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Power series and conformal mappings in one boundary value problem for monogenic functions of the biharmonic variable

Dedicated to the 80 th anniversary of Prof. D.Ya. Petrina

Considered a boundary value problem (BVP) for monogenic functions of biharmonic variable taking values in a two-dimensional commutative Banach algebra. This BVP is associated with the main biharmonic problem for biharmonic functions of two real variables. Developing a reduction's scheme for this BVP for monogenic functions to BVP in a disk by using of expansions in power series and conformal mappings in the complex plane. For some particular cases this problem is solved in a complete form.

Розглядається крайова задача для моногенних функцій бігармонічної змінної зі значеннями в двовимірній комутативній алгебрі. Дана задача асоційована з основною бігармонічною задачею на площині. Розробляється схема редукції цієї задачі для моногенних в однозв'язних областях функцій до відповідної крайової задачі в крузі бігармонічної площини, застосовуючи розвинення в степеневий ряд аналітичних функцій комплексної змінної та конформні відображення комплексної площини. Наведено частинні випадки, коли дана задача розв'язується у замкненій формі.

1. Introduction. Monogenic functions in a biharmonic plane. We say that an associative commutative two-dimensional algebra $\mathbb{B}$ with the unit 1 over the field of complex numbers $\mathbb{C}$ is *biharmonic* if in $\mathbb{B}$ there exists a *biharmonic basis*, i.e., a basis $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \qquad e_1^2 + e_2^2 \neq 0, \tag{1}$$

In [1], I. P. Mel'nichenko proved that there exists the unique biharmonic algebra $\mathbb{B}$ and all biharmonic bases form an infinite collection belonging to the algebra $\mathbb{B}$, moreover, $\mathbb{B}$ is generated by a non-biharmonic bases $\{e_1, \rho\}$, where $\rho^2 = 0$.

Here and elsewhere we mean by the biharmonic bases $\{e_1, e_2\}$ the following:

$$e_1 = 1, \quad e_2 = i - \frac{i}{2}\rho, \tag{2}$$

where i is an imaginary complex unit. Therefore, we have equalities $e_2^2 = 1 + 2ie_2$ and

$$\rho = 2 + 2ie_2. \tag{3}$$

Consider a biharmonic plane $\mu := \{\zeta = x\,e_1 + y\,e_2 : x, y \in \mathbb{R}\}$ which is a linear span of the elements e_1, e_2 of biharmonic basis over the field of real numbers $\mathbb{R}$.

Let D be a domain in the Cartesian plane xOy and $D_\zeta := \{\zeta = x + ye_2 : (x, y) \in D\}$ be a domain in μ, and $D_z := \{z = x + iy : (x, y) \in D\}$ be a domain in the complex plane $\mathbb{C}$. In what follows, $\zeta = x + ye_2, z = x + iy$ and $x, y \in \mathbb{R}$.

Inasmuch as divisors of zero do not belong to the biharmonic plane, one can define the derivative $\Phi'(\zeta)$ of the function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ in the same way as in the complex plane:

$$\Phi'(\zeta) := \lim_{h \to 0,\, h \in \mu} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1}. \tag{4}$$

We say that a function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ if and only if its derivative $\Phi'(\zeta)$ exists in every point $\zeta \in D_\zeta$. Note, that the limit (4) can be considered according to the euclidian norm $a := \sqrt{|z_1|^2 + |z_2|^2}$, where $a = z_1 + z_2 e_2 \in \mathbb{B}$, z_1 and z_2 in $\mathbb{C}$.

In [2], it is established that a function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ is monogenic in a domain D_ζ if and only if the following Cauchy–Riemann condition is satisfied:

$$\frac{\partial \Phi(\zeta)}{\partial y} = \frac{\partial \Phi(\zeta)}{\partial x}\, e_2 \;\; \forall \zeta = x + e_2 y \in D_\zeta. \tag{5}$$

Note, that in [2] the condition (5) is written in an equivalent form by each component.

In [3], [4], there were established basic analytic properties of monogenic functions similar to properties of holomorphic functions of the complex variable: the Cauchy integral theorem and integral formula, the Morera theorem, the uniqueness theorem, the Taylor and Laurent expansions, a property of monogenic functions to be infinitely times monogenic.

Any function of a type $\Phi : D_\zeta \longrightarrow \mathbb{B}$ is expressed in the form

$$\Phi(\zeta) = U_1(x,y) + U_2(x,y)\, i + U_3(x,y)\, e_2 + U_4(x,y)\, ie_2, \quad \zeta = x + ye_2, \tag{6}$$

where $U_k\colon D \longrightarrow \mathbb{R}$, $k = \overline{1,4}$.

Every component U_k, $1 \le k \le 4$, of monogenic function (6) satisfies in the domain D the biharmonic equation

$$(\Delta_k)^m\, u(x,y) \equiv \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) u(x,y) = 0,\ m = k = 2, \tag{7}$$

due to the relations (1), an existence of derivatives $\Phi^{(k)}$ of the order k, $1 \le k \le 4$, and the equality $(\Delta_2)^2\, \Phi(\zeta) = (e_1^2 + e_2^2)^2 \Phi^{(4)}(\zeta)$.

