Advances in Astronomy and Space Physics, **3**, 58-62 (2013)

# Damping of magnetospheric toroidal Alfvén eigenmodes due to phase mixing

*S. O. Cheremnykh, A. S. Parnowski**

Space Research Institute, Akad. Glushkova ave., 40, build. 4/1, 03680 MSP Kyiv-187, Ukraine

We demonstrate that toroidal shear Alfvén eigenmodes in dipolar magnetostatic plasma equilibrium are subject to damping due to phase mixing. This result is of particular interest for the investigation of Pc5-6 geomagnetic pulsations.

**Key words:** methods: analytical, MHD waves and instabilities, kinetic and MHD theory

## INTRODUCTION

The study of the generation of geomagnetic pulsations includes three distinct research topics: the spectrum and the stability criterion, the propagation mechanism, and the energy balance (source vs. sink). The first topic was addressed in [1, 2, 3, 6, 7, 8, 17], the second one was studied e. g. by Klimushkin [13]. The third topic was considered by many authors with respect to the source, but the damping mechanisms were studied not so widely. The decay due to the resistance of the ionosphere was considered in [3, 7, 11, 12, 17], and some other effects in e. g. [9, 10, 18, 14, 19].

We demonstrate that there is another damping mechanism, never considered before in magnetosphere studies – the damping due to phase mixing. The physics behind this damping can be explained in simple words in the following way. Consider a small volume of plasma spanning across the magnetic surfaces, so that the eigenmode frequency is slightly different at its two edges. Then the phase shift between the oscillations at these two edges will grow, and with time they will oscillate in opposite directions, effectively cancelling each other. Such effects are well-known in radioelectronics with regard to open resonators, see e. g. [21, 22].

We study this effect only in the inner magnetosphere, which is well represented by an axially symmetric static plasma equilibrium, formed by a point dipole in the origin and a toroidal ring current with some spatial distribution (see Fig. 1). We also assume that the field lines rest upon a perfectly conductive ionosphere, which we place at the ground level.

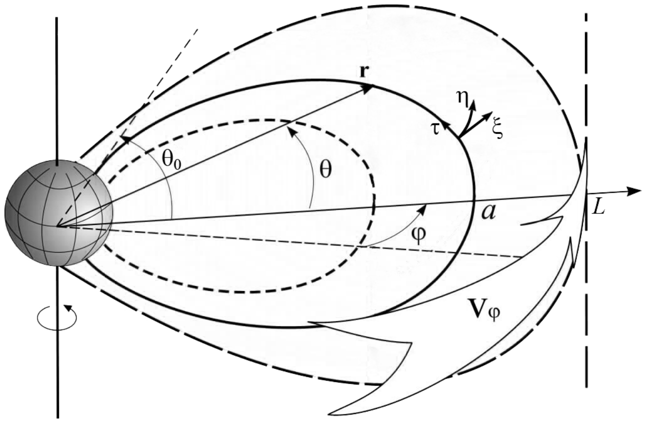


Fig. 1: A dipolar model of the inner magnetosphere.

As it was shown in [1], Alfvén waves in such geometry can be polarised either toroidally (in this case they are torsional) or poloidally (in this case they are compressional). We demonstrate that Alfvén waves are damped due to phase mixing, which is caused by the continuity of their spectrum. In this article we consider only toroidally polarised waves for simplicity. The principal result holds for poloidally polarised waves as well; this will be the subject of the next article.

The article is organised in three sections. Firstly, we briefly summarize the derivation of the equations of small oscillations for ULF MHD eigenmodes. Then we explain the phase mixing phenomenon and apply it to the consider problem. And in the last section we summarize the obtained results.

