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V.I. Gerasimenko

Institute of Mathematics of NAS of Ukraine, Kyiv

gerasym@imath.kiev.ua

Mean field asymptotic behavior of quantum particles with initial correlations

Dedicated to the 80 [th] anniversary of Prof. D.Ya. Petrina

In the paper we consider the problem of the rigorous description of the kinetic evolution in the presence of initial correlations of quantum particles. One developed approach consists on the description of the evolution of quantum many-particle systems within the framework of marginal observables in mean field scaling limit. Another method based on the possibility to describe the evolution of states within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation in case of initial states specified by a one-particle marginal density operator and correlation operators.

В роботі розглядається проблема строгого опису кінетичної еволюції за наявності початкових кореляцій квантових частинок. Один з розвинутих підходів полягає в описі еволюції квантових систем багатьох частинок в термінах маргінальних спостережуваних в скейлінґовій границі середнього поля. Ще один метод ґрунтується на можливості опису еволюції станів за допомогою одночастинкового маргінального оператора густини, який визначається узагальненим квантовим кінетичним рівнянням у випадку початкових станів заданих одночастинковим маргінальним оператором густини і кореляційними операторами.

1   Introduction

As is known the collective behavior of quantum many-particle systems can be effectively described within the framework of a one-particle marginal density operator governed by the kinetic equation in a suitable scaling limit of underlying dynamics. At present the considerable advances in the rigorous derivation of the quantum kinetic equations in the mean (self-consistent) field scaling limit is observed [1]-[6]. In particular, the nonlinear Schrödinger equation [3]-[10] and the Gross–Pitaevskii equation [7]-[15] was justified.

The conventional approach to this problem is based on the consideration of an asymptotic behavior of a solution of the quantum BBGKY hierarchy for marginal density operators constructed within the framework of the theory of perturbations in case of initial data specified by one-particle marginal density operators without correlations, i.e. such that satisfy a chaos condition [16],[17]. We note, that for the first time a perturbative solution of the quantum BBGKY hierarchy was constructed by D. Petrina [18] (see also [19]).

In paper [20] it was developed more general method of the derivation of the quantum kinetic equations. By means of a non-perturbative solution of the quantum BBGKY hierarchy constructed in [21] it was established that, if initial data is completely specified by a one-particle marginal density operator, then all possible states of many-particle systems at arbitrary moment of time can be described within the framework of a one-particle density operator governed by the generalized quantum kinetic equation (see also [22]). Then the actual quantum kinetic equations can be derived from the generalized quantum kinetic equation in appropriate scaling limits, for example, a mean field limit [23].

Another approach to the description of the many-particle evolution is given within the framework of marginal observables governed by the dual quantum BBGKY hierarchy [24]. In paper [25] a rigorous formalism for the description of the kinetic evolution of observables of quantum particles in a mean field scaling limit was developed.

In this paper we consider the problem of the rigorous description of the kinetic evolution in the presence of initial correlations of quantum particles. Such initial states are typical for the condensed states of quantum gases in contrast to the gaseous state. For example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition specified by correlations of the condensed state [26]. Thus, our goal consists in the

derivation of the mean field quantum kinetic equation including initial correlations.

We outline the structure of the paper. In section 2, we establish the mean field asymptotic behavior of marginal observables governed by the dual quantum BBGKY hierarchy. The limit dynamics is described by the set of recurrence evolution equations, namely by the dual quantum Vlasov hierarchy. Furthermore, the links of the dual quantum Vlasov hierarchy for the limit marginal observables and the quantum Vlasov-type kinetic equation with correlations are established. In section 3, we consider the relationships of dynamics described by marginal observables and within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation including initial correlations. In section 4, we develop one more approach to the description of the quantum kinetic evolution with correlations in the mean field limit. We prove that a solution of the generalized quantum kinetic equation with correlations is governed by the quantum Vlasov-type equation with correlations. The property of the propagation of initial correlations is also established. Finally, in section 5, we conclude with some perspectives for future research.

2 The kinetic evolution within the framework of marginal observables

The kinetic evolution of many-particle systems can be described within the framework of observables. We consider this problem on an example of the mean field asymptotic behavior of a non-perturbative solution of the dual quantum BBGKY hierarchy for marginal observables. Moreover, we establish the links of the dual quantum Vlasov hierarchy for the limit marginal observables with the quantum Vlasov-type kinetic equation in the presence of initial correlations.

2.1 Many-particle dynamics of observables

We consider a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics in the space $\mathbb{R}^3$. We will use units where $h = 2\pi\hbar = 1$ is a Planck constant, and $m = 1$ is the mass of particles.

Let the space $\mathcal{H}$ be a one-particle Hilbert space, then the n-particle space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ is a tensor product of n Hilbert spaces $\mathcal{H}$. We adopt the usual convention that $\mathcal{H}^{\otimes 0} = \mathbb{C}$. The Fock space over the Hilbert space $\mathcal{H}$ we denote by $\mathcal{F}_{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$.

The Hamiltonian H_n of the n-particle system is a self-adjoint operator with the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$

$$H_n = \sum_{i=1}^{n} K(i) + \epsilon \sum_{i_1 < i_2 = 1}^{n} \Phi(i_1, i_2), \tag{1}$$

where $K(i)$ is the operator of a kinetic energy of the i particle, $\Phi(i_1, i_2)$ is the operator of a two-body interaction potential and $\epsilon > 0$ is a scaling parameter. The operator $K(i)$ acts on functions ψ_n, that belong to the subspace $L_0^2(\mathbb{R}^{3n}) \subset \mathcal{D}(H_n) \subset L^2(\mathbb{R}^{3n})$ of infinitely differentiable functions with compact supports, according to the formula: $K(i)\psi_n = -\frac{1}{2}\Delta_{q_i}\psi_n$. Correspondingly, we have: $\Phi(i_1, i_2)\psi_n = \Phi(q_{i_1}, q_{i_2})\psi_n$, and we assume that the function $\Phi(q_{i_1}, q_{i_2})$ is symmetric with respect to permutations of its arguments, translation-invariant and bounded function.

