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Mathematical Simulation of the Influence of Delay Factors on the Oscillations of Non-ideal Pendulum Systems

Dynamic system "pendulum - source of limited excitation"with taking into account the various factors of delay is considered. Different approaches to write a mathematical model of this system using three- or fifteen-dimensional systems of differential equations without delay is suggested. It is established that for small values of the delay it is sufficient to use three-dimensional mathematical model, whereas for relatively large values of the delay the fifteen-dimensional mathematical model should be used.
Genesis of deterministic chaos is studied in detail. Maps of dynamic regimes, phase portraits of attractors of systems, phase-parametric characteristics, Poincare sections and maps are constructed and analyzed. The scenarios of transition from steady-state regular regimes to chaotic ones are identified. It is shown, that in some cases the delay is the main reason of origination of chaos in the system "pendulum - source of limited excitation".

1. Introduction. In mathematical modeling of oscillatory processes a mathematical model of a relatively simple dynamical system is often used to study the dynamics of much more complex systems. A typical example of this approach is the extensive use of pendulum models to study the dynamics of systems of an entirely different nature. Pendulum mathematical models are widely used to describe the dynamics of various technical constructions, machines and mechanisms, in the study of cardiovascular system, financial markets, etc. Such widespread use of pendulum models makes it relevant to study directly the dynamics of pendulum systems.

The study of the non-ideal by Zommerfeld–Kononenko [1] dynamical system "pendulum–electric motor"in the absence of any delay factors was initiated in [2], [3]. In this system the existence of deterministic chaos was identified and studied. It was proved that limited excitation is the main cause of chaos in this system.

In this paper the oscillations of "pendulum–electric motor"system with taking into account various factors of delay are considered. The delay factors are always present in rather extended systems due to the limitations of signal transmission speed: waves of compression, stretching, bending, current strength, etc. The aim of this work is to study the influence of various factors of delay on steady-state regimes of this system.

2. Delay factors in "Pendulum–electric motor"system. In the absence of any delay factors the equations of motion of the system "pendulum–electric motor"were obtained in [2]:

$$\begin{cases} \dfrac{dy_1}{d\tau} = Cy_1 - y_2 y_3 - \dfrac{1}{8}(y_1^2 y_2 + y_2^3); \\ \dfrac{dy_2}{d\tau} = Cy_2 + y_1 y_3 + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1; \\ \dfrac{dy_3}{d\tau} = Dy_2 + Ey_3 + F; \end{cases} \tag{1}$$

where phase variables y_1, y_2 describe the pendulum deviation from the vertical and phase variable y_3 is proportional to the rotation speed of the motor shaft. The system parameters are defined by

$$C = -\delta_1 \varepsilon^{-2/3} \omega_0^{-1}, D = -\frac{2ml^2}{I}, F = 2\varepsilon^{-2/3}(\frac{N_0}{\omega_0} + E) \tag{2}$$

where m - the pendulum mass, l - the reduced pendulum length, ω_0 - natural frequency of the pendulum, a - the length of the electric motor crank, $\varepsilon = \dfrac{a}{l}$, δ_1 - damping coefficient of the medium resistance force, I - the electric motor moment of inertia, E, N_0 - constants of the electric motor static characteristics.

Let us consider the following system of equations [4, 5]:

$$\begin{cases} \dfrac{dy_1(\tau)}{d\tau} = Cy_1(\tau-\delta) - y_2(\tau)y_3(\tau-\gamma) - \dfrac{1}{8}(y_1^2(\tau)y_2(\tau) + y_2^3(\tau)); \\ \dfrac{dy_2(\tau)}{d\tau} = Cy_2(\tau-\delta) + y_1(\tau)y_3(\tau-\gamma) + \\ \qquad\qquad + \dfrac{1}{8}(y_1^3(\tau) + y_1(\tau)y_2^2(\tau)) + 1; \\ \dfrac{dy_3(\tau)}{d\tau} = Dy_2(\tau-\gamma) + Ey_3(\tau) + F. \end{cases} \tag{3}$$

