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Generalized kinetic equations for dense gases and liquids far from equilibrium in Renyi statistics

Dedicated to the 80 th anniversary of Prof. D.Ya. Petrina

Based on the Zubarev nonequilibrium statistical operator method and the maximum entropy principle for the Renyi entropy the nonequilibrium statistical operator and the generalized kinetic equations for the nonequilibrium one- and two-particle distribution functions are obtained for description of kinetic processes in gases and liquids far from equilibrium.

Для опису кінетичних процесів у газах та рідинах далеких від рівноваги на основі методу нерівноважного статистичного оператора Зубарєва та принципу максимуму ентропії Рені отримано нерівноважний статистичний оператор та узагальнені кінетичні рівняння для нерівноважних одночастинкової та двочастинкової функцій розподілу частинок.

1 Introduction

The ideas of D.Ya. Petrina for investigation of urgent problems in statistical theory of many-particle system based on the strict mathematical approach to the Bogoliubov equations [1, 2, 3, 4] remain important nowadays. The study of nonequilibrium processes in gases and liquids far from equilibrium or in finite quantum systems (nanosystems) are among them. These investigations are actively developed by the followers of D.Ya. Petrina [5, 6, 7, 8].

Different models and approaches are applied for the study of nonlinear kinetic fluctuations in dense gases, liquids and plasma far from equilibrium with the typical long-range interactions which remains urgent in statistical theory of nonequilibrium processes [9, 10, 11, 12, 13, 14].

In the present paper, for description of nonequilibrium processes in dense gases and liquids we propose to use the Renyi entropy which depends on parameter $q\,(0 < q \leqslant 1)$ and coincide with the Shannon-Gibbs entropy at $q = 1$. In reference [14] based on the Zubarev nonequilibrium statistical operator (NSO) method [15, 16] and the maximum entropy principle for the Renyi entropy there were obtained the NSO and the generalized transport equations for the parameters of the reduced-description of nonequilibrium processes in extensive statistical mechanics. Here, we apply this approach to description of kinetic processes in dense gases and liquids far from equilibrium, when the nonequilibrium one- and two-particle distribution functions are chosen for the reduced-description parameters.

2 Zubarev nonequilibrium statistical operator in Renyi statistics

Within the framework of the Zubarev NSO method, when the basic parameters of a reduced description $\langle \hat{P}_n \rangle^t$ are selected according to N.N. Bogoliubov, the nonequilibrium statistical operator $\rho\,(x_1, \ldots, x_N; t) = \rho\left(x^N; t\right)$ can be presented in general form as a solution of Liouville equation with taking into account a projection [15, 16]:

$$\begin{aligned}\rho(x^N;t) &= \rho_{rel}(x^N;t) \\ &- \int_{-\infty}^{t} e^{\varepsilon(t'-t)} T(t,t')[1-P_{rel}(t')] iL_N \rho_{rel}(x^N;t')dt'.\end{aligned} \tag{1}$$

Here, $T\,(t,t') = \exp_+ \left\{ -\int_{t'}^{t} [1 - P_{rel}\,(t')]\, iL_N dt' \right\}$ is the evolution operator with regard to projection, $\exp_+$ is the ordered exponential, iL_N is the Liouville operator for a system of interacting particles, which in classical case has the following form:

$$iL_N = \sum_{j=1}^{N} \frac{\vec{p}_j}{m} \cdot \frac{\partial}{\partial \vec{r}_j} - \frac{1}{2} \sum_{l \neq j=1}^{N} \frac{\partial}{\partial \vec{r}_j} \Phi\left(r_{lj}\right) \left(\frac{\partial}{\partial \vec{p}_j} - \frac{\partial}{\partial \vec{p}_l} \right).$$

We use the following notations: $x_j = \{\vec{p}_j, \vec{r}_j\}$ are the phase variables of the particle j, $\Phi\left(r_{lj}\right)$ is the interaction energy of two particles, $\vec{p}_j$ is the j-particle momentum and m its mass, $r_{lj} = |\vec{r}_l - \vec{r}_j|$ the distance between a pair of interacting particles. $P_{rel}(t')$ is the generalized Kawasaki–Gunton projection operator whose structure depends on the form of the relevant statistical operator:

$$P_{rel}\rho' = \left(\rho_{rel}(t) - \sum_n \frac{\delta \rho_{rel}(t)}{\delta \langle \hat{P}_n \rangle^t} \langle \hat{P}_n \rangle^t \right) \int d\Gamma_N \rho' + \sum_n \frac{\delta \rho_{rel}(t)}{\delta \langle \hat{P}_n \rangle^t} \int d\Gamma_N \hat{P}_n \rho'.$$