In [5], there were introduced hyperanalytic functions taking values in real Clifford algebras of an arbitrary dimension, so-called, holomorphic Cliffordian functions. Any real component of holomorphic Cliffordian function (similar to U_k in (6)) satisfies the polyharmonic equation of the type (7) with some $m \ge 2$ and $k = 2m$.

In [6], V.V.Karachic and N.A. Antropova used Almansi representation formula for solving the inhomogeneous Dirichlet problem for the homogeneous biharmonic equation with polynomial boundary data.

2. Statement of (1-3)-Problem for monogenic functions.

Consider the following boundary value problem: to find a monogenic function $\Phi\colon D_\zeta \longrightarrow \mathbb{B}$ which is continuous in the closure $\overline{D_\zeta}$ of the domain D_ζ by given boundary values u_1, u_3, respectively, of the first and the third components of the expansion (6):

$$U_1(x,y) = u_1(\zeta),\, U_3(x,y) = u_3(\zeta) \qquad \forall\, \zeta = x + e_2 y \in \partial D_\zeta. \tag{8}$$

Problems of this type was first considered by V.F. Kovalev [7] and was called as the biharmonic Schwarz problem because it is analogous in a certain sense to the classical Schwarz problem on finding an analytic function of complex variable when values of its real part are given on the boundary of domain. Note that V.F. Kovalev stated only a sketch of solving of the biharmonic Schwarz problems in an integral form for semi-plane and discussed a possibility of the reduction this problem for an arbitrary domain to an integro-differential equation without investigation conditions of solvability of these problems.

Certain relation between the (1-3)-problem and Theory of 2D-elasticity is discussed in [8] for a case of a disk. Dwell on a case of an arbitrary simply connected domain $D \in \mathbb{R}^2$ corresponding to the domain D_ζ in the biharmonic plane μ. The main biharmonic problem (see, for example, [9, p. 202]) is to find a biharmonic function $V : D \longrightarrow \mathbb{R}$ by given limiting values of its partial derivatives

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0),\,(x,y)\in D} \frac{\partial V(x,y)}{\partial x} &= u_1(x_0,y_0), \\ \lim_{(x,y)\to(x_0,y_0),\,(x,y)\in D} \frac{\partial V(x,y)}{\partial y} &= u_3(x_0,y_0) \qquad \forall\,(x_0,y_0)\in\partial D. \end{aligned} \tag{9}$$

In [7], there was considered a reduction scheme of the main biharmonic problem to the (1-3)-problem. Consider a modification of this scheme.

Let Φ_1 is monogenic in D_ζ function

$$\Phi_1(\zeta) := V(x,y)\,e_1 + V_2(x,y)\,ie_1 + V_3(x,y)\,e_2 + V_4(x,y)\,ie_2,$$

which has as the first component the required biharmonic function $V(x,y)$. It follows from the Cauchy–Riemann condition (5) with $\Phi = \Phi_1$ that $\partial V_3(x,y)/\partial x = \partial V(x,y)/\partial y$. Therefore,

$$\Phi_1'(\zeta) = \frac{\partial V(x,y)}{\partial x}\,e_1 + \frac{\partial V_2(x,y)}{\partial x}\,ie_1 + \frac{\partial V(x,y)}{\partial y}\,e_2 + \frac{\partial V_4(x,y)}{\partial x}\,ie_2\,,$$

and the main biharmonic problem with the boundary conditions (9) can be reduced to the (1-3)-problem on finding a monogenic in D_ζ function $\Phi(\zeta) := \Phi_1'(\zeta)$, then, solving the latter problem, we recover functions $M(x,y) := \frac{\partial V(x,y)}{\partial x}$ and $N(x,y) := \frac{\partial V(x,y)}{\partial y}$ defined in D. In a conclusion, obtain a solution of the main biharmonic problem in the form of the following curvilinear integral

$$V(x,y) = \int_{(x_0,y_0)}^{(x,y)} M(x,y)\,dx + N(x,y)\,dy,$$

where (x_0, y_0) is a fixed point in D, integration means along any piecewise smooth curve, which joints this point with a point with variable coordinates (x, y).

In [8], investigated the (1-3)-problem for a case, when D_ζ is an upper semi-plane of the biharmonic plane or a unit disk $\{\zeta \in \mu : \|\zeta\| \leq 1\}$. Solutions of these problems are found in an explicit form by means of some integrals similar to a classis Schwarz integral in the complex plane.

