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Exponentially convergent method for the final value problem for the first order differential equation in Banach space

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Розглянуто зворотну в часі задачу для диференціального рівняння першого порядку з необмеженим операторним коефіцієнтом у банаховому просторі. Ця задача пов'язана зі зворотною у часі задачею теплопровідності, яка відноситься до некоректно поставлених за Адамаром задач. Її розв'язок (якщо існує) не залежить неперервно від початкових даних. Для регуляризації використовується метод Латеса-Ліонса, який є стійким щодо до точного розв'язку. До регуляризованого розв'язку застосовується зображення за допомогою інтеграла Данфорда–Коші і експоненціально збіжна Sinc-квадратурна формула.

Рассмотрено обратную по времени задачу для дифференциального уравнения первого порядка с неограниченным оператрным коэффициентом в банаховом пространстве. Эта задача связана с обратной по времени задачей теплопроводности, которая является некоректной по Адамару задачей. Ее решение (если существует) не зависит непрерывно от начальных условий. Для регуляризации используется метод Латесса-Лионса, который есть устойчивым по отношению к точному решению. К регуляризированному решению применяется представление с помощью интеграла Данфорда–Коши и экспоненциально сходимая Sinc-квадратурная формула.

1. Introduction

In the last years, the field of inverse problems has certainly been one of the fastest growing areas in applied mathematics. This growth has largely been driven by the needs of applications in industry and sciences. Inverse problems typically lead to mathematical models that are not well-posed in the sense of Hadamard, i.e., to ill-posed problems. This means especially that their solution is unstable under data perturbations. Numerical methods that can cope with this problem are the so-called regularization methods [1–3]. In many practical problems it is required to reconstruct the distribution of the temperature of a body at a certain instant of time $t \in (0, T)$ from the temperature measured at $t = T$. Such problems are called retrospective, initial boundary value problems for the heat conduction equation with reverse time or final value problem. Since industry requires fast for given accuracy and simple algorithms for the solution of a wide variety of inverse problems, this implies a growing need for constructing such numerical methods. Exponentially convergent algorithms are widely developed in recent years. It is known that they are optimal (or near optimal) for analytic solutions [4].

Exponentially convergent algorithms were proposed recently for various problems [4–7]. The corresponding analysis is often carried out in an abstract setting. This means that the initial value and boundary value problems of parabolic, hyperbolic and elliptic types are formulated as abstract differential equations with an operator coefficient A in Banach space [3, 4, 8].

The main goal of this paper is to construct exponentially convergent method for the following final value problem:

$$\begin{aligned} &\frac{du}{dt} + Au = 0, \quad t \in [0, T), \\ &u(T) = u_T, \end{aligned} \tag{1}$$

where $u_T \in X$. The operator A with the domain $D(A)$ in a Banach space X is assumed to be densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin. The resolvent of A decays inversely proportional to $|z|$ at the infinity

$$\|R_A(z)\| = \left\|(zI - A)^{-1}\right\| \le \frac{M}{1 + |z|}. \tag{2}$$

To construct exponentially convergent method we use Lattès-Lions quasi-reversibility method [3, 9] with representation of solution by Danford-Cauchy integral and then by applying sinc-quadrature rule.

2. Regularization and representation of the solution

Inhomogeneous problem related to (1) can be reduced to the homogeneous one by change of function in the following way. If we have

$$\begin{aligned} &\frac{du}{dt} + Au = f(x), \quad x \in [0, X], \\ &u(T) = u_T, \end{aligned} \tag{3}$$

with $f(x)$ being vector-valued function in the Banach space X then we consider a function

$$v_1(t) = \int_0^t \mathrm{e}^{-A(t-s)} f(s) ds,$$

that can be efficiency calculated using algorithm from [4] and a function $v(t)$

$$\begin{aligned} &\frac{dv}{dt} + Av = 0, \quad t \in [0, T), \\ &v(T) = u_T - v_1(T). \end{aligned}$$

Further

$$u(t) = v(t) + v_1(t).$$

So, let us consider problem (1). Suppose, as it is common in the theory of ill-posed problems, that for some exactly given final value u_T there exists a solution to problem (1) for $t = 0$. Let $u(0) \in D(A^\sigma)$. Using theory of operator-valued functions we have from (1)

$$u(t) = \mathrm{e}^{-A(t-T)} u_T.$$

So, we have $\|A^\sigma u(0)\| = \left\|\mathrm{e}^{AT} A^\sigma u_T\right\| \le C < \infty$.