Positive constant parameter γ was introduced to account the delay effects of electric motor impulse on the pendulum. We assume that the delay of the electric motor response to the impact of the pendulum inertia force is also equal to γ. Taking into account the delay γ conditioned by the fact that the wave velocity perturbations on the elements of the construction has a finite value that depends on the properties of external fields, for instance, the temperature field. In turn, the constant positive parameter δ characterizes the delay of the medium reaction on the dynamical state of the pendulum. This delay is due to the limited sound velocity in that medium.

Assuming a small delay, we can write

$$y_i(\tau-\gamma) = y_i(\tau) - \frac{y_1(\tau)}{d\tau}\gamma + ..., \quad i = 2, 3$$

$$y_i(\tau-\delta) = y_i(\tau) - \frac{y_1(\tau)}{d\tau}\delta + ..., \quad i = 1, 2$$

Then, if $C\delta \neq -1$, we get the following system of equations [4, 5]:

$$\begin{cases} \dot{y_1} = \dfrac{1}{1+C\delta}\Bigg(Cy_1 - y_2\left[y_3 - \gamma\left[(Dy_2 + Ey_3 + F)\right] - \right. \\ \qquad\qquad\qquad\qquad \left. - \dfrac{1}{8}(y_1^2 y_2 + y_2^3)\right)\Bigg); \\ \dot{y_2} = \dfrac{1}{1+C\delta}\Bigg(Cy_2 + y_1 y_3 - y_1\gamma(Dy_2 + Ey_3 + F) + \\ \qquad\qquad\qquad\qquad + \dfrac{1}{8}(y_1^3 + y_1 y_2^2) + 1\Bigg); \\ \dot{y_3} = (1 - C\gamma)Dy_2 - \dfrac{D\gamma}{8}(y_1^3 + y_1 y_2^2 + 8y_1 y_3 + 8) + Ey_3 + F. \end{cases} \tag{4}$$

The obtained system of equations is already a system of ordinary differential equations. Delays are included in this system as additional parameters.

In order to approximate the system (3) another, more precise, method can be used [6], [7]. If $\gamma > 0,\ \delta > 0$ let us divide each of the segments $[-\gamma;\ 0]$ and $[-\delta;\ 0]$ into m equal parts. We introduce the following notation

$$y_1(\tau - \frac{i\delta}{m}) = y_{1i}(\tau),\ y_2(\tau - \frac{i\gamma}{m}) = y_{2i}(\tau),\ y_2(\tau - \frac{i\delta}{m}) = \tilde{y}_{2i}(\tau),$$
$$y_3(\tau - \frac{i\gamma}{m}) = y_{3i}(\tau), i = \overline{0, m}.$$

Then, using difference approximation of derivative [6], [7] the system of equations with delay (3) can be reduced to the following system of equations without delay:

$$\begin{cases} \dfrac{dy_{10}(\tau)}{d\tau} = Cy_{1m}(\tau) - y_{20}(\tau)y_{3m}(\tau) - \dfrac{1}{8}(y_{10}^2(\tau)y_{20}(\tau) + y_{20}^3(\tau)); \\ \dfrac{dy_{20}(\tau)}{d\tau} = C\tilde{y}_{2m}(\tau) + y_{10}(\tau)y_{3m}(\tau) + \\ \qquad\qquad \frac{1}{8}(y_{10}^3(\tau) + y_{10}(\tau)y_{20}^2(\tau)) + 1; \\ \dfrac{dy_{30}(\tau)}{d\tau} = Dy_{2m}(\tau) + Ey_{30}(\tau) + F; \\ \dfrac{dy_{1i}(\tau)}{d\tau} = \dfrac{m}{\delta}(y_{1\ i-1}(\tau) - y_{1i}(\tau)), \quad i = \overline{1,m}; \\ \dfrac{dy_{2i}(\tau)}{d\tau} = \dfrac{m}{\gamma}(y_{2\ i-1}(\tau) - y_{2i}(\tau)), \quad i = \overline{1,m}; \\ \dfrac{d\tilde{y}_{2i}(\tau)}{d\tau} = \dfrac{m}{\delta}(\tilde{y}_{2\ i-1}(\tau) - \tilde{y}_{2i}(\tau)), \quad i = \overline{1,m}; \\ \dfrac{dy_{3i}(\tau)}{d\tau} = \dfrac{m}{\gamma}(y_{3\ i-1}(\tau) - y_{3i}(\tau)), \quad i = \overline{1,m}. \end{cases} \tag{5}$$