$\rho_{rel}(x^N; t')$ is the relevant statistical operator which is equal to $\rho(x^N; t)$ at the initial moment of time. We will determine $\rho_{rel}(x^N; t')$ using the Lagrange method from the condition of entropy functional maximum for the Renyi entropy [14]

$$S_R(\rho) = \frac{1}{1-q} \ln \int d\Gamma_N \rho^q(t).$$

The corresponding functional at fixed parameters of the reduced description, taking into account the normalization condition, has the following form:

$$L_R(\rho) = \frac{1}{1-q} \ln \int d\Gamma_N \rho^q(t) - \alpha \int d\Gamma_N \rho(t) - \sum_n F_n(t) \int d\Gamma_N \hat{P}_n \rho(t),$$

where, $F_n(t)$ are the Lagrange multipliers. Equalizing its functional derivative to zero $\frac{\delta L_R(\rho)}{\delta \rho} = 0$ and determining parameter $\alpha = \frac{q}{1-q} - \sum_n F_n(t) \langle \hat{P}_n \rangle^t$ we obtain the relevant statistical operator in the form:

$$\rho_{rel}(t) = \frac{1}{Z_R(t)} \left[1 - \frac{q-1}{q} \sum_n F_n(t) \delta \hat{P}_n \right]^{\frac{1}{q-1}}, \tag{2}$$

where

$$Z_R(t) = \int d\Gamma_N \left[1 - \frac{q-1}{q} \sum_n F_n(t) \delta \hat{P}_n \right]^{\frac{1}{q-1}}$$

is the partition function, $\delta\hat{P}_n = \hat{P}_n - \langle\hat{P}_n\rangle^t$. The Lagrange multipliers $F_n(t)$ are defined from the self-consistency conditions:

$$\langle\hat{P}_n\rangle^t = \langle\hat{P}_n\rangle^t_{rel}. \tag{3}$$

The variation derivative of the relevant statistical operator can be presented in the form:

$$\frac{\delta\rho_{rel}(t)}{\delta\langle\hat{P}_m\rangle^t} = \rho_{rel}(t)\delta\left[\frac{1}{q}\psi^{-1}(t)\left(F_m(t) - \sum_n \frac{\delta F_n(t)}{\delta\langle\hat{P}_m\rangle^t}\delta\hat{P}_n\right)\right],$$

where $\delta\,[\ldots] = [\ldots] - \langle[\ldots]\rangle^t_{rel}$ and we use the notation

$$\psi(t) = 1 - \frac{q-1}{q}\sum_n F_n(t)\delta\hat{P}_n. \tag{4}$$

We calculate the derivatives of the Lagrange multipliers with regard to the reduced-description parameters in the following way:

$$\frac{\delta F_n(t)}{\delta\langle\hat{P}_m\rangle^t} = \left(\frac{\delta\langle\hat{P}_m\rangle^t}{\delta F_n(t)}\right)^{-1}.$$

This can be done in general case. Thus,

$$\frac{\delta\langle\hat{P}_m\rangle^t}{\delta F_n(t)} = \int d\Gamma_N \hat{P}_m \frac{\delta\rho_{rel}(t)}{\delta F_n(t)}$$

and after calculating $\frac{\delta\rho_{rel}(t)}{\delta F_n(t)}$ in the right-hand side of relation we obtain the set of equations for the desired derivatives

$$\frac{\delta\langle\hat{P}_m\rangle^t}{\delta F_n(t)} = \langle\delta\hat{P}_m\frac{1}{q}\psi^{-1}(t)\rangle^t_{rel}\sum_l \frac{\delta\langle\hat{P}_l\rangle^t}{\delta F_n(t)} - \langle\delta\hat{P}_m\frac{1}{q}\psi^{-1}(t)\delta\hat{P}_n\rangle^t_{rel}.$$

The solution in matrix form is

$$\frac{\delta\langle\hat{P}\rangle^t}{\delta F(t)} = -\left[I - \langle\delta\hat{P}\frac{1}{q}\psi^{-1}(t)\rangle^t_{rel}F(t)\right]^{-1}\langle\delta\hat{P}\frac{1}{q}\psi^{-1}(t)\delta\hat{P}\rangle^t_{rel} = f(t),$$

where

$$\frac{\delta\langle\hat{P}_m\rangle^t}{\delta F_n(t)} = \left(\frac{\delta\langle\hat{P}\rangle^t}{\delta F(t)}\right)_{mn} = f_{mn}(t).$$