*parnowski@ikd.kiev.ua

## EQUATIONS OF SMALL PERTURBATIONS IN DIPOLAR FIELD

To obtain the equations of small perturbations in the dipolar magnetic field we use the equations derived by Cheng and Chance [5] in the form given by Cheremnykh and Danilova [6]:

$$\frac{\rho}{|\nabla a|^2}\frac{\partial^2\xi}{\partial t^2}+\vec{B}\cdot\nabla\left(\frac{1}{|\nabla a|^2}\vec{B}\cdot\nabla\xi\right)+\frac{s}{\alpha_s}(\gamma_s-s)\times \times\xi+2\left(\delta p_1+p'\xi+\gamma p\,\mathrm{div}\vec{\xi}\right)\frac{\vec{\chi}\cdot\nabla a}{|\nabla a|^2}+\frac{(s-\gamma_s)}{\alpha_s}\times \times\vec{B}\cdot\nabla\eta=\frac{\nabla a\cdot\nabla\delta p_1}{|\nabla a|^2}, \quad (1)$$

$$\frac{\rho}{\alpha_s}\frac{\partial^2\eta}{\partial t^2}+\vec{B}\cdot\nabla\left(\frac{1}{\alpha_s}+\vec{B}\cdot\nabla\eta\right)+2\left(\delta p_1+p'\xi+ +\gamma p\,\mathrm{div}\vec{\xi}\right)\frac{\vec{\chi}\cdot\left[\vec{B}\times\nabla a\right]}{\left|\vec{B}\right|^2}=\vec{B}\cdot\nabla\left(\frac{s}{\alpha_s}\xi\right)- -\frac{\gamma_s}{\alpha_s}\vec{B}\cdot\nabla\xi+\frac{\left[\vec{B}\times\nabla a\right]\cdot\nabla\delta p_1}{\left|\vec{B}\right|^2}, \quad (2)$$

$$\rho\frac{\partial^2\tau}{\partial t^2}+\gamma p\vec{B}\cdot\nabla\mathrm{div}\vec{\xi}=0. \quad (3)$$

Here the following standard notations were used: $\rho$ is the plasma density, $p$ is the plasma pressure, $\gamma$ is the ratio of specific heats, $\vec{j}$ is the current density, $\vec{B}$ is the magnetic induction, $\alpha_s=\frac{\left|\vec{B}\right|^2}{|\nabla a|^2}$, $\gamma_s=\frac{\vec{j}\cdot\vec{B}}{|\nabla a|^2}$ is the longitudinal current,

$$\vec{\xi}=\xi\frac{\nabla a}{|\nabla a|^2}+\eta\frac{\left[\vec{B}\times\nabla a\right]}{\left|\vec{B}\right|^2}+\tau\frac{\vec{B}}{\left|\vec{B}\right|^2}$$

is the displacement vector of an elementary volume of plasma, $\vec{\chi}=\left(\frac{\vec{B}}{\left|\vec{B}\right|}\cdot\nabla\right)\frac{\vec{B}}{\left|\vec{B}\right|}$ is the curvature vector of the magnetic field, $s=\frac{\left[\vec{B}\times\nabla a\right]}{\left|\vec{B}\right|^2}\cdot\mathrm{rot}\frac{\left[\vec{B}\times\nabla a\right]}{|\nabla a|^2}$ is the shear of the magnetic field,

$$\delta p_1=-\gamma p\,\mathrm{div}\vec{\xi}-\left|\vec{B}\right|^2\left(\mathrm{div}\vec{\xi}_\perp+2\vec{\chi}\cdot\vec{\xi}_\perp\right)$$

is the total perturbed plasma pressure, $(\cdots)'=\frac{\partial}{\partial a}(\cdots)$, subscript $\perp$ denotes a vector, perpendicular to the magnetic field line.

The quantity $a$ is called a label of the magnetic surface and satisfies the following conditions:

$$\vec{B}\cdot\nabla a=0,\ \vec{j}\cdot\nabla a=0. \quad (4)$$

The magnetic surface [15] is a surface, which contains magnetic field lines and current lines. At a given magnetic surface, the value of $a$ is constant.