Below we consider the (1-3)-problem for a sufficiently large class of domains D_ζ using the technique of conformal mappings D_ζ to the disk $\mathcal{D}_1 := \{\zeta \in \mu : \|\zeta\| < 1\}$, which is generated by a conformal mapping of D_z to the unit disk in $\mathbb{C}$. We notice some sufficient condition to the domain D_ζ and boundary data u_1 and u_3 for a reduction of the (1-3)-problem to a suitable boundary value problem on finding some $\mathbb{B}$ – valued function defined in $\mathcal{D}_1$. For some particular cases of domains D_ζ this reduction recover a solution of the (1-3)-problem in an explicit form.

Proposed method of solving boundary value problems for monogenic functions of the biharmonic variable analogous to the method of N. I. Muskhelishvili of solving boundary value problems of 2D-Elasticity based on using a technique of conformal mappings of complex plane and power series expansions of analytic functions of complex variable (cf., e.g., [10, §63]).

3. Using technique of conformal mappings for (1-3)-problem in a simply connected domain. There is an expression (cf., e.g., [3] – [12]) of an arbitrary monogenic function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ via two analytic functions F, F_0 of complex variable $z \in D_z$:

$$\Phi(\zeta) = F(z)e_1 - \left(\frac{iy}{2}\, F'(z) - F_0(z)\right)\rho \qquad \forall \zeta \in D_\zeta. \tag{10}$$

Consider a problem on solving of the (1-3)-problem in a domain D_ζ, which is congruent to a simply connected domain D_z. Let $\mathbb{N}$ be a set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ be a set of integer numbers. Taking into account any conformal mapping of the type $\omega\colon \mathcal{D}_1 \longrightarrow D_z$, we generale a domain D_ζ. Denote $\Gamma_1 := \{z \in \mathbb{C} : |z| = 1\}$. For any complex-valued function of the type $G(z)$, $z = \omega(\tau)$, $\tau \in \mathcal{D}_1$, we will denote by $\widetilde{G}(\tau)$ an expression $G(\omega(\tau))$. For any τ in the disk $\mathcal{D}_1$ denote by (η, φ) its polar coordinates, i.e., $\tau = \eta \exp\{i\varphi\}$, by $(1, \theta)$ we will denote polar coordinates of points $\sigma = \exp\{i\theta\} \in \Gamma_1$. Obviously, that if a function G is analytic in D_z, then $\widetilde{G}$ is analytic in $\mathcal{D}_1$.

For any $z \in \mathbb{C}$ by Rez and Imz we mean, accordingly, real and imaginary parts of z: $z = \mathrm{Re}z + i\,\mathrm{Im}z$. Denote $\Phi_*(\tau) := \Phi\left(\mathrm{Re}\,\omega(\tau) + \mathrm{Im}\,\omega(\tau)\,e_2\right)$, $\tau \in \mathcal{D}_1$. Then the equality (10) transforms to the form

$$\Phi_*(\tau) = \widetilde{F}(\tau) - \left(\frac{i}{2}\,Y(\tau)\,\widetilde{F}'(\tau) - \widetilde{F}_0(\tau)\right)\rho \qquad \forall\tau \in \mathcal{D}_1, \tag{11}$$

where

$$Y(\tau) := \frac{\mathrm{Im}\,\omega(\tau)}{\omega'(\tau)}. \tag{12}$$

Therefore, receive that the (1-3)-problem for monogenic function Φ reduced to an ***auxiliary (1-3)-problem*** on finding the first, V_1, and the third component, V_3, for a function (11) ($\widetilde{F}$, $\widetilde{F}_0$ are unknown analytic in $\mathcal{D}_1$ functions of complex variable τ):

$$\Phi_*(\tau) = V_1(\tau) + V_2(\tau)i + V_3(\tau)e_2 + V_4(\tau)ie_2, \tag{13}$$

where $\tau = \tau_1 + i\tau_2$, $\tau_k \in \mathbb{R}$, $k = 1, 2$, $V_k : \mathcal{D}_1 \longrightarrow \mathbb{R}$, $k = \overline{1, 4}$, furthermore, we assume, that Φ_* is continuous in $\overline{\mathcal{D}_1}$ and the following boundary conditions fulfilled

$$V_k(\sigma) = \widetilde{u}_k(\sigma),\; k = 1, 3, \qquad \forall\,\sigma \in \Gamma_1, \tag{14}$$

where $\widetilde{u}_k : \Gamma_1 \longrightarrow \mathbb{R}$ are given continuous functions. Boundary functions $\widetilde{u}_k, k = 1, 2$, are connected with boundary data u_1 and u_3 (see (8)) of the (1-3)-problem for a function (6), which is monogenic in D_ζ, by means of the following relations:

$$\widetilde{u}_k(\sigma) = u_k(\zeta),\; \omega(\sigma) = z,\; k = 1, 3, \tag{15}$$

where $\sigma \in \Gamma_1$, $z = \omega(\sigma) := x + iy \in \mathbb{C}$, $\zeta := x + e_2 y \in \partial D_\zeta$.