Thus, the functional derivative can be written in the form:

$$\frac{\delta\rho_{rel}(t)}{\delta\langle\hat{P}_m\rangle^t} = \rho_{rel}(t)\delta\left[\frac{1}{q}\psi^{-1}(t)\left(F_m(t) - \sum_n f^{-1}_{mn}\delta\hat{P}_n\right)\right].$$

Then, the Kawasaki–Gunton projection operator has the following structure:

$$\begin{aligned} P_{rel}(t)\rho' &= \rho_{rel}(t)\int d\Gamma_N\rho' \\ &+ \sum_m \rho_{rel}(t)\delta\left[\frac{1}{q}\psi^{-1}(t)\left(F_m(t) + \sum_n f^{-1}_{mn}(t)\delta\hat{P}_n\right)\right] \\ &\times \left(\int d\Gamma_N\hat{P}_m\rho' - \langle\hat{P}_m\rangle^t\int d\Gamma_N\rho'\right). \end{aligned}$$

It is further necessary to explore an action of the operators $P_{rel}(t)iL_N$ on the relevant statistical operator. Since

$$iL_N\rho_{rel}(t) = -\rho_{rel}(t)\frac{1}{q}\psi^{-1}(t)\sum_n F_n(t)\dot{\hat{P}}_n = A(t)\rho_{rel}(t),$$

then $P_{rel}(t)iL_N\rho_{rel}(t) = P_{rel}(t)A(t)\rho_{rel}(t) = [P(t)A(t)]\,\rho_{rel}(t)$, where $P(t)$ is the projection operator which now acts on dynamic variables:

$$\begin{aligned} P(t)\ldots &= \langle\ldots\rangle^t_{rel} \\ &+ \sum_m \delta\left[\frac{1}{q}\psi^{-1}(t)\left(F_m(t) + \sum_n f^{-1}_{mn}(t)\delta\hat{P}_n\right)\right]\langle\ldots\delta\hat{P}_m\rangle^t_{rel}. \end{aligned}$$

Since

$$A(t) = -\frac{1}{q}\psi^{-1}(t)\sum_n F_n(t)\dot{\hat{P}}_n,$$

we can present $[1-P_{rel}(t)]\,iL_N\rho_{rel}(t)$ as follows:

$$\begin{aligned} [1-P_{rel}(t)]\,iL_N\rho_{rel}(t) &= [1-P(t)]\,iL_N\rho_{rel}(t) \\ &= -\sum_n I_n(t)F_n(t)\rho_{rel}(t), \end{aligned} \tag{5}$$

where

$$I_n(t) = [1-P(t)]\,\frac{1}{q}\psi^{-1}(t)\dot{\hat{P}}_n$$

are the generalized flows. Taking into account (5), we can now write down an explicit expression for the nonequilibrium statistical operator (1):

$$\begin{aligned} \rho(x^N;t) &= \rho_{rel}(x^N;t) \\ &- \sum_n \int_{-\infty}^{t} e^{\varepsilon(t'-t)} T(t,t') I_n(t') F_n(t') \rho_{rel}(x^N;t') dt'. \end{aligned} \tag{6}$$

This allows us to obtain the generalized transport equations for the reduced-description parameters. They can be presented in the form:

$$\frac{\partial}{\partial t}\langle \hat{P}_m \rangle^t = \langle \dot{\hat{P}}_m \rangle^t_{rel} + \sum_n \int_{-\infty}^{t} e^{\varepsilon(t'-t)} \varphi_{mn}(t,t') F_n(t') dt', \tag{7}$$

where

$$\varphi_{mn}(t,t') = \int d\Gamma_N \dot{\hat{P}}_m T(t,t') I_n(t') \rho_{rel}(t') \tag{8}$$

are the generalized transport kernels (memory functions) – the time correlation functions describing the dissipative processes in the system. They are built on the generalized flows $I_n(t)$. Transport equations (7) describe non-Markovian processes and when $\varphi_{mn}(t,t') \approx \varphi_{mn}\delta(t-t')$ describe Markovian processes. The set of transport equations is not closed. The nonequilibrium Lagrange multipliers in it (the nonequilibrium thermodynamic parameters in the case of hydrodynamic description) are determined from the self-consistency conditions (3). From this point of view, the set of transport equations is closed. Nonequilibrium statistical operator (6) and transport equations (7) compose a complete instrument for description of nonequilibrium processes when the reduced-description parameters $\langle \hat{P}_n \rangle^t$ are selected.

In the following section we apply the presented approach to description of nonlinear kinetic fluctuations in gases and liquids far from equilibrium.