When deriving the equations (1)–(3) we took into account that the vectors $\nabla a$, $\left[\vec{B}\times\nabla a\right]$ and $\vec{B}$ are orthogonal. For simplicity, we used Lorentz-Heavyside units for $\vec{B}$ and $\vec{j}$, normalising the latter by $c$ as well:

$$\vec{B}=\frac{\vec{B}^{(CGS)}}{\sqrt{4\pi}},\ \vec{j}=\frac{\sqrt{4\pi}}{c}\vec{j}^{(CGS)}, \quad (5)$$

where $\vec{B}^{(CGS)}$ and $\vec{j}^{(CGS)}$ are the magnetic induction and the current density respectively in Gauss units.

Equations (1) – (3) do not depend on the particular coordinate system, since they were obtained using the general properties of differential operators. For this reason they are exact and represent arbitrary MHD perturbations in ideal plasma and do not impose any restrictions on the pressure, current and electromagnetic fields.

In spherical coordinates ($r$, $\theta$, $\varphi$) with $\theta=0$ at the equator an axially symmetric dipolar magnetic field can be defined as

$$\vec{B}=\left[\nabla\psi\times\nabla\varphi\right], \quad (6)$$

where $\psi=M\cos^2\theta/r$ is the poloidal magnetic flux, $M$ is the terrestrial magnetic dipolar momentum, $\varphi$ is the toroidal (longitudinal) angle. The equation (6) is the simplest curvilinear three-dimensional model of the geomagnetic field. The field line equation of the dipolar field is

$$r=L\cos^2\theta, \quad (7)$$

where $L$ is the equatorial distance to the field line measured in terrestrial radii [16].

Choosing the function $\psi$ as the magnetic surface label, and considering that for such a field the following conditions hold true:

$$\gamma_s=0,\ \vec{\chi}\cdot\left[\vec{B}\times\nabla\psi\right]=0, \quad (8)$$

from the equations (1)–(3) and (8) we get a set of equations of small perturbations [1]:

$$\frac{\rho}{|\nabla\psi|^2}\frac{\partial^2\xi}{\partial t^2}+\vec{B}\cdot\nabla\left(\frac{1}{|\nabla\psi|^2}\vec{B}\cdot\nabla\xi\right)+2\left(\delta p_1+p'\xi+ +\gamma p\,\mathrm{div}\vec{\xi}\right)\frac{\vec{\chi}\cdot\nabla\psi}{|\nabla\psi|^2}=\frac{\nabla\psi\cdot\nabla\delta p_1}{|\nabla\psi|^2}, \quad (9)$$

$$\frac{\rho}{\alpha_s}\frac{\partial^2\eta}{\partial t^2}+\vec{B}\cdot\nabla\left(\frac{1}{\alpha_s}\vec{B}\cdot\nabla\eta\right)=\frac{\left[\vec{B}\times\nabla\psi\right]\cdot\nabla\delta p_1}{\left|\vec{B}\right|^2}, \tag{10}$$

$$\rho\frac{\partial^2\tau}{\partial t^2}+\gamma p\vec{B}\cdot\nabla\mathrm{div}\vec{\xi}=0. \tag{11}$$

These equations describe all three MHD modes – a shear Alfvén mode, a compressional Alfvén mode, which splits into a shear Alfvén mode and a slow magnetoacoustic mode in the limit of zero curvature, and a fast magnetoacoustic mode.

Setting $\delta p_1 = 0$ in equations (9)–(11), we drop out the fast magnetoacoustic mode, since the remaining two modes do not perturb the total pressure. This yields a condition

$$\mathrm{div}\vec{\xi}=\frac{1}{1+\beta}\left\{\vec{B}\cdot\nabla\left(\frac{\tau}{\left|\vec{B}\right|^2}\right)-\frac{2\vec{\chi}\cdot\nabla\psi}{\left|\nabla\psi\right|^2}\xi\right\}, \tag{12}$$

where $\beta=\dfrac{\gamma p}{\left|\vec{B}\right|^2}$. In the $\beta \ll 1$ limit the equation (12) tends to a well-known relation $\mathrm{div}\vec{\xi}_\perp \sim -2\vec{\chi}\cdot\vec{\xi}_\perp$, which is commonly used in energy analysis to cancel the fast magnetoacoustic mode [4].