Let a sequence $g = (g_0, g_1, \ldots, g_n, \ldots)$ be an infinite sequence of self-adjoint bounded operators g_n defined on the Fock space $\mathcal{F}_{\mathcal{H}}$. An operator g_n defined on the n-particle Hilbert space $\mathcal{H}_n = \mathcal{H}^{\otimes n}$ will be also denoted by the symbol $g_n(1, \ldots, n)$. Let the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$ be the space of sequences $g = (g_0, g_1, \ldots, g_n, \ldots)$ of bounded operators g_n defined on the Hilbert space $\mathcal{H}_n$ that satisfy symmetry condition: $g_n(1, \ldots, n) = g_n(i_1, \ldots, i_n)$, for arbitrary $(i_1, \ldots, i_n) \in (1, \ldots, n)$, equipped with the operator norm $\|\cdot\|_{\mathfrak{L}(\mathcal{H}_n)}$. We will also consider a more general space $\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ with the norm

$$\|g\|_{\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})} \doteq \max_{n \geq 0} \frac{\gamma^n}{n!} \|g_n\|_{\mathfrak{L}(\mathcal{H}_n)},$$

where $0 < \gamma < 1$. We denote by $\mathfrak{L}_{\gamma,0}(\mathcal{F}_{\mathcal{H}}) \subset \mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ the everywhere dense set in the space $\mathfrak{L}_{\gamma}(\mathcal{F}_{\mathcal{H}})$ of finite sequences of degenerate operators with infinitely differentiable kernels with compact supports.

For $g_n \in \mathfrak{L}(\mathcal{H}_n)$ it is defined the one-parameter mapping

$$\mathbb{R}^1 \ni t \mapsto \mathcal{G}_n(t)g_n \doteq e^{itH_n} g_n e^{-itH_n}, \tag{2}$$

where the Hamilton operator H_n has the structure (1). On the space $\mathfrak{L}(\mathcal{H}_n)$ one-parameter mapping (2) is an isometric $*$-weak continuous group of

operators. The infinitesimal generator $\mathcal{N}_n$ of this group of operators is a closed operator for the $*$-weak topology, and on its domain of the definition $\mathcal{D}(\mathcal{N}_n) \subset \mathfrak{L}(\mathcal{H}_n)$ it is defined in the sense of the $*$-weak convergence of the space $\mathfrak{L}(\mathcal{H}_n)$ by the operator

$$\mathrm{w}^* - \lim_{t \to 0} \frac{1}{t}\big(\mathcal{G}_n(t)g_n - g_n\big) = -i(g_n H_n - H_n g_n) \doteq \mathcal{N}_n g_n, \qquad (3)$$

where H_n is the Hamiltonian (1) and the operator $\mathcal{N}_n g_n$ is defined on the domain $\mathcal{D}(H_n) \subset \mathcal{H}_n$. Therefore on the space $\mathfrak{L}(\mathcal{H}_n)$ a unique solution of the Heisenberg equation for observables of a n-particle system is determined by group (2) [22].

In what follows we shall hold abridged notations: $Y \equiv (1, \ldots, s)$, $X \equiv (j_1, \ldots, j_n) \subset Y$, and $\{Y \setminus X\}$ is the set, consisting of a single element $Y \setminus X = (1, \ldots, s) \setminus (j_1, \ldots, j_n)$, thus, the set $\{Y \setminus X\}$ is a connected subset of the set Y.

To describe the evolution within the framework of marginal observables we introduce a notion of the $(1+n)th$-order ($n \geq 0$) cumulant of groups of operators (2) as follows [21]

$$\mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X) \doteq \qquad\qquad\qquad\qquad\qquad (4)$$
$$\doteq \sum_{\mathrm{P}:\, (\{Y \setminus X\}, X) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}| - 1)! \prod_{X_i \subset \mathrm{P}} \mathcal{G}_{|\theta(X_i)|}(t, \theta(X_i)),$$

where the symbol $\sum_{\mathrm{P}}$ means the sum over all possible partitions P of the set $(\{Y \setminus X\}, j_1, \ldots, j_n)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset (\{Y \setminus X\}, X)$, and $\theta(\cdot)$ is the declusterization mapping defined as follows: $\theta(\{Y \setminus X\}, X) = Y$. For example,

$$\mathfrak{A}_1(t, \{Y\}) = \mathcal{G}_s(t, Y),$$
$$\mathfrak{A}_2(t, \{Y \setminus (j)\}, j) = \mathcal{G}_s(t, Y) - \mathcal{G}_{s-1}(t, Y \setminus (j))\mathcal{G}_1(t, j).$$

In terms of observables the evolution of quantum many-particle systems is described by the sequence $B(t) = (B_0, B_1(t, 1), \ldots, B_s(t, 1, \ldots, s), \ldots)$ of marginal observables (or s-particle observables) $B_s(t, 1, \ldots, s)$, $s \geq 1$, determined by the following expansions [24]:

$$B_s(t, Y) = \sum_{n=0}^{s} \frac{1}{n!} \sum_{j_1 \neq \ldots \neq j_n = 1}^{s} \mathfrak{A}_{1+n}(t, \{Y \setminus X\}, X)\, B_{s-n}^{0,\epsilon}(Y \setminus X), \quad (5)$$

where $B(0) = (B_0, B_1^{0,\epsilon}(1), \ldots, B_s^{0,\epsilon}(1, \ldots, s), \ldots) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ is a sequence of initial marginal observables, and the generating operator $\mathfrak{A}_{1+n}(t)$ of expansion (5) is the $(1+n)th$-order cumulant of groups of operators (2) defined by expansion (4). The simplest examples of marginal observables (5) are given by the expressions:

$$B_1(t, 1) = \mathfrak{A}_1(t, 1)B_1^{0,\epsilon}(1),$$
$$B_2(t, 1, 2) = \mathfrak{A}_1(t, \{1, 2\})B_2^{0,\epsilon}(1, 2) + \mathfrak{A}_2(t, 1, 2)(B_1^{0,\epsilon}(1) + B_1^{0,\epsilon}(2)).$$

If $\gamma < e^{-1}$, for the sequence of operators (5) the estimate is true: $\left\|B(t)\right\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})} \leq e^2(1 - \gamma e)^{-1}\left\|B(0)\right\|_{\mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})}$.