When the final value is given with an error we assume that

$$\|u_T - u_\delta\| \le \delta,$$

and consider the following problem instead of (1)

$$\begin{aligned} &\frac{du_\delta(t)}{dt} + Au_\delta(t) = 0, \quad t \in [0, T), \\ &u_\delta(T) = u_\delta. \end{aligned} \tag{4}$$

This problem is ill-posed. We must to use regularization method to find solution to this problem. One of the ways to do that is to change the operator A in (4) by some "close" operator A_ε such that our new problem is well posed and its solution $u_\varepsilon(t) \to u(t)$ for $\varepsilon \to 0$. Besides this, $\varepsilon \to 0$ for $\delta \to 0$.

We use Lattes-Lions method of regularization for problem (4) in the case when the spectrum of operator A is situated in a sector of angle $\frac{\pi}{2}$ in the right-half plane. Let us suppose that spectrum of the operator A is inside of

$$G_1 = \left\{ z \in \mathbb{C} : \; Re\, z \geq a_0, \; |arg\, z| < \varphi < \frac{\pi}{4} \right\}.$$

Due to Lattes-Lions regularization method we consider the following problem instead of (4).

$$\begin{aligned} &\frac{du_{\varepsilon,\delta}(t)}{dt} + \left(A - \varepsilon A^2\right) u_{\varepsilon,\delta}(t) = 0, \quad t \in [0,T), \\ &u_{\varepsilon,\delta}(T) = u_\delta, \end{aligned} \tag{5}$$

where $\varepsilon > 0$. For the error estimate we obtain

$$\|u(t) - u_{\varepsilon,\delta}(t)\| \leq \|u(t) - u_\varepsilon(t)\| + \|u_\varepsilon(t) - u_{\varepsilon,\delta}(t)\| = \Delta_1 + \Delta_2,$$

where $u_\varepsilon(t)$ is a solution to problem (5) with a final value $u_\varepsilon(T) = u_T$.

From (1) and (5) we have

$$\begin{aligned} \Delta_1 &= \left\| \left[\mathrm{e}^{-A(t-T)} - \mathrm{e}^{-(A-\varepsilon A^2)(t-T)} \right] u_T \right\| = \\ &= \left\| \left[I - \mathrm{e}^{-\varepsilon A^2)(T-t)} \right] A^{-\sigma} \mathrm{e}^{-A(t-T)} A^\sigma u_T \right\| \leq \\ &\leq \left\| \left[I - \mathrm{e}^{-\varepsilon A^2)(T-t)} \right] A^{-\sigma} \right\| \left\| \mathrm{e}^{-A(t-T)} A^\sigma u_T \right\| \leq \varepsilon(T-t) C_1 C, \end{aligned}$$

for $\sigma \geq 2$. Here we have used our assumption about regularity of u_T. We use an estimate form [3] for Δ_2:

$$\Delta_2 \leq L\delta \exp\left\{ \frac{T}{4\varepsilon(1 - \tan^2\varphi)} \right\}, \quad L = M^2.$$

So, we have

$$\max_{t\in[0,T]} \|u(t) - u_{\varepsilon,\delta}(t)\| \leq \varepsilon C + L\delta \exp\left\{ \frac{T}{4\varepsilon(1 - \tan^2\varphi)} \right\} = \varepsilon C + L\delta \mathrm{e}^{\frac{Q}{\varepsilon}},$$

$$Q = \frac{T}{4(1-\tan^2\varphi)}.$$

Further we describe how to chose the parameter ε for an arbitrary δ. Recall that we must choose $\varepsilon(\delta)$ such that $\varepsilon(\delta) \to 0$, for $\delta \to 0$. We propose to choose

$$\varepsilon = \arg\min_{\varepsilon} \{f(\varepsilon)\}, \tag{6}$$

where $f(\varepsilon) = \varepsilon C + L\delta e^{\frac{Q}{\varepsilon}}$ for an arbitrary $\delta \geq 0$. The function $f(\varepsilon)$ is continuous for $\varepsilon > 0$. It is easy to see that

$$f''(\varepsilon) = \frac{L\delta Q e^{\frac{Q}{\varepsilon}}(2\varepsilon + Q)}{\varepsilon^4} > 0.$$

So, it is convex downward and, therefore, it has a unique minimum. The minimum of the function f can be found from the equation $f'(\varepsilon) = 0$ or $C - L\delta \frac{Q}{\varepsilon^2} e^{\frac{Q}{\varepsilon}} = 0$.