Using polar coordinates, deliver equivalent denotations for boundary functions $\widetilde{u}_1$ and $\widetilde{u}_3$:

$$\widetilde{u}_k(\theta) \equiv \widetilde{u}_k(\cos\theta + \sin\theta\, e_2),\; k = 1, 3,\; 0 \le \theta \le 2\pi. \tag{16}$$

Let l_1 is a totaly of consequences of the type $(\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots)$, where $\alpha_k \in \mathbb{R}$, $k = 1, 2, \ldots$, and $\sum_{k=1}^{\infty} |\alpha_k| < \infty$. Denote by $\{\alpha\}_m$, $m \in \mathbb{N}_0$, any consequence of the type $(\alpha_m, \alpha_{m+1}, \ldots) \in l_1$, and conversely.

We say, that the ordered quadruple of consequences $\left(\{\alpha^{(0)}\}_0, \{\beta^{(0)}\}_0, \{\alpha\}_0, \{\beta\}_1\right)$ *belongs to the class* $\mathcal{E}$ if and only if there exists a constant $M > 0$, natural number p and a sequence $\{\upsilon\}_p$, for which the following inequality fulfilled

$$|\alpha_k| + |\beta_k| \le M\frac{|\upsilon_k|}{k} \quad \forall k \ge p. \tag{17}$$

This definition can be naturally generalized to an ordered quadruple of consequences of the type $\big(\{\alpha^{(0)}\}_{N_1}, \{\beta^{(0)}\}_{N_2}, \{\alpha\}_{N_3}, \{\beta\}_{N_4}\big)$, $N_m \in \mathbb{N}_0$, $m = \overline{1,4}$.

Theorem 2. *Let the following conditions fulfilled*:

1* *conformal mapping* $\omega\colon \mathcal{D}_1 \longrightarrow D_z$ *is such, that the series*

$$\frac{\operatorname{Im}\omega(\sigma)}{\omega'(\sigma)} = \sum_{n=-\infty}^{\infty} \delta_n \sigma^n \quad \forall \sigma \in \Gamma_1, \tag{18}$$

is absolutely convergent on Γ_1,

2* *boundary functions* $\widetilde{u}_1$, $\widetilde{u}_3$ *of the auxiliary (1-3)-problem expressed by absolutely and uniformly convergent on the segment* $[0, 2\pi]$ *the Fourier series*:

$$\widetilde{u}_1(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k\theta + b_k \sin k\theta\right), \tag{19}$$

$$\widetilde{u}_3(\theta) = \frac{a'_0}{2} + \sum_{k=1}^{\infty} \left(a'_k \cos k\theta + b'_k \sin k\theta\right). \tag{20}$$

3* *The system of equations*

$$\alpha_0 + 2\alpha_0^{(0)} + \sum_{k=0}^{\infty}(k+1)\Big(\alpha_{k+1}\delta''_{-k} + \beta_{k+1}\delta'_{-k}\Big) = \frac{a_0}{2}, \tag{21}$$

$$\alpha_n + 2\alpha_n^{(0)} + \sum_{k=0}^{\infty}(k+1)\Big(\alpha_{k+1}\Lambda^{+}_{2,n,k} + \beta_{k+1}\Lambda^{+}_{1,n,k}\Big) = a_n \;\forall n \in \mathbb{N}, \tag{22}$$

$$-\beta_n - 2\beta_n^{(0)} + \sum_{k=0}^{\infty}(k+1)\Big(\alpha_{k+1}\Lambda^{-}_{1,n,k} - \beta_{k+1}\Lambda^{-}_{2,n,k}\Big) = b_n \;\forall n \in \mathbb{N}, \tag{23}$$

$$-2\beta_0^{(0)} + \sum_{k=0}^{\infty}(k+1)\Big(\alpha_{k+1}\delta'_{-k} - \beta_{k+1}\delta''_{-k}\Big) = \frac{a'_0}{2}, \tag{24}$$

$$-2\beta_n^{(0)}+\sum_{k=0}^{\infty}(k+1)\Big(\alpha_{k+1}\Lambda^{+}_{1,n,k}-\beta_{k+1}\Lambda^{+}_{2,n,k}\Big)=a'_n\quad\forall n\in\mathbb{N},\ (25)$$

$$-2\alpha_n^{(0)}-\sum_{k=0}^{\infty}(k+1)\Big(\alpha_{k+1}\Lambda^{-}_{2,n,k}+\beta_{k+1}\Lambda^{-}_{1,n,k}\Big)=b'_n\quad\forall n\in\mathbb{N},\ (26)$$

where δ'_n,δ''_n are, respectively, a real and an imaginary parts of coefficients δ_n in expression (18)*: $\delta_n =: \delta'_n + i\delta''_n$ for all $n\in\mathbb{Z}$; $\Lambda^{\pm}_{1,n,k} := \delta'_{n-k}\pm\delta'_{-n-k}$, $\Lambda^{\pm}_{2,n,k} := \delta''_{n-k}\pm\delta''_{-n-k}$ for $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$, is solvable and its general solution belongs to the class $\mathcal{E}$, if, besides, the system of equations* (21) $-$ (26) *with $a_k=a'_k=b_{k+1}=b'_{k+1}=0$, $k=0,1,\ldots,$ is solvable and its general solution belongs to the class $\mathcal{E}$.*