Should be noted that the main variables in this system are only y_{10}, y_{20}, y_{30}. In other words the solutions y_1, y_2, y_3 of the system (3) are described by the functions y_{10}, y_{20}, y_{30} of the system (5).

The system (5) is a system of ordinary differential equations of $(4m + 3)$-th order. Choosing a sufficiently large m in the system (5), the system (3) will be very well approximated by the system (5) [6]. In this paper the system of equation (5) was considered at $m = 3$. In this case, the system (5) has 15 equations. The calculations of cases $m > 3$, with a significant increase the number of equations, were also carried out. It was established, that increasing the number of equations has practically no effect on identification and description of steady-state regimes of “pendulum–electric motor"system. But it significantly increases the complexity of constructing characteristics, which are necessary for study the steady-state regimes of oscillations. Therefore, the use of mathematical model (5) at $m = 3$ is the most optimal for studying the influence of delay on regular and chaotic dynamics of “pendulum–electric motor"system.

3. Maps of dynamic regimes. Therefore, we obtained three-

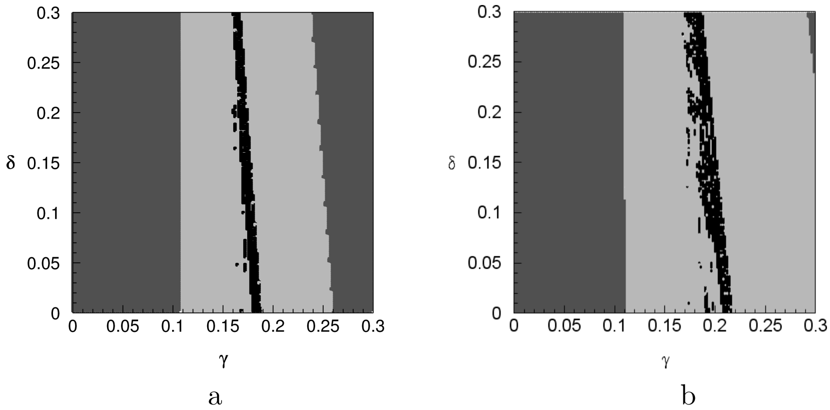


Fig. 1: Maps of dynamic regimes

dimensional (4) and fifteen-dimensional (5) models each describing the system of equations with delay (3). These models are the systems of non-linear differential equations, so in general the study of steady-state regimes can be carried out only by using numerical methods and algorithms. The methodology of such studies is described in detail in [2].

In the study of dynamical systems the information about the type of steady-state regime of the the system depending on its parameters is crucial. This information can provide a map of dynamic regimes. It is a diagram on the plane, where two parameters are plotted on axes and the boundaries of different dynamic regimes areas are shown. The construction of dynamic regimes maps is based on analysis and processing of spectrum of Lyapunov characteristic exponents [2,8]. Where necessary, for more accurate determination of steady-state regime of the system, we study other characteristics of attractors: phase portraits, Poincare sections and maps, Fourier spectrums and distributions of the invariant measure.