Let ε_0 be a minimum of the function $f(\varepsilon)$ for an arbitrary δ_0. Further let us show that $\varepsilon_0 \to 0$ for $\delta_0 \to 0$. We suppose that $\delta_2 < \delta_1$ and let $\varepsilon_\alpha = \arg\min_\varepsilon \{f_\alpha(\varepsilon)\}$, where $f_\alpha(\varepsilon) = \varepsilon C + L\delta_\alpha e^{\frac{Q}{\varepsilon}}$, $\alpha = 1, 2$. ε_α is a solution to equation $f'_\alpha(\varepsilon) = 0$. Namely, ε_α satisfies the equation $C - L\delta_\alpha \frac{Q}{\varepsilon_\alpha^2} e^{\frac{Q}{\varepsilon_\alpha}} = 0$. We must to show that $\varepsilon_2 < \varepsilon_1$. Indeed, we have from the equation that $\delta_\alpha = \frac{C}{LQ}\varepsilon_\alpha^2 e^{-\frac{Q}{\varepsilon_\alpha}}$. So, due to our assumption we have $\varepsilon_2 e^{-\frac{Q}{\varepsilon_2}} < \varepsilon_1 e^{-\frac{Q}{\varepsilon_1}}$. The function $g(x) = xe^{-\frac{Q}{x}}$ is monotonically increasing for $x > 0$ because $g'(x) = \left(1 + \frac{Q}{x}\right) e^{-\frac{Q}{x}} > 0$. Therefore we obtain $\varepsilon_2 < \varepsilon_1$.

Further let us show that $f(\varepsilon_*) \to 0$ for $\delta_* \to 0$ in the case when $\varepsilon_* = \arg\min_\varepsilon \{f(\varepsilon)\}$, with $f(\varepsilon) = \varepsilon C + L\delta_* e^{\frac{Q}{\varepsilon}}$, for an arbitrary δ_*. Really, we have that $\delta_* = \frac{C}{LQ}\varepsilon_*^2 e^{-\frac{Q}{\varepsilon_*}}$. We obtain after it substitution into the function f that

$$f(\varepsilon_*) = \varepsilon_* C + \varepsilon_*^2 \frac{C}{Q}.$$

Due to the previously proved fact that $\varepsilon_* \to 0$ for $\delta_* \to 0$ we have that $f(\varepsilon_*) \to 0$ for $\delta_* \to 0$.

For example let us consider the case when $C = L = Q = 1$. The plot of $f(\varepsilon)$ for various δ is presented on figure 1. Results of calculation are shown in table 1.

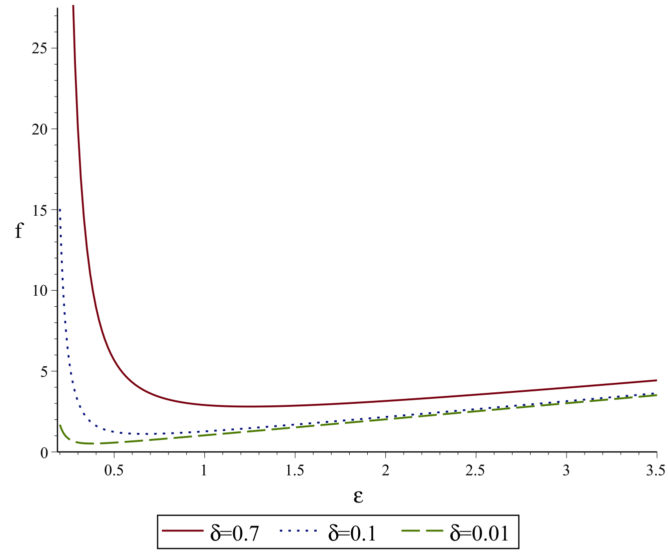

Fig 1. Plot of $f(\varepsilon)$ for various δ.

Table 1. Choice of ε

δ	ε_*	$f(\varepsilon_*)$
0.7	1.248678557	2.807876694
0.1	0.6682589247	1.114828915
0.01	0.3768679464	0.5188973953

3. Operator representation of solution

Further we change $u_{\varepsilon,\delta}(t)$ by $v(t)$ for convenience. Then a solution to problem (5) can be written as follows:

$$v(t) = \mathrm{e}^{-(A-\varepsilon A^2)(t-T)} u_\delta.$$

We write the solution of this problem using Danford-Cauchy presentation for operator-valued functions in the form

$$v(t) = \frac{1}{2\pi i} \int_\Gamma \mathrm{e}^{-(z-\varepsilon z^2)(t-T)} R_A(z) u_\delta dz,$$

where integrating path envelopes spectrum of the operator A.