Then a general solution of of the auxiliary (1-3)-problem is expressed by the following formula

$$\Phi_*(\tau)=\widetilde{F}(\tau)-\left(\frac{i}{2}\frac{\operatorname{Im}\omega(\tau)}{\omega'(\tau)}\,\widetilde{F}'(\tau)-\widetilde{F}_0(\tau)\right)\rho\qquad\forall\tau\in\mathcal{D}_1,\qquad(27)$$

where

$$\widetilde{F}(\tau)=\sum_{n=0}^{\infty}c_n\tau^n,\quad \widetilde{F}_0(\tau)=\sum_{n=0}^{\infty}c_n^{(0)}\tau^n\quad\forall\tau\in\mathcal{D}_1,\qquad(28)$$

and an ordered quadruple of consequences $\big(\{\alpha^{(0)}\}_0,\{\beta^{(0)}\}_0,\{\alpha\}_0,\{\beta\}_1\big)$, formed by real components α_n, $\alpha_n^{(0)}$, $\beta_n^{(0)}$, β_{n+1}, $n=0,1,\ldots,$ of complex coefficients $c_n=\alpha_n+i\beta_n$, $c_n^{(0)}=\alpha_n^{(0)}+i\beta_n^{(0)}$ in the expression (28)*, is a general solution of the system* (21) $-$ (26)*.*

Proof. Expansions (28) of the Taylor series hold for functions $\widetilde{F}$ and $\widetilde{F}_0$ in the disk $\mathcal{D}_1$ with unknown coefficients $c_n=\alpha_n+i\beta_n$, $c_n^{(0)}=\alpha_n^{(0)}+i\beta_n^{(0)}$, where $\alpha_n=\operatorname{Re}c_n$, $\beta_n=\operatorname{Im}c_n$, $\alpha_n^{(0)}=\operatorname{Re}c_n^{(0)}$, $\beta_n^{(0)}=\operatorname{Im}c_n^{(0)}$, $n=0,1,\ldots.$

It follows from (28) that

$$\widetilde{F}'(\tau)=\sum_{n=0}^{\infty}(n+1)c_{n+1}\,\tau^n\ \forall\tau\in\mathcal{D}_1.\qquad(29)$$

Assume, that the series (28) and (29) are absolutely and uniformly convergent on $\overline{\mathcal{D}_1}$, and further, verify the validity of our assumption.

Using the following equalities for products of absolutely convergent on Γ_1 series $\sum_{k=0}^{\infty} a_k\sigma^k$, $\sum_{k=0}^{\infty} b_k\sigma^k$, $\sum_{k=1}^{\infty} h_{-k}\sigma^{-k}$:

$$\sum_{k=0}^{\infty} a_k\sigma^k \sum_{k=0}^{\infty} b_k\sigma^k = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)\sigma^n,$$

$$\sum_{k=1}^{\infty} a_k\sigma^k \sum_{k=1}^{\infty} h_{-k}\sigma^{-k} = \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_k h_{-k-n}\Big)\sigma^{-n}+\sum_{n=0}^{\infty}\Big(\sum_{k=n}^{\infty} a_{k+1} h_{n-k-1}\Big)\sigma^n,$$

multiply series (29) and (18), obtain the equality

$$\widetilde{F}'(\sigma)\frac{\operatorname{Im}\omega(\sigma)}{\omega'(\sigma)} = \sum_{n=-\infty}^{\infty} c_n^*\sigma^n \quad \forall\sigma\in\Gamma_1, \tag{30}$$

where for any integer n:

$$c_n^* := \sum_{k=0}^{\infty}(k+1)c_{k+1}\delta_{n-k} =: c_{n,1}^* + ic_{n,2}^*,$$

$$c_{n,1}^* := \sum_{k=0}^{\infty}(k+1)\left(\alpha_{k+1}\delta'_{n-k} - \beta_{k+1}\delta''_{n-k}\right), \tag{31}$$

$$c_{n,2}^* := \sum_{k=0}^{\infty}(k+1)\left(\alpha_{k+1}\delta''_{n-k} + \beta_{k+1}\delta'_{n-k}\right). \tag{32}$$

Using the Moivre formula rewrite the equality (30) in the form

$$\widetilde{F}'(\sigma)\frac{\operatorname{Im}\omega(\sigma)}{\omega'(\sigma)} = c_{0,1}^* + \sum_{n=1}^{\infty}\left(c_{-n,1}^* + c_{n,1}^*\right)\cos n\theta +$$

$$+\sum_{n=1}^{\infty}\left(c_{-n,2}^* - c_{n,2}^*\right)\sin n\theta +$$

$$+ic_{0,2}^* + i\sum_{n=1}^{\infty}\left(c_{-n,2}^* + c_{n,2}^*\right)\cos n\theta +$$

$$+i\sum_{n=1}^{\infty}\left(c_{n,1}^* - c_{-n,1}^*\right)\sin n\theta \quad \forall\sigma=\exp\{i\theta\}\in\Gamma_1. \tag{33}$$