We note that a sequence of marginal observables (5) is the non-perturbative solution of recurrence evolution equations known as the dual quantum BBGKY hierarchy [24].

2.2 A mean field asymptotic behavior of marginal observables

A mean field asymptotic behavior of marginal observables (5) is described by the following statement [25].

Theorem 1 *Let for $B_n^{0,\epsilon} \in \mathfrak{L}(\mathcal{H}_n)$, $n \geq 1$, in the sense of the $*$-weak convergence on the space $\mathfrak{L}(\mathcal{H}_s)$ it holds:* $\mathrm{w}^* - \lim_{\epsilon \to 0}(\epsilon^{-n}B_n^{0,\epsilon} - b_n^0) = 0$, *then for arbitrary finite time interval there exists the mean field limit of marginal observables (5):* $\mathrm{w}^* - \lim_{\epsilon \to 0}(\epsilon^{-s}B_s(t) - b_s(t)) = 0$, $s \geq 1$, *that are determined by the following expansions:*

$$b_s(t, Y) = \sum_{n=0}^{s-1} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \tag{6}$$

$$\times \sum_{i_1 \neq j_1 = 1}^{s} \mathcal{N}_{\mathrm{int}}(i_1, j_1) \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \ldots$$

$$\prod_{l_n \in Y \setminus (j_1, \ldots, j_{n-1})} \mathcal{G}_1(t_{n-1} - t_n, l_n) \sum_{\substack{i_n \neq j_n = 1, \\ i_n, j_n \neq (j_1, \ldots, j_{n-1})}}^{s} \mathcal{N}_{\mathrm{int}}(i_n, j_n)$$

$$\times \prod_{l_{n+1} \in Y \setminus (j_1, \ldots, j_n)} \mathcal{G}_1(t_n, l_{n+1})b_{s-n}^0(Y \setminus (j_1, \ldots, j_n)).$$

In expansion (6) we denote by the symbol $\mathcal{N}_{\text{int}}(i_1, j_2)$ the operator defined on $g_n \in \mathfrak{L}(\mathcal{H}_n)$

$$\mathcal{N}_{\text{int}}(i_1, j_2)g_n \doteq -i(g_n \Phi(i_1, j_2) - \Phi(i_1, j_2)g_n).$$

The proof of Theorem 1 is based on formulas for cumulants of asymptotically perturbed groups of operators (2).

For arbitrary finite time interval the asymptotically perturbed group of operators (2) has the following scaling limit in the sense of the $*$-weak convergence on the space $\mathfrak{L}(\mathcal{H}_s)$:

$$\text{w}^*- \lim_{\epsilon \to 0} \big(\mathcal{G}_s(t, Y) - \prod_{j=1}^{s} \mathcal{G}_1(t, j)\big)g_s = 0. \tag{7}$$

Taking into account analogs of the Duhamel equations for cumulants of asymptotically perturbed groups of operators, in view of formula (7) we have

$$\text{w}^*- \lim_{\epsilon \to 0} \Big(\epsilon^{-n}\frac{1}{n!}\mathfrak{A}_{1+n}\big(t, \{Y \setminus X\}, j_1, \ldots, j_n\big) -$$

$$- \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1) \sum_{i_1 \neq j_1 = 1}^{s} \mathcal{N}_{\text{int}}(i_1, j_1)$$

$$\times \prod_{l_2 \in Y \setminus (j_1)} \mathcal{G}_1(t_1 - t_2, l_2) \ldots \prod_{l_n \in Y \setminus (j_1, \ldots, j_{n-1})} \mathcal{G}_1(t_{n-1} - t_n, l_n)$$

$$\times \sum_{\substack{i_n \neq j_n = 1, \\ i_n, j_n \neq (j_1, \ldots, j_{n-1})}}^{s} \mathcal{N}_{\text{int}}(i_n, j_n) \prod_{l_{n+1} \in Y \setminus (j_1, \ldots, j_n)} \mathcal{G}_1(t_n, l_{n+1})\Big)g_{s-n} = 0,$$

where we used notations accepted in formula (6) and $g_{s-n} \equiv g_{s-n}((1, \ldots, s)\setminus(j_1, \ldots, j_n))$, $n \geq 1$. As a result of this equality we establish the validity of Theorem 1 for expansion (5) of marginal observables.