Let

$$\Gamma_0 = \{z(s) = a_0 \cosh s - ib_0 \sinh s : \ s \in (-\infty, \infty), \ b_0 = a_0 \tan \varphi\} \quad (7)$$

be a spectral hyperbola. It has a vertex at $(a_0, 0)$ and asymptotes that are parallel to the rays of the spectral angle of Σ.

We choose a path of integration Γ_I and replace resolvent $R_A(z)$ by $R_A^1(z)$ (see [4] for details) where

$$\Gamma_I = \{z(s) = a_I \cosh s - ib_I \sinh s : \ s \in (-\infty, \infty)\}, \quad (8)$$

$$R_A^1(z) = R_A(z) - \frac{1}{z}.$$

Then $v(t)$ can be written down after parametrization as follows

$$v(t) = \frac{1}{2\pi i} \int_\infty^\infty \mathrm{e}^{-(z(s)-\varepsilon z(s)^2)(t-T)} z'(s) R_A^1(s) u_T ds = \int_{-\infty}^\infty \mathcal{F}(t,s) ds, \quad (9)$$

with

$$z'(s) = a_I \sinh s - ib_I \cosh s.$$

The next step toward a numerical algorithm is an approximation of (9) by an efficient quadrature formula. For this purpose we need to estimate the width of a strip around the real axis where the integrand in (9) admits analytical extension (with respect to s). The integration hyperbola Γ_I will be translated into the parametric set of hyperbolas with respect to ν after changing s to $s + i\nu$

$$\Gamma(\nu) = \{z(s,\nu) = a_I \cosh(s + i\nu) - ib_I \sinh(s + i\nu) : \ s \in (-\infty, \infty)\} =$$
$$= \{z(s,\nu) = a(\nu) \cosh s - ib(\nu) \sinh s : \ s \in (-\infty, \infty)\},$$

with

$$a(\nu) = a_I \cos\nu + b_I \sin\nu = \sqrt{a_I^2 + b_I^2}\,\sin(\nu + \phi/2),$$
$$b(\nu) = b_I \cos\nu - a_I \sin\nu = \sqrt{a_I^2 + b_I^2}\,\cos(\nu + \phi/2),$$
$$\cos\frac{\phi}{2} = \frac{b_I}{\sqrt{a_I^2 + b_I^2}},\ \sin\frac{\phi}{2} = \frac{a_I}{\sqrt{a_I^2 + b_I^2}}\ .$$

The analyticity of the integrand in the strip

$$D_{d_1} = \{(s,\nu) :\ s \in (-\infty,\infty), |\nu| < d_1/2\},$$

with some d_1 could be violated if the resolvent become unbounded or exponential function increases. To avoid this we have to choose d_1 in a way such that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola $\Gamma(\nu)$ remains in the right half-plane of the complex plane. For $\nu = -d_1/2$ the corresponding hyperbola has asymptotically an angle $\frac{\pi}{2}$. For $\nu = d_1/2$ it coincides with the spectral hyperbola. Therefore for all $\nu \in (-d_1/2, d_1/2)$ the set $\Gamma(\nu)$ does not intersect the spectral sector. For $\nu = 0$ we have $\Gamma(0) = \Gamma_I$.

Such requirements for $\Gamma(\nu)$ are fulfilled when

$$\begin{cases} a_I \cos\frac{d_1}{2} + b_I \sin\frac{d_1}{2} = a_0, \\ -a_I \sin\frac{d_1}{2} + b_I \cos\frac{d_1}{2} = b_0, \\ a_I \cos\frac{d_1}{2} - b_I \sin\frac{d_1}{2} = a_I \sin\frac{d_1}{2} + b_I \cos\frac{d_1}{2}. \end{cases}$$

The solution of this system is

$$a_I = a_0 \frac{\cos\left(\frac{d_1}{2} + \varphi\right)}{\cos\varphi},\ b_I = a_0 \frac{\sin\left(\frac{d_1}{2} + \varphi\right)}{\cos\varphi},\ d_1 = \frac{\pi}{4} - \varphi. \tag{10}$$

Taking into account (10) we can similarly write the equations for $a(\nu)$, $b(\nu)$ on the whole interval $-\frac{d_1}{2} \le \nu \le \frac{d_1}{2}$

$$a(\nu) = a_I \cos\nu + b_I \sin\nu = \frac{a_0}{\cos\varphi} \cos\left(\frac{d_1}{2} + \varphi - \nu\right),$$
$$b(\nu) = b_I \cos\nu - a_I \sin\nu = \frac{a_0}{\cos\varphi} \sin\left(\frac{d_1}{2} + \varphi - \nu\right),$$

with d_1, defined by (10).