Let us consider the behavior of the systems (4) and (5) when the parameters are $C = -0.1$, $D = -0.6$, $E = -0.44$, $F = 0.3$. In fig. 1 the maps of dynamic regimes are shown. The map in fig. 1a was built for three-dimensional model (4) and the map in fig. 1b was built for fifteen-dimensional model (5). These figures illustrate the effect of delays γ and δ on changing the type of steady-state regime of the systems. The dark-grey areas of the maps correspond to equilibrium positions of the system.

The light-grey areas of the maps correspond to limit cycles of the system. And finally, the black areas of the maps correspond to chaotic attractors.

We can notice a certain similarity the maps in fig.1a, b. At small values of the delays in these systems, the steady-state regime is stable equilibrium position. With an increase of the delay of the medium δ the type of steady-state regime of the systems (4) and (5) does not change. It still remains an equilibrium position (dark-grey areas in the figures). However, with an increase of the delay of interaction between pendulum and electric motor γ, the equilibrium position is replaced by the area of limit cycles with "mounted"area of chaos. With further increase of the delay γ, the attractor of both systems is again equilibrium position.

Let us study the dynamics of the system (4) and (5) at other values of the parameters. The maps of dynamical regimes of respectively the system (4) and the system (5) at $C = -0.1$, $D = -0.6$, $E = -0.7$, $F = -0.4$ are built in fig. 2a, b. At small values of the delays the steady-state regime of both systems is limit cycle (light-grey areas in the figures). With a further increase of the delay values the maps in fig. 2a, b are certainly different. In fig. 2a there are narrow area in which the limit cycle is replaced by an equilibrium position, as well as by a chaotic attractor. Whereas in 2b these narrow area is almost missing. Further in both figures there are a rather wide area of periodic regimes, which with further increase of the delay is replaced by chaos area. Moreover, in this rather wide area of chaos fairly narrow strips of periodic regimes are built in.

In fig. 2c, d the maps of dynamic regimes of respectively the system (4) and the system (5) at $C = -0.1$, $D = -0.53$, $E = -0.6$, $F = 0.19$ are constructed. At small values of the delays both systems have chaotic attractors (black areas in the figures). With an increase of the delay values the region of chaos is replaced by the region of periodic regimes. Then again chaos arises in the system. Further this area is replaced by the area of limit cycles.

As seen from the constructed maps of dynamic regimes, the dynamics of the system (4) and (5) is the same only for small values of the delay γ and δ. With an increase of the delays the differences of the dynamics of these systems is very significant.

4. Regular and chaotic dynamics. Let us study the types of regular and chaotic attractors that exist in the systems (4) and (5). We implement a horizontal section of the maps of dynamic regimes in fig.2c, d along the delay γ at $\delta = 0.15$. In other words, let us consider the behavior of the systems (4) and (5) when parameters are $C = -0.1$, $D = -0.53$,