3 Generalized kinetic equations in Renyi statistics

For description of kinetic processes in classical gases and liquids far from equilibrium the nonequilibrium one- and two-particle distribution functions can be selected as the basic parameters of the reduced description

$$f_1(x;t) = \langle \hat{n}_1(x) \rangle^t, \qquad f_2(x,x';t) = \langle \hat{n}_2(x,x') \rangle^t, \tag{9}$$

where

$$\hat{n}_1(x) = \sum_{j=1}^{N} \delta(x - x_j), \qquad \hat{n}_2(x, x') = \sum_{j=1}^{N} \sum_{l=1}^{N} \delta(x - x_j)\delta(x' - x_l)$$

are the microscopic phase densities of N particles in volume V. The latter completely satisfy conservation laws of particles density, momentum and energy since they define microscopic densities of particles number, momentum and energy:

$$\hat{n}(\vec{r}) = \int d\vec{p}\, \hat{n}_1(\vec{r}, \vec{p}), \qquad \hat{\vec{p}}(\vec{r}) = \int d\vec{p}\, \hat{n}_1(\vec{r}, \vec{p})\, \vec{p},$$

$$\hat{\varepsilon}^{kin}(\vec{r}) = \int d\vec{p}\, \hat{n}_1(\vec{r}, \vec{p}) \frac{p^2}{2m},$$

$$\hat{\varepsilon}^{int}(\vec{r}) = \frac{1}{2} \int d\vec{p} \int d\vec{p}' \int d\vec{r}'\, \Phi(|\vec{r} - \vec{r}'|) \hat{n}_2(\vec{r}, \vec{p}; \vec{r}', \vec{p}')$$

and

$$\langle \hat{n}(\vec{r}) \rangle^t = \int d\vec{p} f_1(\vec{r}, \vec{p}; t), \qquad \langle \hat{\vec{p}}(\vec{r}) \rangle^t = \int d\vec{p} f_1(\vec{r}, \vec{p}; t) \vec{p},$$

$$\langle \hat{\varepsilon}^{kin}(\vec{r}) \rangle^t = \int d\vec{p} f_1(\vec{r}, \vec{p}; t) \frac{p^2}{2m},$$

$$\langle \hat{\varepsilon}^{int}(\vec{r}) \rangle^t = \frac{1}{2} \int d\vec{p} \int d\vec{p}' \int d\vec{r}'\, \Phi(|\vec{r} - \vec{r}'|) f_2(\vec{r}, \vec{p}; \vec{r}', \vec{p}'; t).$$

Conservation laws for average particles number, momentum and total energy have the following form:

$$\frac{\partial}{\partial t} \langle \hat{n}(\vec{r}) \rangle^t = -\frac{1}{m} \vec{\nabla} \cdot \langle \hat{\vec{p}}(\vec{r}) \rangle^t, \tag{10}$$

$$\frac{\partial}{\partial t} \langle \hat{\vec{p}}(\vec{r}) \rangle^t = -\vec{\nabla} : \left(\langle \hat{\overleftrightarrow{T}}^{kin}(\vec{r}) \rangle^t + \langle \hat{\overleftrightarrow{T}}^{int}(\vec{r}) \rangle^t \right),$$

$$\frac{\partial}{\partial t} \langle \hat{\varepsilon}(\vec{r}) \rangle^t = -\vec{\nabla} \cdot \left(\langle \hat{\vec{j}}_E{}^{kin}(\vec{r}) \rangle^t + \langle \hat{\vec{j}}_E{}^{int}(\vec{r}) \rangle^t \right),$$

where $\langle \hat{\varepsilon}(\vec{r}) \rangle^t = \langle \hat{\varepsilon}^{kin}(\vec{r}) \rangle^t + \langle \hat{\varepsilon}^{int}(\vec{r}) \rangle^t$ is the nonequilibrium average value of total energy density and $\vec{\nabla} = \frac{\partial}{\partial \vec{r}}$.