4. Uniform numerical algorithm for a regular perturbation

Further we estimate the function $\mathcal{F}(t,s)$ from (9). In the case when $u_\delta \in D(A^\alpha)$ we have (see [4])

$$\left\|z'(s)R_A^1(z(s))u_\delta\right\| \le (1+M)K\frac{b_I}{a_I}\left(\frac{2}{a_I}\right)^\alpha \mathrm{e}^{-\alpha|s|}\left\|A^\alpha u_\delta\right\|,$$

$$\left|\mathrm{e}^{-(z(s)-\varepsilon z(s)^2)(t-T)}\right| = \mathrm{e}^{-\left(\varepsilon\left(a_I^2\cosh^2 s - b_I^2\sinh^2 s\right)-a_I\cosh s\right)(T-t)} =$$

$$= \exp\left\{-\left(\varepsilon m_1\cos(2\psi)\cosh^2 s - \cos(\psi)\cosh s + \varepsilon m_1\sin^2(\psi)\right)m_1(T-t)\right\},$$

where

$$m_1 = \frac{a_0}{\cos(\varphi)}, \quad \psi = \frac{\varphi}{2}+\frac{\pi}{8}.$$

Therefore, we obtain

$$\|\mathcal{F}(t,s)\| \le C(\varphi,\alpha)\mathrm{e}^{-\left(\varepsilon g_1\cosh^2 s - g_2\cosh s\right)(T-t)-\alpha|s|}\left\|A^\alpha u_\delta\right\|, \tag{11}$$

where

$$g_1 = m_1^2\cos(2\psi), \quad g_2 = m_1\cos(\psi),$$

$$C(\varphi,\alpha) = (1+M)K\frac{b_I}{a_I}\left(\frac{2}{a_I}\right)^\alpha \mathrm{e}^{-\varepsilon m_1^2\sin^2(\psi)(T-t)} =$$

$$= (1+M)K\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}\right)\left(\frac{2\cos\varphi}{a_0\cos\left(\frac{\varphi}{2}+\frac{\pi}{8}\right)}\right)^\alpha \mathrm{e}^{-\varepsilon m_1^2\sin^2(\psi)(T-t)}.$$

We choose $d = d_1 - \epsilon_1$ for an arbitrarily small positive ϵ_1 and for $w \in D_d$ get the estimate

$$\|\mathcal{F}(t,w)\| \le (1+M)K\frac{b(\nu)}{a(\nu)}\left(\frac{2}{a(\nu)}\right)^\alpha \mathrm{e}^{-\varepsilon b^2(\nu)(T-t)}\times$$

$$\times\mathrm{e}^{-\left\{\varepsilon\left(a^2(\nu)-b^2(\nu)\right)\cosh^2 s - a(\nu)\cosh s\right\}(T-t)-\alpha|s|}\left\|A^\alpha u_\delta\right\| \le$$

$$\le (1+M)K\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}-\nu\right)\left(\frac{2\cos\varphi}{a_0\cos\left(\frac{\varphi}{2}+\frac{\pi}{8}-\nu\right)}\right)^\alpha \mathrm{e}^{-\varepsilon b^2(\nu)(T-t)}\times$$

$$\times\mathrm{e}^{-\left\{\varepsilon\left(a^2(\nu)-b^2(\nu)\right)\cosh^2 s - a(\nu)\cosh s\right\}(T-t)-\alpha|s|}\left\|A^\alpha u_\delta\right\| \le$$

$$\leq (1+M)K\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}-\nu\right)\left(\frac{2\cos\varphi}{a_0\cos\left(\frac{\varphi}{2}+\frac{\pi}{8}-\nu\right)}\right)^{\alpha}\times$$

$$\times\, \mathrm{e}^{\left\{-\varepsilon b^2(\nu)+\frac{a^2(\nu)}{4\varepsilon(a^2(\nu)-b^2(\nu))}\right\}(T-t)-\alpha|s|}\left\|A^{\alpha}u_{\delta}\right\|, \qquad \forall w\in D_d.$$