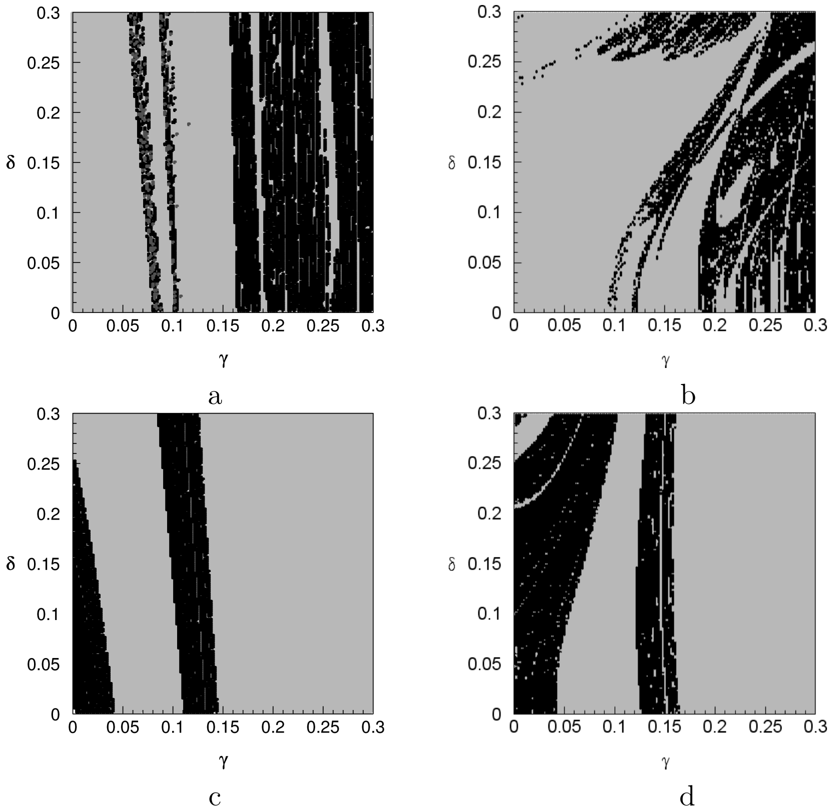
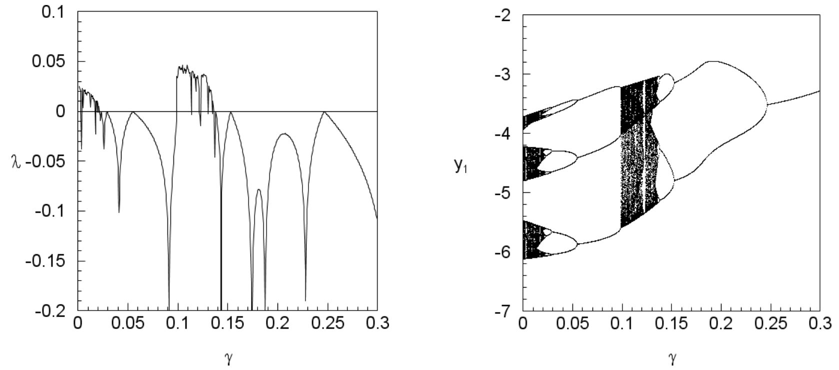


Fig. 2: Maps of dynamic regimes

$E = -0.6$, $F = 0.19$ and the delays $\delta = 0.15$ and $0 \leq \gamma \leq 0.3$.

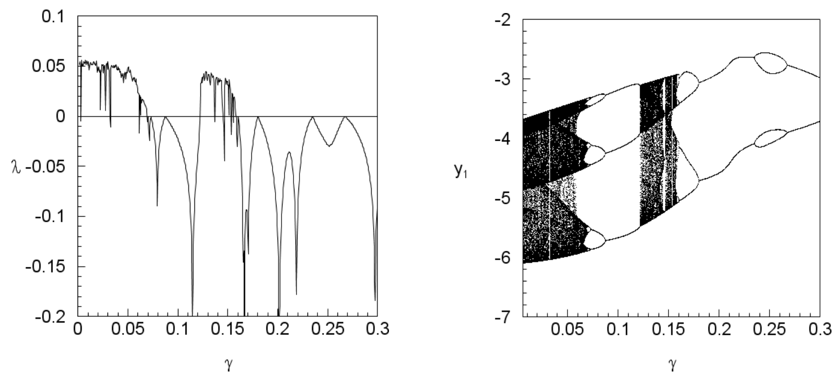
In fig. 3a,b the dependence of maximum non-zero Lyapunov's characteristic exponent and phase-parametric characteristic of three-dimensional system (4) are shown respectively. These figures illustrate the influence of the delay of interaction between pendulum and electric motor γ on chaotization of the system (4).