$$\langle \hat{\overleftrightarrow{T}}^{kin}(\vec{r}) \rangle^t = \int d\vec{p}\, \frac{\vec{p}\,\vec{p}}{m} f_1(\vec{r}, \vec{p}; t)$$

is the nonequilibrium average value of the kinetic part of stress tensor density,

$$\langle \hat{\overleftrightarrow{T}}^{int}(\vec{r})\rangle^t = \frac{1}{2}\int d\vec{p}\int d\vec{p}'\int d\vec{r}'\frac{\partial}{\partial|\vec{r}-\vec{r}'|}\Phi(|\vec{r}-\vec{r}'|) \times \frac{(\vec{r}-\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}f_2(\vec{r},\vec{r}';\vec{p},\vec{p}';t) \qquad (11)$$

is the nonequilibrium average value of the potential part of stress tensor density,

$$\langle \hat{\vec{j}}_E^{kin}(\vec{r})\rangle^t = \int d\vec{p}\,\frac{p^2}{2m}\,\vec{p}\,f_1(\vec{r},\vec{p};t)$$

is the nonequilibrium average value of the kinetic part of energy flow density,

$$\langle \hat{\vec{j}}_E^{int}(\vec{r})\rangle^t = \int d\vec{p}\int d\vec{p}'\int d\vec{r}'\left[\frac{\vec{p}}{m}\Phi(|\vec{r}-\vec{r}'|) -\Phi(|\vec{r}-\vec{r}'|)\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\right]f_2(\vec{r},\vec{r}';\vec{p},\vec{p}';t)$$

is the nonequilibrium average value of the potential part of energy flow density. It follows from the above relations that the nonequilibrium one-particle distribution function defines the macroscopic nonequilibrium densities of particles number, momentum as well as kinetic part of total energy, stress tensor and energy flow. Whereas, the two-particle nonequilibrium distribution function defines potential part of total energy, stress tensor and energy flow. Thus, in systems far from equilibrium the nonlinear hydrodynamic fluctuations are caused by the nonlinear fluctuations of nonequilibrium one- and two-particle distribution functions for which the kinetic equations should be built. Therefore, in the case when the nonequilibrium one- and two-particle distribution functions $f_1(x;t) = \langle\hat{n}_1(x)\rangle^t$ and $f_2(x,x';t) = \langle\hat{n}_2(x,x')\rangle^t$ are selected as the parameters of the reduced description, according to (2) the relevant distribution function has the following form:

$$\rho_{rel}(t) = \frac{1}{Z_R(t)}\left\{1 - \frac{q-1}{q}\left[\int dx a(x;t)\delta\hat{n}_1(x;t) + \int dx\int dx' b(x,x';t)\delta\hat{n}_2(x,x';t)\right]\right\}^{\frac{1}{q-1}}, \qquad (12)$$

where

$$Z_R(t) = \int d\Gamma_N \Big\{ 1 \quad - \quad \frac{q-1}{q} \Big[\int dx a(x;t) \delta\hat{n}_1(x;t) + \int dx \int dx' b(x,x';t) \delta\hat{n}_2(x,x';t) \Big] \Big\}^{\frac{1}{q-1}}$$

is the partition function of the relevant distribution function. The parameters $a(x;t)$ and $b(x,x';t)$ are determined from the self-consistency conditions:

$$\langle \hat{n}_1(x) \rangle^t = \langle \hat{n}_1(x) \rangle^t_{rel}, \qquad \langle \hat{n}_2(x,x') \rangle^t = \langle \hat{n}_2(x,x') \rangle^t_{rel}. \tag{13}$$

The relevant distribution function (12) can be presented in a slightly different way

$$\rho_{rel}(t) = \frac{1}{Z_R(t)} \Big\{ 1 \quad - \quad \frac{q-1}{q} \Big[\int dx a'(x;t) \hat{n}_1(x) + \int dx \int dx' b'(x,x';t) \hat{n}_2(x,x') \Big] \Big\}^{\frac{1}{q-1}} \tag{14}$$

writing down the Lagrange parameters in the form:

$$a'(x;t) = a(x;t) \Big\{ 1 + \frac{q-1}{q} \times \Big[\int dx a(x;t) f_1(x;t) + \int dx \int dx' b(x,x';t) f_2(x,x';t) \Big] \Big\}^{-1},$$

$$b'(x,x';t) = b(x,x';t) \Big\{ 1 + \frac{q-1}{q} \times \Big[\int dx a(x;t) f_1(x;t) + \int dx \int dx' b(x,x';t) f_2(x,x';t) \Big] \Big\}^{-1}.$$

It is important to note that in the case of $q = 1$, $a'(x;t) = a(x;t)$, $b'(x,x';t) = b(x,x';t)$ and we obtain the relevant distribution function corresponding to Gibbs statistics.