Taking into account that the integrals over the vertical sides of the rectangle $D_d(\epsilon_1)\{z\in\mathbb{C}:|\Re z|<1/\epsilon_1, |\Im z|<(1-\epsilon_1)d/2\}$ vanish as $\epsilon_1\to 0$ $(D_d(\epsilon_1)\to D_d)$ this estimate leads us to

$$\|\mathcal{F}(t,\cdot)\|_{\mathbf{H}^1(D_d)}\leq C(\varphi,\alpha,\epsilon_1,\varepsilon)\|A^{\alpha}u_{\delta}\|, \tag{12}$$

with

$$C(\varphi,\alpha,\epsilon_1,\varepsilon)=\frac{2}{\alpha}[C_+(\varphi,\alpha,\epsilon_1,\varepsilon)+C_-(\varphi,\alpha,\epsilon_1,\varepsilon)],$$

$$C_{\pm}(\varphi,\alpha,\epsilon_1,\varepsilon)=(1+M)K\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}\pm\nu\right)\left(\frac{2\cos\varphi}{a_0\cos\left(\frac{\varphi}{2}+\frac{\pi}{8}\pm\nu\right)}\right)^{\alpha}\times$$

$$\times\, \mathrm{e}^{\left(-\varepsilon\frac{a_0^2\cos^2\left(\frac{\varphi}{2}+\frac{\pi}{8}\pm\nu\right)}{\cos^2(\varphi)}+\frac{\cos^2\left(\frac{\varphi}{2}+\frac{\pi}{8}\pm\nu\right)\cos(\varphi)}{4\varepsilon a_0\cos(2\varphi+d\pm2\nu)}\right)(T-t)}.$$

Note that $C(\varphi,\alpha,\epsilon_1,\varepsilon)$ tends to ∞ if $\alpha\to 0$ or $\epsilon_1\to 0,\ \varphi\to\pi/4$.

We approximate integral (9) by the following Sinc-quadrature

$$v_N(t)=\frac{h}{2\pi i}\sum_{k=-N}^{N}\mathcal{F}(t,z(kh)), \tag{13}$$

with the error

$$\|\eta_N(\mathcal{F},h)\|=\|v(t)-v_N(t)\|\leq$$

$$\leq\|v(t)-\frac{h}{2\pi i}\sum_{k=-\infty}^{\infty}\mathcal{F}(t,z(kh))\|+\|\frac{h}{2\pi i}\sum_{|k|>N}\mathcal{F}(t,z(kh))\|\leq$$

$$\leq\frac{1}{2\pi}\frac{\mathrm{e}^{-\pi d/h}}{2\sinh(\pi d/h)}\|\mathcal{F}\|_{\mathbf{H}^1(D_d)}+$$

$$+\frac{C(\varphi,\alpha)h\|A^{\alpha}u_{\delta}\|}{2\pi}\sum_{k=N+1}^{\infty}\mathrm{e}^{-\left(\varepsilon g_1\cosh^2(kh)-g_2\cosh(kh)\right)(T-t)-\alpha kh}\leq$$

$$\leq\frac{c\|A^{\alpha}u_0\|}{\alpha}\left\{\frac{\mathrm{e}^{-\pi d/h}}{\sinh(\pi d/h)}+\right.$$

$$+\,\mathrm{e}^{-\left(\varepsilon g_1 \cosh^2((N+1)h) - g_2 \cosh((N+1)h)\right)(T-t) - \alpha(N+1)h}\Big\},$$

where the constant c does not depend on h, N, t. Equalizing the both exponentials for $t = T$ by $\frac{2\pi d}{h} = \alpha(N+1)h$, we get for the step size

$$h = \sqrt{\frac{2\pi d}{\alpha(N+1)}}.$$

With this step-size the following error estimate holds true

$$\begin{gathered}\|\eta_N(\mathcal{F}, h)\| \leq \frac{c}{\alpha}\mathrm{e}^{-\sqrt{\frac{\pi d\alpha(N+1)}{2}}} \times \\ \times \mathrm{e}^{-\left(\varepsilon g_1 \cosh^2 \sqrt{\frac{2\pi d(N+1)}{\alpha}} - g_2 \cosh \sqrt{\frac{2\pi d(N+1)}{\alpha}}\right)(T-t)} \|A^\alpha u_\delta\|,\end{gathered} \tag{14}$$

with a constant c independent of t, N. We must to take N taking into account that maximum of function $f(s) = -\varepsilon g_1 \cosh^2 s + g_2 \cosh s$ is $\frac{g_2^2}{4\varepsilon g_1}$, for $\cosh s = \frac{g_2}{2\varepsilon g_1}$. So, we have the following condition:

$$N > \operatorname{arccosh} \frac{g_2}{2\varepsilon g_1}. \tag{15}$$

Theorem 4.1. *Let A be a densely defined strongly positive operator and $u_\delta \in D(A^\alpha)$, $\alpha \in (0,1)$, then Sinc-quadrature* (13) *represents an approximate solution of the regularized final value problem* (5) *and possesses a uniform with respect to $t \leq T$ exponential convergence rate with estimate* (14) *with condition* (15).