Let us construct the same characteristics at the same values of the parameters for fifteen-dimensional system (5). In fig. 4a,b respectively the dependence of maximum non-zero Lyapunov's characteristic exponent and phase-parametric characteristic are shown.



a b

Fig. 3: The dependence of maximal non-zero Lyapunov's characteristic exponent (a), phase-parametric characteristic (b) of three-dimensional system (4)



a b

Fig. 4: The dependence of maximal non-zero Lyapunov's characteristic exponent (a), phase-parametric characteristic (b) of fifteen-dimensional system (5)

In fig.3a, 4a we can clearly see the presence of intervals γ in which maximum Lyapunov exponent of the systems is positive. In these intervals the systems have chaotic attractors. The area of existence of chaos is clearly seen in phase-parametric characteristics of the systems. The areas of chaos in the bifurcation trees are densely filled with points. A careful examination of the obtained images allows not only to identify the origin of chaos in the systems, but also to describe the scenario of transition to chaos. So with a decrease of γ there are the transitions to chaos by Feigenbaum scenario (infinite cascade of period-doubling bifurcations of a limit cycle). Bifurcation points for the delay γ are clearly visible in each figures. These points are the points of approaches of the Lyapunov's exponent graph to the zero line (fig.3a, 4a) and the points of splitting the branches of the bifurcation tree (fig.3b, 4b). In turn, the transition to chaos with an increase of the delay happens under the scenario of Pomeau-Manneville, in a single bifurcation, through intermittency.

A careful analysis of these figures allows to see qualitative similarity of the respective characteristics of the systems (4) and (5). However, with increasing the delay the differences in the dynamics of these systems become very significant. So for instance at $\gamma = 0.05$ the steady–state regime of the system (4) is limit cycle. While at this value of the delay the attractor of the system (5) is chaotic attractor. Conversely, for example at $\gamma = 0.11$ the system (4) has steady–state chaotic regime. While at this value of the delay the system (5) has periodic regime of oscillations.

This suggests that three-dimensional system of equations (4) should be used to study the system (3) only at very small values of the delay. With increasing values of the delay to study regular and chaotic oscillations of "pendulum–electric motor"system, fifteen-dimensional system of equations (5) should be used.

5. Conclusion. Various factors of delay have significant influence on the dynamics of “pendulum–electric motor"system. The presence of delay in such systems can affect the type of steady-state regime change. It is shown that for small values of the delay it is sufficient to use three-dimensional mathematical model, whereas for relatively high values of the delay the fifteen-dimensional mathematical model should be used.

In future research is planned to construct and research mathematical models of “pendulum–electric motor"system in the presence of variable in time delay factors.

References

[1] Kononenko V. O. Vibrating System with a Limited Power-supply. Iliffe, London. — 1969. — 236 p.

[2] Krasnopolskaya T.S., Shvets A.Yu. Regular and chaotical dynamics of systems with limited excitation. R&C Dynamics, Moscow. — 2008. — 280 p.

[3] Shvets A.Yu., Makaseyev A.M. Chaotic Oscillations of Nonideal Plane Pendulum Systems // Chaotic Modeling and Simulation (CMSIM). — 2012. — No. 1. — P. 195–204.

[4] Shvets A.Yu., Makaseyev A.M. Delay Factors and Chaotization of Non-ideal Pendulum Systems // Chaotic Modeling and Simulation (CMSIM). — 2012. — No. 4. — P. 633–642.

[5] Shvets A.Yu., Makaseyev A.M. The influence of delay factors on regular and chaotic oscillations of plane pendulum // Proceedings of Institute of Mathematics of NAS of Ukraine. — 2012. — **9**, No. 1. — P. 365-377.

[6] Magnizkiy N.A., Sidorov S.V. New methods of chaotic dynamics. Editorial URSS. — 2004. — 320 p.

[7] Samarskiy A.A., Gulin A.V. Numerical methods, Nauka. — 1989. — 436 p.

[8] Kouznetsov S.P. Dynamic chaos. Physmatlit, Moscow. — 2001. — 296 p.