Now we can present the nonequilibrium statistical operator as follows:

$$\begin{aligned}\rho(t) \quad = \quad & \rho_{rel}(t) + \int dx' \int_{-\infty}^{t} e^{\varepsilon(t'-t)} T(t,t') a(x';t') I_n^{(1)}(x';t') \rho_{rel}(t) dt' \qquad (15) \\ + \quad & \int dx' \int dx'' \int_{-\infty}^{t} e^{\varepsilon(t'-t)} T(t,t') b(x',x'';t') I_n^{(2)}(x',x'';t') \rho_{rel}(t) dt'.\end{aligned}$$

Here,

$$I_n^{(1)}(x;t) = [1 - P(t)] \frac{1}{q} \psi^{-1}(t) i L_N \hat{n}_1(x),$$
$$I_n^{(2)}(x,x';t) = [1 - P(t)] \frac{1}{q} \psi^{-1}(t) i L_N \hat{n}_2(x,x')$$

are the generalized flows in which the function $\psi(t)$ equals to

$$\psi(t) = 1 - \frac{q-1}{q} \left[\int dx\, a(x;t) \delta\hat{n}_1(x;t) + \int dx \int dx' b(x,x';t) \delta\hat{n}_2(x,x';t) \right].$$

Using the NSO (15) we obtain a set of the generalized kinetic equations for the reduced-description parameters (9) $f_1(x;t) = \langle \hat{n}_1(x) \rangle^t$ and $f_2(x,x';t) = \langle \hat{n}_2(x,x') \rangle^t$ according to (7):

$$\begin{aligned}\frac{\partial}{\partial t} \langle \hat{n}_1(x) \rangle^t = & \int dx' \Phi_{nn}^{11}(x,x';t) a(x';t) \qquad (16) \\ & + \int dx' \int dx'' \Phi_{nn}^{12}(x;x',x'';t) b(x',x'';t) \\ & + \int dx' \int_{-\infty}^{t} e^{\varepsilon(t'-t)} \varphi_{nn}^{11}(x,x';t,t') a(x';t') dt' \\ & + \int dx' \int dx'' \int_{-\infty}^{t} e^{\varepsilon(t'-t)} \varphi_{nn}^{12}(x;x',x'';t,t') b(x',x'';t') dt',\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t} \langle \hat{n}_2(x,x') \rangle^t = & \int dx'' \Phi_{nn}^{21}(x,x';x'';t) a(x'';t) \qquad (17) \\ & + \int dx'' \int dx''' \Phi_{nn}^{22}(x;x';x'',x''';t) b(x'',x''';t) \\ & + \int dx'' \int_{-\infty}^{t} e^{\varepsilon(t'-t)} \varphi_{nn}^{21}(x,x';x''t,t') a(x'';t') dt' \\ & + \int dx'' \int dx''' \int_{-\infty}^{t} e^{\varepsilon(t'-t)} \varphi_{nn}^{22}(x;x';x'',x''';t,t') b(x'',x''';t') dt'.\end{aligned}$$

Here,

$$\Phi^{\alpha\beta}_{pp}(x,x';t) = \int d\Gamma_N p_\alpha(x)\frac{1}{q}\psi^{-1} iL_N p_\beta(x')\rho_{rel}(x^N;t), \tag{18}$$

$$\varphi^{\alpha\beta}_{I_pI_p}(x;x';t,t') = \int d\Gamma_N iL_N p_\alpha(x) T(t,t') I^\beta_p(x';t')\rho_{rel}(x^N;t'), \tag{19}$$

are the kinetic transport kernels, where we use the notation $p_\alpha(x) = \{\hat{n}_1(x), \hat{n}_2(x,x')\}$. Neglecting the two-particle correlation at $q = 1$ the generalized kinetic equation in Renyi statistics transforms into the kinetic equation within Gibbs statistics [16] with the transport kernel calculated using the relevant distribution function $\rho_{rel}(t) = \prod_{j=1}^N \frac{f_1(x_j;t)}{e}$. In this case, at $q = 1$, within the NSO method [15, 16] the Liouville equation should be solved with the boundary condition

$$\frac{\partial}{\partial t}\rho(x^N;t) + iL_N\rho(x^N;t) = -\varepsilon\left(\rho(x^N;t) - \prod_{j=1}^N \frac{f_1(x_j;t)}{e}\right),$$

that corresponds to the Bogolyubov hypothesis of weakening of correlations between particles.