Then using the equalities (3) deliver formulas for components V_1 and V_3 from the expression (13) on Γ_1:

$$V_1(\theta) := V_1(\sigma) = \alpha_0 + 2\alpha_0^{(0)} + c^*_{0,2} + \sum_{n=1}^{\infty} \left(\alpha_n + 2\alpha_n^{(0)} + c^*_{-n,2} + c^*_{n,2}\right) \cos n\theta +$$

$$+ \sum_{n=1}^{\infty} \left(-\beta_n - 2\beta_n^{(0)} + c^*_{n,1} - c^*_{-n,1}\right) \sin n\theta, \qquad (34)$$

$$V_3(\theta) := V_3(\sigma) = c^*_{0,1} - 2\beta_0^{(0)} + \sum_{n=1}^{\infty} \left(-2\beta_n^{(0)} + c^*_{-n,1} + c^*_{n,1}\right) \cos n\theta +$$

$$+ \sum_{n=1}^{\infty} \left(-2\alpha_n^{(0)} + c^*_{-n,2} - c^*_{n,2}\right) \sin n\theta. \qquad (35)$$

Equating coefficients near $\cos n\theta$ and $\sin n\theta$, respectively, in the qualities (34) and (19), (35) and (20), receive, using the denotations (31) and (32), a system of equations (21) — (26) according to coefficients of required series (28).

Summarize obtained results, we have that restricting a solvability of the system (21) — (26) in the class $\mathcal{E}$, that means, firstly, a condition to the geometry of the domain D_ζ and, secondly, a condition to the choice of the boundary functions u_1 and u_3, obtain, that the series (28) and (29) are absolutely and uniformly convergent on $\overline{\mathcal{D}_1}$ and a function (27) is a general solution of the auxiliary (1-3)-problem. The theorem is proved.

Remark. A choice of the class $\mathcal{E}$ can be done by any another way, choosing conditions for functions of the class for which series (28) and (29) are absolutely convergent on Γ_1.

Theorem 3. *Let conditions of Theorem 1 are satisfied, then the formula*

$$\Phi(\zeta) \equiv \Phi_*(\tau) \quad \forall \zeta = x + ye_2 \in D_\zeta, \ \tau \in \mathcal{D}_1 : \omega(\tau) = z := x + ie_2 \in \mathbb{C} \qquad (36)$$

generates a general solution of the (1-3)*-problem.*

Examples.

1. Let a domain D_ζ be a unit disk $\mathcal{D}_1$. Then a mapping ω is the identity mapping, i.e., $\omega(z) = z$ for all $z \in D_z$, $\operatorname{Im}\omega(\sigma) = \sin\theta$. The auxiliary (1-3)-problem coincides with the the (1-3)-problem for $\mathcal{D}_1$. Furthermore, $\delta''_{-1} = \frac{1}{2}$, $\delta''_1 = -\frac{1}{2}$, $\delta''_n = 0$ for another integer n, and $\delta'_k = 0$ for all integer k. It is easy to check, that for this particular case the system of equations (21) — (26) transforms to the system (22) — (31) from the paper [13] for $r = 1$, a condition of solvability of which can be written in the form $b_1 = a'_1$. The proposed method gives a required solution of the (1-3)-problem, for example, if boundary functions u_1 and u_3 satisfy conditions of Theorem 1 in the paper [13]. Note, that for our case in (17): $\upsilon_k = k^{-(1+\alpha)}$, $k = 1, 2, \ldots$, $\alpha > 0$, and a general solution of the (1-3)-problem with zero data $u_1 = u_3 \equiv 0$ is a function $\Phi(\zeta) = i\,(b - ae_2 + c\zeta)$, where a, b and c are arbitrary real numbers.

2. Let a boundary of the domain D_ζ be the Pascal's limacon $D_\zeta := \{x + e_2 y\}$, where $x = R\left(\cos\theta + \frac{1}{4}\cos 2\theta\right)$, $y = R\left(\sin\theta + \frac{1}{4}\sin 2\theta\right)$, $0 \le \theta < 2\pi$, and any fixed constant $R > 0$. Then a function ω has a form

$$\omega(\zeta) = R\left(\zeta + \frac{1}{4}\zeta^2\right) \quad \forall\zeta \in \overline{D_\zeta}. \tag{37}$$

Consider the (1-3)-problem with boundary conditions, for which

$$a_k = a'_k = 0, k = 0, 1, 2,\ \ b_k = b'_k = 0, k = 1, 2, \tag{38}$$

and the following inclusions holds

$$\big(\{a'\}_0, \{b\}_1, \{a\}_0, \{b'\}_1\big) \in \mathcal{E},\ \big(\{a\}_0, \{b'\}_1, \{a'\}_0, \{b\}_1,\big) \in \mathcal{E}. \tag{39}$$