If $b^0 \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$, then the sequence $b(t) = (b_0, b_1(t), \ldots, b_s(t), \ldots)$ of limit marginal observables (6) is a generalized global solution of the Cauchy problem of the dual quantum Vlasov hierarchy

$$\frac{\partial}{\partial t}b_s(t, Y) = \sum_{j=1}^{s} \mathcal{N}(j)\, b_s(t, Y) + \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{N}_{\text{int}}(j_1, j_2)\, b_{s-1}(t, Y \setminus (j_1)), \tag{8}$$

$$b_s(t)\,|_{t=0} = b_s^0, \quad s \geq 1, \tag{9}$$

where the symbol $\mathcal{N}(j)$ denotes the infinitesimal generator defined on $g_n \in \mathfrak{L}_0(\mathcal{H}_n)$ of the group of operators $\mathcal{G}_1(t, j)$ of j particle

$$\mathcal{N}(j)g_n \doteq -i(g_n K(j) - K(j)g_n).$$

It should be noted that equations set (8) has the structure of recurrence evolution equations. We give several examples of the evolution equations of the dual quantum Vlasov hierarchy (8) in terms of operator kernels of the limit marginal observables

$$i\frac{\partial}{\partial t}b_1(t, q_1; q_1') = -\frac{1}{2}(-\Delta_{q_1} + \Delta_{q_1'})b_1(t, q_1; q_1'),$$

$$i\frac{\partial}{\partial t}b_2(t, q_1, q_2; q_1', q_2') = \big(-\frac{1}{2}\sum_{i=1}^{2}(-\Delta_{q_i} + \Delta_{q_i'}) +$$

$$+(\Phi(q_1' - q_2') - \Phi(q_1 - q_2)))b_2(t, q_1, q_2; q_1', q_2') +$$

$$+\big(\Phi(q_1' - q_2') - \Phi(q_1 - q_2)\big)\big(b_1(t, q_1; q_1') + b_1(t, q_2; q_2')\big).$$

We consider the mean field limit of a particular case of marginal observables, namely the additive-type marginal observables $B^{(1)}(0) = (0, B_1^{0,\epsilon}(1), 0, \ldots)$. We remark that the k-ary marginal observables are represented by the sequence $B^{(k)}(0) = \big(0, \ldots, 0, B_k^{0,\epsilon}(1, \ldots, k), 0, \ldots\big)$. In case of additive-type marginal observables expansions (5) the following form:

$$B_s^{(1)}(t, Y) = \mathfrak{A}_s(t)\sum_{j=1}^{s}B_1^{0,\epsilon}(j), \quad s \geq 1, \tag{10}$$

where $\mathfrak{A}_s(t)$ is s-order cumulant (4) of groups of operators (2).

Corollary 1 *If for the additive-type marginal observable $B_1^{0,\epsilon} \in \mathfrak{L}(\mathcal{H})$, it holds $\mathrm{w}^* - \lim_{\epsilon \to 0}(\epsilon^{-1}B_1^{0,\epsilon} - b_1^0) = 0$, then, according to the statement of Theorem 1, for additive-type marginal observable (10) we have $\mathrm{w}^* - \lim_{\epsilon \to 0}(\epsilon^{-s}B_s^{(1),\epsilon}(t) - b_s^{(1)}(t)) = 0$, $s \geq 1$, where the limit additive-type marginal observable $b_s^{(1)}(t)$ is determined by a special case of expansion (6)*

$$b_s^{(1)}(t, Y) = \int_0^t dt_1 \ldots \int_0^{t_{s-2}} dt_{s-1} \prod_{l_1 \in Y} \mathcal{G}_1(t - t_1, l_1)\sum_{i_1 \neq j_1 = 1}^{s}\mathcal{N}_{\mathrm{int}}(i_1, j_1) \tag{11}$$

$$\times \prod_{l_2 \in Y \setminus (j_1)}\mathcal{G}_1(t_1 - t_2, l_2)\ldots\prod_{l_{s-1} \in Y \setminus (j_1, \ldots, j_{s-2})}\mathcal{G}_1(t_{s-2} - t_{s-1}, l_{s-1})$$

$$\times \sum_{\substack{i_{s-1} \neq j_{s-1} = 1, \\ i_{s-1}, j_{s-1} \neq (j_1, \ldots, j_{s-2})}}^{s} \mathcal{N}_{\text{int}}(i_{s-1}, j_{s-1})$$

$$\times \prod_{l_s \in Y \setminus (j_1, \ldots, j_{s-1})} \mathcal{G}_1(t_{s-1}, l_s) b_1^0(Y \setminus (j_1, \ldots, j_{s-1})).$$

We make several examples of expansions (11) for the limit additive-type marginal observables

$$b_1^{(1)}(t, 1) = \mathcal{G}_1(t, 1) \, b_1^0(1),$$

$$b_2^{(1)}(t, 1, 2) = \int_0^t dt_1 \prod_{i=1}^2 \mathcal{G}_1(t - t_1, i) \mathcal{N}_{\text{int}}(1, 2) \sum_{j=1}^2 \mathcal{G}_1(t_1, j) \, b_1^0(j).$$

Thus, for arbitrary initial states in the mean field scaling limit the kinetic evolution of quantum many-particle systems is described in terms of limit marginal observables (6) governed by the dual quantum Vlasov hierarchy (8).

2.3 The derivation of the quantum Vlasov-type kinetic equation with correlations

Furthermore, the relationships between the evolution of observables and the kinetic evolution of states described in terms of a one-particle marginal density operator are considered.

Let initial states specified by the one-particle marginal density operator $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ in the presence of correlations, i.e. initial state is defined by the following sequence of density operators:

$$F^c = \left(1, F_1^{0,\epsilon}(1), g_2 \prod_{i=1}^2 F_1^{0,\epsilon}(i), \ldots, g_n \prod_{i=1}^n F_1^{0,\epsilon}(i), \ldots \right), \qquad (12)$$

where the bounded operators $g_n \equiv g_n(1, \ldots, n) \in \mathfrak{L}(\mathcal{H}_n)$, $n \geq 2$, are specified initial correlations. We note that such assumption about initial states is intrinsic for the kinetic description of a gas. On the other hand, initial data (12) is typical for the condensed states of quantum gases, for example, the equilibrium state of the Bose condensate satisfies the weakening of correlation condition with the correlations which characterize the condensed state [26].

