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On point-local deformations of minimal extensions of non-serial Dynkin diagrams

In this paper we study point-local deformations of Tits quadratic forms of finite graphs. We describe all P-limiting numbers of minimal extensions of non-serial Dynkin dyagrams in the case when these extensions are neither usual neither extended Dynkin diagrams.

1. Introduction. Let

$$f(z) = f(z_1, \dots, z_n) := \sum_{i=1}^{n} f_{ii} z_i^2 + \sum_{i<j} f_{ij} z_i z_j$$

be a qudratic form over the field of real numbers $\mathbb{R}$. By the definition from [1], a quadratic form of the form

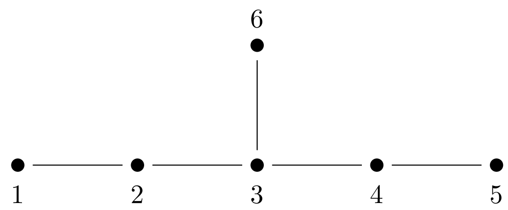
$$f^{(s)}(z,t) = t f_{ss} z_s^2 + \sum_{i \neq s} f_{ii} z_i^2 + \sum_{i<j} f_{ij} z_i z_j \ \text{ with } \ f_{ss} \neq 0$$

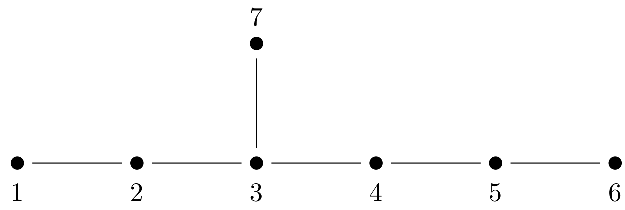
where t is a parameter running $\mathbb{R}$, is called the *local deformation of* $f(z)$ *with respect to* z_s or the s*-deformation of* $f(z)$.

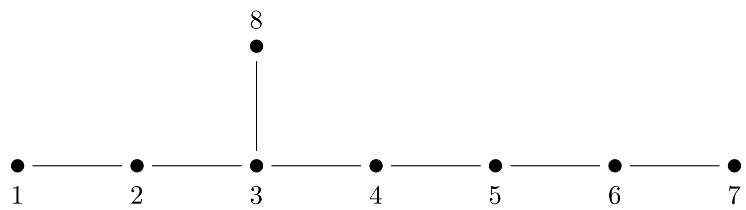
Let now $f_{ss} > 0$. Denote by $F_+^{(s)}$ the set of all $a \in \mathbb{R}$ such that the quadratic form $f^{(s)}(z, a)$ is positive definite, and put $F_-^{(s)} = \mathbb{R} \setminus F_+^{(s)}$. Obviously, $F_-^{(s)} \neq \varnothing$ (since $f^{(s)}(z, 0)$ is not positive definite), and if $F_-^{(s)} \neq \mathbb{R}$ then $m_f^{(s)} = \sup F_-^{(s)} \in F_-^{(s)}$ is called the *P-limiting number of $f(z)$ for z_s* or the *s-th P-limiting number of $f(z)$*. In the case $F_-^{(s)} = \mathbb{R}$ we put $m_f^{(s)} = \infty$. Concerning general properties of P-limiting numbers see in [1, 2, 3]. Deformations considered above were called *point-local deformations of $f(z)$* in [3]. This paper is devoted to study of point-local deformations of the Tits quadratic form of quivers.

2. Minimal extensions of graphs. Elsewhere in the paper all graphs are finite and non-oriented. Sets of vertices and edges of a graph X are denoted by X_0 and X_1, respectively. A graph $G = (G_0, G_1)$ is said to be a *minimal extension of a graph* $Q = (Q_0, Q_1)$ if $G_0 = S_0 \cup d$ with $d \notin Q_0$ and $G_1 = Q_1 \cup (d, j)$ with $j \in Q_0$. The vertex d is said to be the *added vertex of* G. In this case we write $G = Q \cup (d, j)$. If Q is a Dynkin diagram, the most interesting from the point of view of deformations (as we can see in the next section) is the case when the graph G is neither an usual not an extended Dynkin diagram. We call such G an *essential minimal extension of* Q.

We consider minimal extensions of the non-serial Dynkin diagtams, i. e. the diagrams E_6, E_7, E_8:







Directly from the definitions we have the following statement.