In this case an expression (18) has a form

$$\frac{\operatorname{Im}\omega(\sigma)}{\omega'(\sigma)} = i\frac{1}{8}\,\sigma^{-2} + i\frac{7}{16}\,\sigma^{-1} - i\frac{7}{32} - i\frac{25}{64}\,\sigma + i\frac{9}{32}\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n \sigma^n\,\forall\sigma \in \Gamma_1.$$

Then

$$\delta'_k = 0\ \forall k \in \mathbb{Z},\ \delta''_{-n} = 0\ \forall n \in \{3, 4, \ldots\}, \tag{40}$$

$$\delta''_{-2} = \frac{1}{8},\ \delta''_{-1} = \frac{7}{16},\ \delta''_0 = -\frac{7}{32},\ \delta''_1 = -\frac{25}{64}, \tag{41}$$

$$\delta''_n = \frac{9}{32}\left(-\frac{1}{2}\right)^n \ \forall n \in \{2, 3, \ldots\}. \tag{42}$$

Denote for any symbol variable υ the expression

$$\psi_n(\upsilon) := \frac{9}{32}\sum_{k=0}^{n-2}(k+1)\left(-\frac{1}{2}\right)^{n-k}\upsilon_{k+1} - \frac{25}{64}n\upsilon_n -$$

$$-\frac{7}{32}(n+1)\upsilon_{n+1}+\frac{7}{16}(n+2)\upsilon_{n+2}+\frac{1}{8}(n+3)\upsilon_{n+3}\,\forall n\geq 3. \qquad (43)$$

Taking into account relations $\Lambda^{+}_{1,n,k}=\Lambda^{-}_{1,n,k}=0$ for $n\in\mathbb{N}$ and $k\in\mathbb{N}_0$, $\Lambda^{+}_{2,n,k}=\Lambda^{-}_{2,n,k}=\delta''_{n-k}$ for $n\geq 3$ and $k\in\mathbb{N}_0$, formulas (38) — (43), obtain, that the system of equations (21) — (26) transforms to a form

$$\alpha_0+2\alpha_0^{(0)}-\frac{7}{32}\,\alpha_1+\frac{7}{8}\,\alpha_2+\frac{3}{8}\,\alpha_3=0, \qquad (44)$$

$$\frac{67}{64}\,\alpha_1+2\alpha_1^{(0)}-\frac{3}{16}\,\alpha_2+\frac{21}{16}\,\alpha_3+\frac{1}{2}\,\alpha_4=0, \qquad (45)$$

$$\alpha_2+2\alpha_2^{(0)}+\frac{25}{128}\,\alpha_1-\frac{25}{32}\,\alpha_2-\frac{21}{32}\,\alpha_3+\frac{7}{4}\,\alpha_4+\frac{5}{8}\,\alpha_5=0, \qquad (46)$$

$$\alpha_n+2\alpha_n^{(0)}+\psi_n(\alpha)=a_n\,\forall n\in\{3,4,\dots\}, \qquad (47)$$

$$\frac{11}{64}\,\beta_1+2\beta_1^{(0)}-\frac{11}{16}\,\beta_2+\frac{21}{16}\,\beta_3+\frac{1}{2}\beta_4=0, \qquad (48)$$

$$-\frac{7}{128}\,\beta_1+2\beta_2^{(0)}+\frac{7}{32}\,\beta_2-\frac{21}{32}\,\beta_3+\frac{7}{4}\,\beta_4+\frac{5}{8}\,\beta_5=0, \qquad (49)$$

$$\beta_n+2\beta_n^{(0)}+\psi_n(\beta)=-b_n\,\forall n\in\{3,4,\dots\}, \qquad (50)$$

$$2\beta_0^{(0)}-\frac{7}{32}\,\beta_1+\frac{7}{8}\,\beta_2+\frac{3}{8}\,\beta_3=0, \qquad (51)$$

$$2\beta_1^{(0)}+\frac{3}{64}\,\beta_1-\frac{3}{16}\,\beta_2+\frac{21}{16}\,\beta_3+\frac{1}{2}\,\beta_4=0, \qquad (52)$$

$$2\beta_2^{(0)}+\frac{25}{128}\,\beta_1-\frac{25}{32}\,\beta_2-\frac{21}{32}\,\beta_3+\frac{7}{4}\,\beta_4+\frac{5}{8}\,\beta_5=0, \qquad (53)$$

$$2\beta_n^{(0)}+\psi_n(\beta)=-a'_n\,\forall n\in\{3,4,\dots\}, \qquad (54)$$

$$2\alpha_1^{(0)}-\frac{53}{64}\,\alpha_1-\frac{11}{16}\,\alpha_2+\frac{21}{16}\,\alpha_3+\frac{1}{2}\,\alpha_4=0, \qquad (55)$$