For a more detailed calculation of structure of correlation functions (18) and transport kernels (19) let us consider an action of the Liouville operator on $\hat{n}_1(x)$ and $\hat{n}_2(x,x')$:

$$iL_N\hat{n}_1(x) = -\frac{\partial}{\partial\vec{r}}\cdot\frac{1}{m}\hat{\vec{j}}(\vec{r},\vec{p}) + \frac{\partial}{\partial\vec{p}}\cdot\hat{\vec{F}}(\vec{r},\vec{p}), \tag{20}$$

where

$$\hat{\vec{j}}(\vec{r},\vec{p}) = \sum_{j=1}^N \vec{p}_j\delta(\vec{r}-\vec{r}_j)\delta(\vec{p}-\vec{p}_j) \tag{21}$$

is the microscopic momentum density in the space of coordinates and impulses,

$$\hat{\vec{F}}(\vec{r},\vec{p}) = \sum_{l\neq j}\frac{\partial}{\partial\vec{r}_j}\Phi(|\vec{r}_j-\vec{r}_l|)\delta(\vec{r}-\vec{r}_j)\delta(\vec{p}-\vec{p}_j) \tag{22}$$

is the microscopic force density in the space of coordinates and impulses.

$$\begin{aligned} iL_N\hat{n}_2(x,x') \;=\; & -\frac{\partial}{\partial\vec{r}}\cdot\frac{1}{m}\hat{\vec{j}}(\vec{r},\vec{p})\hat{n}_1(x') - \hat{n}_1(x)\frac{\partial}{\partial\vec{r}'}\cdot\frac{1}{m}\hat{\vec{j}}(\vec{r}',\vec{p}') \\ & +\frac{\partial}{\partial\vec{p}}\cdot\hat{\vec{F}}(\vec{r},\vec{p})\hat{n}_1(x') + \hat{n}_1(x)\frac{\partial}{\partial\vec{p}'}\cdot\hat{\vec{F}}(\vec{r}',\vec{p}'). \end{aligned} \tag{23}$$

Taking into account calculations (20)-(23) we obtain, particularly:

$$\Phi_{nn}^{11}(x,x';t)=\left[\Omega_{nj}(x,x';t)\cdot\frac{\partial}{\partial\vec{r}'}-\Omega_{nF}(x,x';t)\cdot\frac{\partial}{\partial\vec{p}'}\right], \tag{24}$$

$$\begin{aligned}\varphi_{nn}^{11}(x,x';t,t')=-\Bigg[&\frac{\partial}{\partial\vec{r}}\cdot D_{jj}(x,x';t,t')\cdot\frac{\partial}{\partial\vec{r}'}\\&-\frac{\partial}{\partial\vec{p}}\cdot D_{Fj}(x,x';t,t')\cdot\frac{\partial}{\partial\vec{r}'}-\frac{\partial}{\partial\vec{r}}\cdot D_{jF}(x,x';t,t')\cdot\frac{\partial}{\partial\vec{p}'}\\&+\frac{\partial}{\partial\vec{p}}\cdot D_{FF}(x,x';t,t')\cdot\frac{\partial}{\partial\vec{p}'}\Bigg],\end{aligned} \tag{25}$$

$$\begin{aligned}&\varphi_{nn}^{22}(x,x',x'',x''';t,t')=\\&-\frac{\partial}{\partial\vec{r}}\cdot\left[D_{jnjn}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}''}+D_{jnnj}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}'''}\right]\\&-\frac{\partial}{\partial\vec{r}'}\cdot\left[D_{njjn}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}''}+D_{njnj}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}'''}\right]\\&+\frac{\partial}{\partial\vec{p}}\cdot\left[D_{Fnjn}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}''}+D_{Fnnj}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}'''}\right]\\&+\frac{\partial}{\partial\vec{p}'}\cdot\left[D_{nFjn}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}''}+D_{nFnj}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{r}'''}\right]\\&+\frac{\partial}{\partial\vec{r}}\cdot\left[D_{jnFn}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{p}''}+D_{jnnF}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{p}'''}\right]\\&+\frac{\partial}{\partial\vec{r}'}\cdot\left[D_{njFn}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{p}''}+D_{njnF}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{p}'''}\right]\\&-\frac{\partial}{\partial\vec{p}}\cdot\left[D_{FnFn}(x,x',x'',x'';t,t')\cdot\frac{\partial}{\partial\vec{p}''}+D_{FnnF}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{p}'''}\right]\\&-\frac{\partial}{\partial\vec{p}'}\cdot\left[D_{nFFn}(x,x',x'',x'';t,t')\cdot\frac{\partial}{\partial\vec{p}''}+D_{nFnF}(x,x',x'',x''';t,t')\cdot\frac{\partial}{\partial\vec{p}'''}\right],\end{aligned} \tag{26}$$

where

$$D_{jj}(x,x';t,t')=\int d\Gamma_N\hat{\vec{j}}(x)T(t,t')(1-P(t'))\frac{1}{q}\psi^{-1}(t)\hat{\vec{j}}(x')\rho_{rel}(x^N;t'),$$