Therefore, we look for an approximate solution to problem (1) with the help of (13) and in the case when $u_\delta \in D(A^\alpha)$, $\alpha \in (0,1)$ we obtain the following error estimate:

$$\begin{gathered}\|u(t) - v_N(t)\| \leq \Delta_1 + \Delta_2 + \|\eta_N(\mathcal{F}, h)\| \leq \varepsilon(\delta)C + L\delta^\alpha + \\ + \frac{c}{\alpha}\mathrm{e}^{-\sqrt{\frac{\pi d\alpha(N+1)}{2}}}\|A^\alpha u_\delta\|,\end{gathered}$$

where ε is defined by (6).

5. Numerical algorithm for a plain perturbation

Further, let us consider the case when u_δ is an arbitrary from Banach space X and $t < T$. Due to [4] we have

$$\left|\frac{z'(s)}{z(s)}\right| \le \frac{b_I}{a_I}.$$

$$\|z'(s)R_A(z(s))u_\delta\| \le M\frac{|z'(s)|}{1+|z(s)|}\|u_\delta\| = M\frac{\left|\frac{z'(s)}{z(s)}\right|}{1+\left|\frac{z'(s)}{z(s)}\right|}\|u_\delta\| \le c\,\|u_\delta\|\,.$$

So, we have

$$\left\|z'(s)\left(R_A(z(s))-\frac{1}{z(s)}\right)u_\delta\right\| \le c(1+\frac{a_I}{b_I})\,\|u_\delta\|\,,$$

$\left|\mathrm{e}^{-(z(s)-\varepsilon z(s)^2)(t-T)}\right|$ is estimated as in the case $u_\delta \in D(A^\alpha)$

Therefore, we obtain

$$\|\mathcal{F}(t,s)\| \le C(\varphi)\mathrm{e}^{-\left(\varepsilon g_1\cosh^2 s - g_2\cosh s\right)(T-t)}\,\|u_\delta\|\,, \tag{16}$$

where

$$g_1 = m_1^2\cos(2\psi), \quad g_2 = m_1\cos(\psi),$$

$$C(\varphi) = c(1+\frac{a_I}{b_I})\mathrm{e}^{-\varepsilon m_1^2\sin^2(\psi)(T-t)} = c\left(1+\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}\right)\right)\mathrm{e}^{-\varepsilon m_1^2\sin^2(\psi)(T-t)}.$$

We choose $d = d_1 - \epsilon_1$ for an arbitrarily small positive ϵ_1 and get the estimate for $w \in D_d$

$$\|\mathcal{F}(t,w)\| \le c\left(1+\frac{b(\nu)}{a(\nu)}\right)\mathrm{e}^{-\left\{\varepsilon\left(a^2(\nu)-b^2(\nu)\right)\cosh^2 s - a(\nu)\cosh s + \varepsilon b^2(\nu)\right\}(T-t)}\,\|u_\delta\|$$

$$\le c\left(1+\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}-\nu\right)\right)\mathrm{e}^{-\varepsilon b^2(\nu)(T-t)}\times$$

$$\times\mathrm{e}^{-\left\{\varepsilon\left(a^2(\nu)-b^2(\nu)\right)\cosh^2 s - a(\nu)\cosh s\right\}(T-t)}\,\|u_\delta\|\,, \qquad \forall w \in D_d.$$

Taking into account that the integrals over the vertical sides of the rectangle $D_d(\epsilon_1)$ vanish as $\epsilon_1 \to 0$ this estimate leads us to

$$\|\mathcal{F}(t,\cdot)\|_{\mathbf{H}^1(D_d)} \le C(\varphi,\epsilon_1,\varepsilon)\|u_\delta\|, \tag{17}$$

with

$$C(\varphi,\epsilon_1,\varepsilon) = C_+(\varphi,\epsilon_1,\varepsilon) + C_-(\varphi,\epsilon_1,\varepsilon),$$

$$C_\pm(\varphi,\epsilon_1,\varepsilon) = c\left(1+\tan\left(\frac{\varphi}{2}+\frac{\pi}{8}\pm\nu\right)\right)\frac{\sqrt{\pi}\mathrm{e}^{-\varepsilon b^2(\pm\nu)(T-t)}\cos(\varphi)}{\sqrt{\varepsilon(T-t)a_0^2\cos(d+2\varphi\pm 2\nu)}}\times$$
$$\times\exp\left\{\frac{(1+\cos(d+2\varphi\pm 2\nu))(T-t)}{4\varepsilon\cos(d+2\varphi\pm 2\nu)}\right\}.$$