We assume that for the initial one-particle marginal density operator $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ exists the mean field limit $\lim_{\epsilon \to 0} \left\| \epsilon F_1^{0,\epsilon} - f_1^0 \right\|_{\mathfrak{L}^1(\mathcal{H})} = 0$, then in the mean field limit initial state is defined by the following sequence of operators:

$$f^c = \left(1, f_1^0(1), g_2 \prod_{i=1}^{2} f_1^0(i), \dots, g_n \prod_{i=1}^{n} f_1^0(i), \dots \right). \tag{13}$$

We consider links of the constructed mean field asymptotic behavior of marginal observables with the nonlinear Vlasov-type kinetic equation in case of initial states (13).

In case of initial states specified by sequence (13) the average values (mean values) of limit marginal observables (6) are determined by the following positive continuous linear functional [22]

$$\left(b(t), f^c \right) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Tr}_{1,\dots,n} b_n(t, 1, \dots, n) \, g_n(1, \dots, n) \prod_{i=1}^{n} f_1^0(i). \tag{14}$$

For $b(t) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ and $f_1^0 \in \mathfrak{L}^1(\mathcal{H})$, functional (14) exists under the condition that $\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})} < \gamma$.

Consequently, for the limit additive-type marginal observables (11) the following equality is true:

$$\left(b^{(1)}(t), f^c \right) = \sum_{s=0}^{\infty} \frac{1}{s!} \operatorname{Tr}_{1,\dots,s} b_s^{(1)}(t, 1, \dots, s) g_s(1, \dots, s) \prod_{i=1}^{s} f_1^0(i) =$$
$$= \operatorname{Tr}_1 b_1^0(1) f_1(t, 1),$$

where the operator $b_s^{(1)}(t)$ is given by expansion (11) and the limit marginal density operator $f_1(t, 1)$ is represented by the series expansion

$$f_1(t, 1) = \sum_{n=0}^{\infty} \int_0^t dt_1 \dots \int_0^{t_{n-1}} dt_n \operatorname{Tr}_{2,\dots,n+1} \mathcal{G}_1(-t + t_1, 1) \tag{15}$$

$$\times \mathcal{N}_{\text{int}}^*(1, 2) \prod_{j_1=1}^{2} \mathcal{G}_1(-t_1 + t_2, j_1) \dots \prod_{i_n=1}^{n} \mathcal{G}_1(-t_n + t_n, i_n)$$

$$\times \sum_{k_n=1}^{n} \mathcal{N}_{\text{int}}^*(k_n, n+1) \prod_{j_n=1}^{n+1} \mathcal{G}_1(-t_n, j_n) g_{1+n}(1, \dots, n+1) \prod_{i=1}^{n+1} f_1^0(i).$$

In series (15) the operator $\mathcal{N}_{\mathrm{int}}^*(j_1, j_2)f_n = -\mathcal{N}_{\mathrm{int}}(j_1, j_2)f_n$ is adjoint operator to operator (3) in the sense of functional (14). For bounded interaction potentials series (15) is norm convergent on the space $\mathfrak{L}^1(\mathcal{H})$ under the condition: $t < t_0 \equiv (2\,\|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)}\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$.

The operator $f_1(t)$ represented by series (15) is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with correlations

$$\frac{\partial}{\partial t}f_1(t,1) = -\mathcal{N}(1)f_1(t,1) + \tag{16}$$

$$+\mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}})(1,2)\prod_{i_1=1}^{2}\mathcal{G}_1(-t,i_1)g_2(1,2)\prod_{i_2=1}^{2}\mathcal{G}_1(t,i_2)f_1(t,1)f_1(t,2),$$

$$f_1(t)|_{t=0} = f_1^0. \tag{17}$$

This fact is proved similarly as in case of a solution of the quantum BBGKY hierarchy represented by the iteration series [18],[22].

Thus, in case of initial states specified by one-particle marginal density operator (13) we establish that the dual quantum Vlasov hierarchy (8) for additive-type marginal observables describes the evolution of a system of quantum particles just as the quantum Vlasov-type kinetic equation with correlations (16).

2.4   The mean field evolution of initial correlations

The property of the propagation of initial chaos is a consequence of the validity of the following equality for the mean value functionals of the limit k-ary marginal observables in case of $k \geq 2$

$$\left(b^{(k)}(t), f^c\right) = \sum_{s=0}^{\infty}\frac{1}{s!}\mathrm{Tr}_{1,\ldots,s}\,b_s^{(k)}(t,1,\ldots,s)g_s(1,\ldots,s)\prod_{j=1}^{s}f_1^0(j) = \tag{18}$$

$$= \frac{1}{k!}\mathrm{Tr}_{1,\ldots,k}\,b_k^0(1,\ldots,k)\prod_{i_1=1}^{k}\mathcal{G}_1(-t,i_1)g_k(1,\ldots,k)$$

$$\times \prod_{i_2=1}^{k}\mathcal{G}_1(t,i_2)\prod_{j=1}^{k}f_1(t,j), \quad k \geq 2,$$

where the limit one-particle marginal density operator $f_1(t,i)$ is defined by series expansion (15) and therefore it is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with correlations (16),(17).

This fact is proved similarly to the proof of a property on the propagation of initial chaos in a mean field limit [25].

Thus, in case of the limit k-ary marginal observables a solution of the dual quantum Vlasov hierarchy (8) is equivalent to a property of the propagation of initial correlations for the k-particle marginal density operator in the sense of equality (18) or in other words the mean field scaling dynamics does not create correlations.

We remark that the general approaches to the description of the evolution of states of quantum many-particle systems within the framework of correlation operators and marginal correlation operators were given in papers [28],[29] and [30], respectively (see also a review [22]).

3   On relationships of dynamics of observables and the kinetic evolution of states

We consider the relationships of dynamics of quantum many-particle systems described in terms of marginal observables and dynamics described within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation in the presence of initial correlations. If initial states is completely specified by a one-particle marginal density operator, using a non-perturbative solution of the quantum dual BBGKY hierarchy we prove that all possible states at arbitrary moment of time can be described within the framework of a one-particle density operator governed by the generalized quantum kinetic equation with correlations.