$$D_{FF}(x,x';t,t') = \int d\Gamma_N \hat{\vec{F}}(x)T(t,t')(1-P(t'))\frac{1}{q}\psi^{-1}(t)\hat{\vec{F}}(x')\rho_{rel}(x^N;t'),$$

are the generalized diffusion and friction coefficients in the spatially-impulse space within Renyi statistics. Herewith,

$$\int d\vec{p}\int d\vec{p}' D_{jj}(x,x';t,t') = D_{jj}(\vec{r},\vec{r}';t,t'),$$

$$\int d\vec{p}\int d\vec{p}' D_{FF}(x,x';t,t') = D_{FF}(\vec{r},\vec{r}';t,t')$$

which at $q=1$ become the generalized diffusion and friction coefficients in Gibbs statistics. The obtained kinetic equations contain correlation functions of the second, the third and the fourth order Ω_{nj}, Ω_{nF}, Ω_{nnj}, Ω_{nnF}, Ω_{nnjn}, Ω_{nnFn} in dynamic variables $\hat{n}(x)$, $\hat{\vec{j}}(x)$, $\hat{\vec{F}}(x)$. Ω are the correlation functions describing nondissipative processes. D are the generalized memory functions – the time correlation functions built on the dynamic variables $\hat{n}(x)$, $\hat{\vec{j}}(x)$, $\hat{\vec{F}}(x)$, $[1-P(t)]\hat{\vec{j}}(x)$, $[1-P(t)]\hat{\vec{F}}(x)$ – and describe non-Markovian dissipative processes in the system. At $q=1$ they transform to the memory function of Gibbs statistics. Memory functions like D_{njnj} and D_{nFnF} have an interesting structure

$$\begin{aligned}
&D_{njnj}(x,x',x'',x''';t,t') = \\
&= \int d\Gamma_N \hat{n}(x)\hat{\vec{j}}(x')T(t,t')[1-P(t')]\frac{1}{q}\psi^{-1}(t)\hat{n}(x'')\hat{\vec{j}}(x''')\rho_{rel}(x^N;t'), \\
&D_{nFnF}(x,x',x'',x''';t,t') = \\
&= \int d\Gamma_N \hat{n}(x)\hat{\vec{F}}(x')T(t,t')[1-P(t')]\frac{1}{q}\psi^{-1}(t)\hat{n}(x'')\hat{\vec{F}}(x''')\rho_{rel}(x^N;t'),
\end{aligned}$$

they can be approximated in the following way:

$$D_{njnj} \approx D_{nn}D_{jj} + D_{nj}D_{jn}, \qquad D_{nFnF} \approx D_{nn}D_{FF} + D_{nF}D_{Fn}.$$

This corresponds to the ideology of the mode-coupling theory.

Generalized kinetic equations (16), (17) with regard to (24)-(26) by their structure are the equations of Fokker-Planck type. They can serve as a basis for transition to the generalized hydrodynamic equations which are based on the set of equations of conservation laws for particles number, momentum and energy densities (10). Indeed, multiplying the set of transport equations (16), (17) by the first moments of the nonequilibrium one-particle distribution function $f_1(\vec{r},\vec{p};t)$: $(1,\vec{p},p^2/2m)$ and by $\frac{1}{2}\Phi(|\vec{r}-\vec{r}'|)$, we obtain the generalized equations of hydrodynamics with the defined generalized viscosity and heat conductivity coefficients having separated kinetic and potential contributions.

4 Summary

By means of the Zubarev NSO method and the maximum entropy principle for the Renyi entropy we obtained the nonequilibrium statistical operator and the generalized kinetic equations for the nonequilibrium one- and two-particle distribution functions $f_1(x;t) = \langle\hat{n}_1(x)\rangle^t$ and $f_2(x,x';t) = \langle\hat{n}_2(x,x')\rangle^t$ for description of kinetic processes in gases and liquids far from equilibrium. We investigated an inner structure of generalized memory functions which permitted to show that the kinetic equations contain correlation functions of the second and higher order (Ω_{nj}, Ω_{nF}, Ω_{nnj}, Ω_{nnF}, Ω_{nnjn}, Ω_{nnFn}) in dynamic variables $\hat{n}(x)$, $\hat{\vec{j}}(x)$, $\hat{\vec{F}}(x)$. By contrast to Ω describing non-dissipative processes, the dissipative processes in the system are described by the memory functions of the kinetic equations D built on the variables $\hat{n}(x)$, $\hat{\vec{j}}(x)$, $\hat{\vec{F}}(x)$, $[1-P(t)]\hat{\vec{j}}(x)$ and $[1-P(t)]\hat{\vec{F}}(x)$.

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