Proposition 1 *An extension $G = E \cup (0, j)$ of a Dynkin diagram $E = E_i$ $(i = 6, 7, 8)$ is essential if and only if one of the following condition holds*: 1) $E = E_6$, $j \neq 1, 5, 6$; 2) $E = E_7$, $j \neq 1, 6$; 3) $E = E_8$, $j \neq 7$.

3. Formulation of the main results. By the definition (see [4]) the Tits quadratic form of a graph $Q = (Q_0, Q_1)$ is the following integral quadratic form:

$$q_Q(z) = q_Q(z_1, z_2, \ldots, z_n) := \sum_{i \in Q_0} z_i^2 - \sum_{\{i-j\} \in Q_1} z_i z_j.$$

It is well-known that $q_Q(z)$ is positive definite if and only if the graph Q is a disjoint union of Dynkin diagrams (see [4]). All P-limiting numbers of such quadratic forms are describes in [2]; they are rational numbers belonging to $[0, 1)$.

Note that formally it is more convenient to say about P-limiting numbers of a graph Q instead of the quadratic form $q_Q(z)$. As in [2], by the P-limiting number of a vertex $i \in Q_0$ we mean the i-th P-limiting number of $q_Q(z)$, and we write $m_Q^{(i)}$ instead of $m_{q_Q(z)}^{(i)}$.

We consider minimal extensions G of Dynkin diagtams. If G is an extended Dynkin diagram then by Theorem 1 in [3] the P-limiting number of the added vertex of G is equal to 1. Therefore, the most interesting is the case of essential extensions. Note that in this case by the same theorem the P-limiting numbers of the added vertices belong to $(1, \infty)$.

Theorem 1 *Let $G = E_6 \cup (0, j)$ be an essential minimal extension of the Dynkin diagram E_6. Then the P-limiting number of the added vertex 0 is* *the following:*

$$m_G^{(0)} = \begin{cases} 1\frac{2}{3}, & \text{if } \ j = 2, 4; \\ 3, & \text{if } \ j = 3. \end{cases}$$

Theorem 2 *Let $G = E_7 \cup (0, j)$ be an essential minimal extension of the Dynkin diagram E_7. Then the P-limiting number of the added vertex 0 is the following:*

$$m_G^{(0)} = \begin{cases} 3, & \text{if } \ j = 2; \\ 6, & \text{if } \ j = 3; \\ 3\frac{3}{4}, & \text{if } \ j = 4; \\ 2, & \text{if } \ j = 5; \\ 1\frac{3}{4}, & \text{if } \ j = 7. \end{cases}$$

Theorem 3 *Let $G = E_8 \cup (0, j)$ be an essential minimal extension of the Dynkin diagram E_8. Then the P-limiting number of the added vertex 0 is the following:*

$$m_G^{(0)} = \begin{cases} 2, & \text{if } \ j = 1; \\ 7, & \text{if } \ j = 2; \\ 15, & \text{if } \ j = 3; \\ 10, & \text{if } \ j = 4; \\ 6, & \text{if } \ j = 5; \\ 3, & \text{if } \ j = 6; \\ 4, & \text{if } \ j = 8. \end{cases}$$

4. Proofs of the theorems. It follows from [1] – [3] that the P-limiting number $m_G^{(0)}$ of a graph $G = E_i \cup (0, j)$ is the root of the equation (linear with respect to t) $|A_i^j(t)| = 0$, where $A_i^j(t)$ is the symmetric matrix of the quadratic form $2q_G^{(0)}(z, t)$. The determinant $|A_i^j(t)|$ is denoted by $D_i^j(t)$. In particular cases, we have:

$$A_6^2(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$D_6^2(t) = 6t - 10;$

$$A_6^3(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

$D_6^3(t) = 6t - 18;$

$$A_7^2(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_7^2(t) = 4t - 12;$

$$A_7^3(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_7^3(t) = 4t - 24;$

$$A_7^4(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_7^4(t) = 4t - 15;$

$$A_7^5(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_7^5(t) = 4t - 8;$

$$A_7^7(t) = \begin{pmatrix} 2t & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_7^7(t) = 4t - 7;$

$$A_8^1(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^1(t) = 2t - 4;$

$$A_8^2(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^2(t) = 2t - 14;$

$$A_8^3(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^3(t) = 2t - 30;$

$$A_8^4(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^4(t) = 2t - 20;$

$$A_8^5(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^5(t) = 2t - 12;$

$$A_8^6(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^6(t) = 2t - 6;$

$$A_8^8(t) = \begin{pmatrix} 2t & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$D_8^8(t) = 2t - 8$.

It follows directly from the quantities of determinants of the above matrices the validity of Theorems 1 – 3.

References

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