3.1   Quantum dynamics of states and correlations

In case of initial states defined by sequence (12) the average values (mean values) of marginal observables (5) are defined by the positive continuous linear functional on the space $\mathfrak{L}(\mathcal{F}_{\mathcal{H}})$

$$\big(B(t), F^c\big) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{Tr}_{1,\ldots,n} B_n(t, 1, \ldots, n)\, g_n(1, \ldots, n) \prod_{i=1}^{n} F_1^{0,\epsilon}(i). \quad (19)$$

For $F_1^{0,\varepsilon} \in \mathfrak{L}^1(\mathcal{H})$ and $B_n^{0,\epsilon} \in \mathfrak{L}(\mathcal{H}_n)$ series (19) exists under the condition that $\|F_1^{0,\varepsilon}\|_{\mathfrak{L}^1(\mathcal{H})} < e^{-1}$.

For mean value functional (19) the following representation holds

$$\big(B(t), F^c\big) = \big(B(0), F(t \mid F_1(t))\big), \tag{20}$$

where $B(0) = (B_0, B_1^{0,\epsilon}(1), \ldots, B_s^{0,\epsilon}(1, \ldots, s), \ldots) \in \mathfrak{L}_\gamma(\mathcal{F}_\mathcal{H})$ is a sequence of initial marginal observables, and $F(t \mid F_1(t)) = (1, F_1(t), F_2(t \mid F_1(t)), \ldots, F_s(t \mid F_1(t)), \ldots)$ is a sequence of explicitly defined marginal functionals $F_s(t \mid F_1(t))$, $s \geq 2$, with respect to the one-particle marginal density operator

$$F_1(t, 1) = \sum_{n=0}^\infty \frac{1}{n!} \operatorname{Tr}_{2,\ldots,1+n} \mathfrak{A}_{1+n}(-t) g_{n+1}(1, \ldots, n+1) \prod_{i=1}^{n+1} F_1^{0,\epsilon}(i). \tag{21}$$

The generating operator $\mathfrak{A}_{1+n}(-t) \equiv \mathfrak{A}_{1+n}(-t, 1, \ldots, n+1)$ of series expansion (21) is the $(1+n)th$-order cumulant of groups of operators $\mathcal{G}_n(-t)$, $n \geq 1$, adjoint to groups (2) in the sense of functional (19).

The marginal functionals of the state $F_s(t \mid F_1(t))$, $s \geq 2$, are represented by the following series expansions:

$$F_s\big(t, Y \mid F_1(t)\big) \doteq \tag{22}$$
$$\doteq \sum_{n=0}^\infty \frac{1}{n!} \operatorname{Tr}_{s+1,\ldots,s+n} \mathfrak{G}_{1+n}\big(t, \{Y\}, s+1, \ldots, s+n\big) \prod_{i=1}^{s+n} F_1(t, i),$$

where the $(1+n)th$-order generating operator $\mathfrak{G}_{1+n}(t)$, $n \geq 0$, is determined by the expansion

$$\mathfrak{G}_{1+n}(t, \{Y\}, X \setminus Y) \doteq \tag{23}$$
$$\doteq n! \sum_{k=0}^n (-1)^k \sum_{n_1=1}^n \ldots \sum_{n_k=1}^{n-n_1-\ldots-n_{k-1}} \frac{1}{(n-n_1-\ldots-n_k)!}$$
$$\times \breve{\mathfrak{A}}_{1+n-n_1-\ldots-n_k}(t, \{Y\}, s+1, \ldots, s+n-n_1-\ldots-n_k)$$
$$\times \prod_{j=1}^k \sum_{\substack{\mathrm{D}_j : Z_j = \bigcup_{l_j} X_{l_j}, \\ |\mathrm{D}_j| \leq s+n-n_1-\cdots-n_j}} \frac{1}{|\mathrm{D}_j|!}$$
$$\times \sum_{i_1 \neq \ldots \neq i_{|\mathrm{D}_j|}=1}^{s+n-n_1-\ldots-n_j} \prod_{X_{l_j} \subset \mathrm{D}_j} \frac{1}{|X_{l_j}|!} \breve{\mathfrak{A}}_{1+|X_{l_j}|}(t, i_{l_j}, X_{l_j}).$$

In formula (23) we denote by $\sum_{\mathrm{D}_j : Z_j = \bigcup_{l_j} X_{l_j}}$ the sum over all possible dissections of the linearly ordered set $Z_j \equiv (s+n-n_1-\ldots-n_j+1, \ldots, s+n-n_1-\ldots-n_{j-1})$ on no more than $s+n-n_1-\ldots-n_j$ linearly ordered subsets and we introduced the $(1+n)th$-order scattering cumulants

$$\breve{\mathfrak{A}}_{1+n}(t, \{Y\}, X \setminus Y) \doteq$$

$$\doteq \mathfrak{A}_{1+n}(-t, \{Y\}, X \setminus Y) g_{s+n}(\theta(\{Y\}), X \setminus Y) \prod_{i=1}^{s+n} \mathfrak{A}_1(t,i),$$

where it is used notations accepted above. We give examples of the scattering cumulants

$$\mathfrak{G}_1(t, \{Y\}) = \breve{\mathfrak{A}}_1(t, \{Y\}) \doteq \mathfrak{A}_1(-t, \{Y\}) g_s(\theta(\{Y\})) \prod_{i=1}^{s} \mathfrak{A}_1(t,i),$$

$$\mathfrak{G}_2(t, \{Y\}, s+1) = \mathfrak{A}_2(-t, \{Y\}, s+1) g_{s+1}(\theta(\{Y\}), s+1) \prod_{i=1}^{s+1} \mathfrak{A}_1(t,i) -$$

$$-\mathfrak{A}_1(-t, \{Y\}) g_s(\theta(\{Y\})) \prod_{i=1}^{s} \mathfrak{A}_1(t,i) \times$$

$$\times \sum_{i=1}^{s} \mathfrak{A}_2(-t, i, s+1) g_2(i, s+1) \mathfrak{A}_1(t,i) \mathfrak{A}_1(t, s+1).$$

If $\|F_1(t)\|_{\mathfrak{L}^1(\mathcal{H})} < e^{-(3s+2)}$, then for arbitrary $t \in \mathbb{R}$ series expansion (20) converges in the norm of the space $\mathfrak{L}^1(\mathcal{H}_s)$ [22].

