achieve the same (i.e. Δ-sets of different dimensions) in the general case. We are going to prove this fact in the following form.

Theorem 4. *Let* X *be a connected ENR and let* $A \subseteq X \times X$ *be an ENR subset whose subspace topological complexity is* $\mathrm{TC}_X(A) = n$. *Then* A *can be decomposed as a disjoint union* $A = X_1 \sqcup \ldots \sqcup X_n$, *where each* X_i *is an ENR* Δ*-set and* $\dim(A) = \dim(X_1) > \dim(X_2) > \ldots \dim(X_n) \geq 0$.

In particular, if X *is a connected ENR whose topological complexity is* $\mathrm{TC}(X) = n$, *then* $X \times X = X_1 \sqcup \ldots \sqcup X_n$, *where* X_i *are ENR* Δ*-sets and* $2\dim(X) = \dim(X_1) > \dim(X_2) > \ldots \dim(X_n) \geq 0$.

The proof of the theorem is based on the following auxiliary result.

Lemma 5. *For every ENR subset* $A \subseteq X \times X$ *there exists an ENR subset* $B \subset X \times X$ *such that* $\mathrm{TC}_X(A) > \mathrm{TC}_X(B)$, $\dim(A) > \dim(B)$ *and* $(A - B)$ *is a* Δ*-set.*

Proof. For $TC_X(A) = 1$ we take $B := \emptyset$.

Let $\mathrm{TC}_X(A) = n$ and assume inductively that the claim holds for all $B \subseteq X \times X$ with $\mathrm{TC}_X(B) < n$. Let $U_1, \ldots, U_n$ be a cover of A by open Δ-sets in X. Then by the normality of X, and by the properties of the small inductive dimension, we can find an open set V_1 in X such that

$$A - U_2 - \ldots - U_n \subseteq V_1 \subseteq \overline{V}_1 \subseteq U_1,$$

and satisfying the requirement $\dim(\overline{V}_1 - V_1) < \dim(A)$. We can furthermore find an open cover $V_2, \ldots, V_n$ of $U_2 \cup \ldots \cup U_n$ such that $\overline{V}_i \subseteq U_i$ and $\dim(\overline{V}_i - V_i) < \dim(A)$.

Define $B := (\overline{V}_1 - V_1) \cup \ldots \cup (\overline{V}_n - V_n)$, so that clearly, $\dim(B) < \dim(A)$. Moreover, B is by the construction contained in the union $U_2 \cup \ldots \cup U_n$, hence $\mathrm{TC}_X(B) < \mathrm{TC}_X(A)$. Each component of $A - B$ is a Δ-set, as it contained in some U_i. Since the components of $A - B$ are separated 3.4 implies that $A - B$ itself is a Δ-set, which concludes the proof. □

Proof. (of Theorem 4)
If $\mathrm{TC}_X(A) = n$ we can inductively apply the above lemma to obtain spaces $A = A_1 \supset A_2 \ldots \supset A_n \supset A_{n+1} = \emptyset$ such that $\dim(A_i) > \dim(A_{i+1})$ and $(A_i - A_{i+1})$ are ENR Δ-sets. To obtain the decomposition stated in the theorem we let $X_i := A_i - A_{i+1}$. Moreover, it is clear that $\dim(A) = \dim X_1$. □

If X is a polyhedron with $\mathrm{TC}(X) = n$ then the above argument can be easily modified to obtain a filtration $\emptyset \leq X_1 \leq \ldots \leq X_n = X \times X$ by polyhedra whose dimension is strictly increasing, and such that each $X_i - X_{i-1}$ is a Δ-set. If X is $(p-1)$-connected then by Cellular approximation theorem every subcomplex of dimension less then p is a Δ-set, which implies that $\dim(X_2) \geq p$. It would be interesting to know (at least for the case when p divides $\dim(X)$) whether we can extend further the analogy with the spheres and obtain a filtration of $X \times X$ as above, by subpolyhedra whose dimensions are multiples of p.

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