$$2\alpha_2^{(0)}-\frac{7}{128}\,\alpha_1-\frac{25}{32}\,\alpha_2-\frac{21}{32}\,\alpha_3+\frac{7}{4}\,\alpha_4+\frac{5}{8}\,\alpha_5=0, \qquad (56)$$

$$2\alpha_n^{(0)}+\psi_n(\alpha)=-b'_n\,\forall n\in\{3,4,\dots\}. \qquad (57)$$

Solving obtained system (44) – (57) conclude, that a solution of the required (1-3)-problem has the form:

$$\Phi(\mathrm{Re}\tau+e_2\mathrm{Im}\tau)=\widetilde{F}(\tau)-\left(\frac{i}{2}\,Y(\tau)\,\widetilde{F}'(\tau)-\widetilde{F}_0(\tau)\right)\rho \qquad \forall\tau\in\mathcal{D}_1, \quad (58)$$

where coefficients of expansions (28) are expressed by the formulas

$$c_0 = -2A + iC \qquad \forall A \,\text{and}\, C \in \mathbb{R},$$

$$c_1 = 4iB,\; c_2 = iB \qquad \forall B \in \mathbb{R},$$

$$c_n = a_n + b'_n + i\left(a'_n - b_n\right), n = 3, 4, \dots ,$$

$$c_0^{(0)} = -\frac{3}{16}\left(a_3 + b'_3\right) + A + i\frac{3}{16}\left(b_3 - a'_3\right) \qquad \forall A \in \mathbb{R},$$

$$c_1^{(0)} = -\frac{21}{32}\left(a_3 + b'_3\right) - \frac{1}{4}\left(a_4 + b'_4\right) + i\left(\frac{21}{32}\left(b_3 - a'_3\right) + \frac{1}{8}\left(b_4 - a'_4\right)\right),$$

$$c_2^{(0)} = \frac{21}{64}\left(a_3 + b'_3\right) - \frac{7}{8}\left(a_4 + b'_4\right) - \frac{5}{16}\left(a_5 + b'_5\right) +$$

$$+i\left(\frac{21}{64}\left(a'_3 - b_3\right) + \frac{7}{8}\left(b_4 - a'_4\right) + \frac{5}{16}\left(b_5 - a'_5\right)\right),$$

$$c_3^{(0)} = \frac{75}{128}\, a_3 + \frac{11}{128}\, b'_3 + \frac{7}{16}\left(a_4 + b'_4\right) - \frac{35}{32}\left(a_5 + b'_5\right) - \frac{3}{8}\left(a_6 + b'_6\right) +$$

$$+i\left(\frac{11}{128}\, a'_3 - \frac{75}{128}\, b_3 + \frac{7}{16}\left(a'_4 - b_4\right) - \frac{35}{32}\left(a'_5 - b_5\right) - \frac{3}{8}\left(a'_6 - b_6\right)\right),$$

$$c_4^{(0)} = -\frac{27}{256}\left(a_3 + b'_3\right) + \frac{25a_4 + 9b'_4}{32} + \frac{35}{64}\left(a_5 + b'_5\right) -$$

$$-\frac{21\left(a_6 + b'_6\right) 7\left(a_7 + b'_7\right)}{16} + i\left(\frac{27}{256}\left(b_3 - a'_3\right) + \frac{11a'_4 - 25b_4}{32} + \right.$$

$$\left. + \frac{35}{64}\left(a'_5 - b_5\right) - \frac{21}{16}\left(a'_6 - b_6\right) - \frac{7}{16}\left(a'_7 - b_7\right)\right),$$

$$c_n^{(0)} = \frac{9}{32}\sum_{k=2}^{n-2}(k+1)\left(-\frac{1}{2}\right)^{n-k+1}\left(a_{k+1} + b'_{k+1}\right) + \frac{25}{128}\, na_n +$$

$$+\frac{25n - 64}{128}\, b'_n + \frac{7}{64}(n+1)\left(a_{n+1} + b'_{n+1}\right) -$$

$$-\frac{7}{32}(n+2)\left(a_{n+2} + b'_{n+2}\right) - \frac{1}{16}(n+3)\left(a_{n+3} + b'_{n+3}\right) +$$

$$+i\left(\frac{9}{32}\sum_{k=2}^{n-2}(k+1)\left(-\frac{1}{2}\right)^{n-k+1}\left(a'_{k+1} - b_{k+1}\right) + \right.$$

$$+\frac{25n-64}{128}\,a'_n-\frac{25}{128}\,nb_n+\frac{7}{64}(n+1)\left(a'_{n+1}-b_{n+1}\right)-$$

$$\left.-\frac{7}{32}(n+2)\left(a'_{n+2}-b_{n+2}\right)-\frac{1}{16}(n+3)\left(a'_{n+3}-b_{n+3}\right)\right)\ \forall n\geq 5.$$

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