We emphasize that marginal functionals of the state (22) characterize the correlations generated by dynamics of quantum many-particle systems in the presence of initial correlations.

3.2   On an equivalence of mean value functional representations

We establish the validity of equality (20) for mean value functional (19).

In a particular case of initial data specified by the additive-type marginal observables, i.e. $B^{(1)}(0) = (0, B_1^{0,\epsilon}(1), 0, \ldots)$, equality (20) takes the form

$$\left(B^{(1)}(t), F^c \right) = \mathrm{Tr}_1 \, B_1^{0,\epsilon}(1) F_1(t,1), \tag{24}$$

where the one-particle marginal density operator $F_1(t)$ is determined by series expansion (21). The validity of this equality is a result of the direct transformation of the generating operators of expansions (10) to adjoint operators in the sense of the functional (19).

In case of initial data specified by the s-ary marginal observables i.e. $B^{(s)}(0) = (0, \ldots, 0, B_s^{0,\epsilon}(1, \ldots, s), 0, \ldots)$, $s \geq 2$, equality (20) takes the following form:

$$\left(B^{(s)}(t), F^c\right) = \frac{1}{s!} \mathrm{Tr}_{1,\ldots,s} \, B_s^{0,\epsilon}(1, \ldots, s) F_s\big(t, 1, \ldots, s \mid F_1(t)\big), \qquad (25)$$

where the marginal functional of the state $F_s(t \mid F_1(t))$ is represented by series expansion (22).

The proof of equality (25) is based on the application of cluster expansions to generating operators (4) of expansions (5) which is dual to the kinetic cluster expansions introduced in [20]. Then the adjoint series expansion can be expressed in terms of one-particle marginal density operator (21) in the form of the functional from the right-hand side of equality (25).

In case of the general type of marginal observables the validity of equality (20) is proven in much the same way as the validity of equalities (24) and (25).

3.3 The generalized quantum kinetic equation with correlations

As a result of the differentiation over the time variable of operator represented by series (21) in the sense of the norm convergence of the space $\mathfrak{L}^1(\mathcal{H}_s)$, then the application of the kinetic cluster expansions [20] to the generating operators of obtained series expansion, for the one-particle marginal density operator we derive the identity [27]

$$\frac{\partial}{\partial t} F_1(t, 1) = -\mathcal{N}(1) F_1(t, 1) + \epsilon \, \mathrm{Tr}_2(-\mathcal{N}_{\mathrm{int}}(1, 2)) F_2(t, 1, 2 \mid F_1(t)), \quad (26)$$

where the collision integral is determined by the marginal functional of the state (22) in case of $s = 2$. This identity we treat as the non-Markovian quantum kinetic equation. We refer to this evolution equation as the generalized quantum kinetic equation with correlations.

We emphasize that the coefficients in an expansion of the collision integral of kinetic equation (26) are determined by the operators specified initial correlations.

We remark, that for initial data (12) specified by a one-particle marginal density operator, the evolution of states described within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation with correlations (26) is dual to the dual quantum BBGKY hierarchy for additive-type marginal observables with respect to bilinear form (19), and it is completely equivalent to the description of states in terms of marginal density operators governed by the quantum BBGKY hierarchy.

Thus, the evolution of quantum many-particle systems described in terms of marginal observables can be also described within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation with correlations in the sense of functional (19).

4   The mean field asymptotic behavior of the generalized quantum kinetic equation

We establish a mean field asymptotics of a solution of the non-Markovian quantum kinetic equation with correlations constructed above. This asymptotics is governed by the quantum Vlasov-type kinetic equation with correlations derived above from the dual quantum Vlasov hierarchy for the limit marginal observables.

4.1   The mean field limit theorem

For solution (21) of the generalized quantum kinetic equation with correlations (26) the following mean field limit theorem is true [23].

Theorem 2 *If for the initial one-particle marginal density operator $F_1^{0,\epsilon} \in \mathfrak{L}^1(\mathcal{H})$ exists the limit $\lim_{\epsilon \to 0} \|\epsilon\, F_1^{0,\epsilon} - f_1^0\|_{\mathfrak{L}^1(\mathcal{H})} = 0$, then for finite time interval $t \in (-t_0, t_0)$, where $t_0 \equiv (2\,\|\Phi\|_{\mathfrak{L}(\mathcal{H}_2)}\|f_1^0\|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$, there exists the mean field limit of solution (21) of the Cauchy problem of the generalized quantum kinetic equation with correlations (26)*

$$\lim_{\epsilon \to 0} \|\epsilon\, F_1(t) - f_1(t)\|_{\mathfrak{L}^1(\mathcal{H})} = 0, \qquad (27)$$

where the operator $f_1(t)$ is represented by series (15) and it is a solution of the Cauchy problem of the quantum Vlasov-type kinetic equation with correlations (16),(17).