We use here the following estimate:

$$\int_{-\infty}^{\infty}\mathrm{e}^{-a\cosh^2(s)+b\cosh(s)}ds = 2\int_0^{\infty}\mathrm{e}^{-a\cosh^2(s)+b\cosh(s)}ds \le$$
$$\le 2\int_0^{\infty}\mathrm{e}^{-a\cosh^2(s)+b\cosh(s)}\sinh(s)ds = 2\int_1^{\infty}\mathrm{e}^{-ax^2+bx}dx =$$
$$= 2\int_{1-\frac{b}{2a}}^{\infty}\mathrm{e}^{-ax^2+\frac{b^2}{2a}}dx \le 2\mathrm{e}^{\frac{b^2}{2a}}\int_{-\infty}^{\infty}\mathrm{e}^{-ax^2}dx = \mathrm{e}^{\frac{b^2}{2a}}\frac{2\sqrt{\pi}}{\sqrt{a}}.$$

Note that the constant $C(\varphi\epsilon_1,\varepsilon)$ tends to ∞ if $\epsilon_1\to 0$, or $\varphi\to\pi/4$.

We approximate integral (9) again by the Sinc-quadrature (13) with the error

$$\|\eta_N(\mathcal{F},h)\| = \|v(t)-v_N(t)\| \le$$
$$\le \|v(t)-\frac{h}{2\pi i}\sum_{k=-\infty}^{\infty}\mathcal{F}(t,z(kh))\| + \|\frac{h}{2\pi i}\sum_{|k|>N}\mathcal{F}(t,z(kh))\| \le$$
$$\le \frac{1}{2\pi}\frac{\mathrm{e}^{-\pi d/h}}{2\sinh(\pi d/h)}\|\mathcal{F}\|_{\mathbf{H}^1(D_d)} +$$
$$+\frac{C(\varphi,\alpha)h\|u_\delta\|}{2\pi}\sum_{k=N+1}^{\infty}\mathrm{e}^{-\left(\varepsilon g_1\cosh^2(kh)-g_2\cosh(kh)\right)(T-t)} \le$$
$$\le c\|u_\delta\|\left\{\frac{\mathrm{e}^{-\pi d/h}}{\sinh(\pi d/h)}+\frac{\mathrm{e}^{-\left(\varepsilon g_1\cosh^2(hN)-g_2\cosh(hN)\right)(T-t)}}{h\sinh(hN)\left(2\varepsilon g_1\cosh(hN)-g_2\right)(T-t)}\right\}$$

where c does not depend on h, N, t.

Equalizing the both exponentials by

$$h = \frac{\ln N}{N}, \tag{18}$$

we obtain

$$\|\eta_N(\mathcal{F},h)\| \le c\|u_\delta\| \left\{ \mathrm{e}^{-\frac{\pi d N}{\ln(N)}} + \frac{c_2 \mathrm{e}^{-c_3\left(\varepsilon g_1 N^2 - g_2 N\right)(T-t)}}{\ln(N)(2\varepsilon g_1 N - g_2)(T-t)} \right\}, \qquad (19)$$

with various c_j independent of t, N. We must to choose N taking into account that maximum of function $f(s) = -\varepsilon g_1 s^2 + g_2 s$ is $\frac{g_2^2}{4\varepsilon g_1}$, for $s = \frac{g_2}{2\varepsilon g_1}$. So, we have the following condition:

$$N > \frac{g_2}{2\varepsilon g_1}. \qquad (20)$$

Theorem 5.1. *Let A be a densely defined strongly positive operator, then Sinc-quadrature* (13) *represents an approximate solution of the regularized final value problem* (5) *and possesses exponential convergence rate with estimate* (19) *for h defined in* (18) *taking into account condition* (20).

Therefore, we look for an approximate solution to problem (1) with the help of (13) and in the case when $u_\delta \in X$, $0 \le t < T$ we obtain the following error estimate:

$$\|u(t) - v_N(t)\| \le \Delta_1 + \Delta_2 + \|\eta_N(\mathcal{F},h)\| \le \varepsilon(\delta)C + L\delta^\alpha + \\ + c\|u_\delta\| \left\{ \mathrm{e}^{-\frac{\pi d N}{\ln(N)}} + \frac{c_2 \mathrm{e}^{-c_3\left(\varepsilon g_1 N^2 - g_2 N\right)(T-t)}}{\ln(N)(2\varepsilon g_1 N - g_2)(T-t)} \right\},$$

where ε is defined by (6).

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