The proof of this theorem is based on formulas of asymptotically perturbed cumulants of groups of operators $\mathcal{G}_n(-t)$, $n \geq 1$, adjoint to groups (2) in the sense of functional (19). Indeed, in a mean field limit for generating evolution operators (23) of series expansion (22) the following equalities are valid:

$$\lim_{\epsilon \to 0} \left\| \frac{1}{\epsilon^n} \mathfrak{G}_{1+n}(t, \{Y\}, X \setminus Y) f_{s+n} \right\|_{\mathfrak{L}^1(\mathcal{H}_{s+n})} = 0, \quad n \geq 1, \qquad (28)$$

and in case of the first-order generating evolution operator we have

$$\lim_{\epsilon \to 0} \left\| \left(\mathfrak{G}_1(t, \{Y\}) - \right. \right. \qquad (29)$$
$$\left. \left. - \prod_{j_1=1}^{s} \mathcal{G}_1(-t, j_1) g_s(\theta(\{Y\})) \prod_{j_2=1}^{s} \mathcal{G}_1(t, j_2) \right) f_s \right\|_{\mathfrak{L}^1(\mathcal{H}_s)} = 0,$$

respectively.

In view that under the condition $t < t_0 \equiv (2 \left\| \Phi \right\|_{\mathfrak{L}(\mathcal{H}_2)} \| \epsilon F_1^{0,\epsilon} \|_{\mathfrak{L}^1(\mathcal{H})})^{-1}$, for a bounded interaction potential the series for the operator $\epsilon F_1(t)$ is norm convergent, then for $t < t_0$ the remainder of solution series (21) can be made arbitrary small for sufficient large $n = n_0$ independently of ϵ. Then, using stated above asymptotic formulas, for each integer n every term of this series converges term by term to the limit operator $f_1(t)$ which is represented by series (15).

As stated above the mean field scaling limit (15) of solution (21) of the generalized quantum kinetic equation in the presence of initial correlations is governed by the quantum Vlasov-type kinetic equation with correlations (16).

Thus, we derived the quantum Vlasov-type kinetic equation with correlations (16) from the generalized quantum kinetic equation (26) in the mean field scaling limit. It is the same as the kinetic equation derived from the dual quantum Vlasov hierarchy for mean field limit marginal observables.

4.2   A mean field limit of marginal functionals of state

As we noted above the all possible correlations of a system of quantum particles are described by marginal functionals of the state (22).

Since solution (21) of initial-value problem of the generalized quantum kinetic equation with correlations (26) converges to solution (15) of initial-value problem of the quantum Vlasov-type kinetic equation with correlations (16) as (27), and equalities (28) and (29) hold, for a mean field asymptotic behavior of marginal functionals of the state (22) is true

$$\lim_{\epsilon \to 0} \big\| \epsilon^s F_s(t, 1, \dots, s \mid F_1(t)) -$$
$$- \prod_{j_1=1}^{s} \mathcal{G}_1(-t, j_1) g_s(1, \dots, s) \prod_{j_2=1}^{s} \mathcal{G}_1(t, j_2) \prod_{l=1}^{s} f_1(t, l) \big\|_{\mathfrak{L}^1(\mathcal{H}_s)} = 0,$$
$$s \geq 2.$$

This equalities mean the propagation of initial correlations in time in the mean field scaling limit.

5 Conclusion and outlook

In the paper the concept of quantum kinetic equations in case of the kinetic evolution, involving correlations of particle states at initial time, for instance, correlations characterizing the condensed states, was considered. Two approaches were developed with a view to this purpose. One approach based on the description of the evolution of quantum many-particle systems within the framework of marginal observables. Another method consists in the possibility in case of initial states specified by a one-particle marginal density operator to describe the evolution of states within the framework of a one-particle marginal density operator governed by the generalized quantum kinetic equation with correlations.

In case of pure states the quantum Vlasov-type kinetic equation with correlations (16) can be reduced to the Gross–Pitaevskii-type kinetic equation. Indeed, in this case the one-particle density operator $f_1(t) = |\psi_t\rangle\langle\psi_t|$ is a one-dimensional projector onto a unit vector $|\psi_t\rangle \in \mathcal{H}$ and its kernel has the following form: $f_1(t, q, q') = \psi(t, q)\psi^*(t, q')$. Then, if we consider particles, interacting by the potential which kernel $\Phi(q) = \delta(q)$ is the Dirac measure, from kinetic equation (16) we derive the Gross–Pitaevskii-type kinetic equation

$$i\frac{\partial}{\partial t}\psi(t, q) = -\frac{1}{2}\Delta_q \psi(t, q) +$$
$$+ \int dq' dq'' \mathfrak{g}(t, q, q; q', q'')\psi(t, q'')\psi^*(t, q)\psi(t, q),$$

where the coupling ratio $\mathfrak{g}(t, q, q; q', q'')$ of the collision integral is the kernel of the scattering length operator $\prod_{i_1=1}^{2} \mathcal{G}_1(-t, i_1) g_2(1, 2) \prod_{i_2=1}^{2} \mathcal{G}_1(t, i_2)$. If we consider a system of quantum particles without initial correlations, then derived kinetic equation is the cubic nonlinear Schrödinger equation.

Observing that on the macroscopic scale of the variation of variables, groups of operators (2) of finitely many particles depend on microscopic time variable $\varepsilon^{-1} t$, where $\varepsilon \geq 0$ is a scale parameter, the dimensionless marginal functionals of the state are represented in the form $F_s(\varepsilon^{-1} t \mid F_1(t))$. As a result of the formal limit processing $\varepsilon \to 0$ in the collision integral, we establish the Markovian kinetic evolution with the corresponding coefficient $\mathfrak{g}(\varepsilon^{-1} t)$.

This paper deals with a quantum system of a non-fixed (i.e. arbitrary but finite) number of identical (spinless) particles obeying Maxwell–Boltzmann statistics. The obtained results can be extended to quantum systems of bosons or fermions [29].

We emphasize, that one of the advantages of the approach to the derivation of the quantum Vlasov-type kinetic equation from underlying dynamics governed by the generalized quantum kinetic equation with correlations enables to construct the higher-order corrections to the mean field evolution of quantum interacting particles.

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