By the definition of divergence we can also write

$$\mathrm{div}\vec{\xi}=\xi\mathrm{div}\left(\frac{\nabla\psi}{\left|\nabla\psi\right|^2}\right)+\frac{\nabla\psi\cdot\nabla\xi}{\left|\nabla\psi\right|^2}+ \\ +\frac{\nabla\varphi\cdot\nabla\eta}{\left|\nabla\varphi\right|^2}+\vec{B}\cdot\nabla\left(\frac{\tau}{\left|\vec{B}\right|^2}\right). \tag{13}$$

Note that the second right-hand term contains a derivative in the $\nabla\psi$ direction, perpendicular to a magnetic surface. Since such a derivative does not appear elsewhere in the equations, the condition (13) can be always satisfied by this term alone by a proper choice of this derivative. Nevertheless, it is important to consider this condition, because it means that for $\mathrm{div}\vec{\xi}$ to be defined we have to consider the three-dimensional vector field $\vec{\xi}$ and not just a single magnetic surface.

As a result, we obtain the exact equations of small perturbations in the dipolar magnetic field:

$$\frac{\rho}{\left|\nabla\psi\right|^2}\frac{\partial^2\xi}{\partial t^2}+\vec{B}\cdot\nabla\left(\frac{1}{\left|\nabla\psi\right|^2}\vec{B}\cdot\nabla\xi\right)+ \\ +2\frac{\vec{\chi}\cdot\nabla\psi}{\left|\nabla\psi\right|^2}\left[p'\xi+\frac{\gamma p}{1+\beta}\left\{\vec{B}\cdot\nabla\left(\frac{\tau}{\left|\vec{B}\right|^2}\right)-\right.\right. \\ \left.\left.-\frac{2\vec{\chi}\cdot\nabla\psi}{\left|\nabla\psi\right|^2}\xi\right\}\right]=0, \tag{14}$$

$$\frac{\rho}{\left|\vec{B}\right|^2}\frac{\partial^2\tau}{\partial t^2}+\beta\vec{B}\cdot\nabla\times \\ \times\left[\frac{1}{1+\beta}\left\{\vec{B}\cdot\nabla\left(\frac{\tau}{\left|\vec{B}\right|^2}\right)-\frac{2\vec{\chi}\cdot\nabla\psi}{\left|\nabla\psi\right|^2}\xi\right\}\right]=0, \tag{15}$$

$$\frac{\rho}{\alpha_s}\frac{\partial^2\eta}{\partial t^2}+\vec{B}\cdot\nabla\left(\frac{1}{\alpha_s}\vec{B}\cdot\nabla\eta\right)=0. \tag{16}$$

These equations coincide with those obtained in the articles [8] using the ballooning approximation. Note that we made only two assumptions: that the ambient magnetic field is dipolar, and that the total pressure is not perturbed. Boundary conditions for this system of equations are determined by ideal conducting ionsphere and by condition

$$\xi\Big|_{\theta=\pm\theta_0}=\eta\Big|_{\theta=\pm\theta_0}=\tau\Big|_{\theta=\pm\theta_0}=0, \tag{17}$$

where $\theta_0=\cos^{-1}\left(L^{-1/2}\right)$ is the latitudinal angle at which the field line crosses the boundary.

## DAMPING DUE TO PHASE MIXING

Equations (12), (13) and (16) can be considered separately from equations (14) and (15) by formally setting $\xi = 0$ and $\tau = 0$. Speaking strictly, they both can be non-zero, but their input in the condition (12) can be negated by a proper choice of the term with a transverse derivative in the equation (13). In this case

$$\frac{\nabla\varphi\cdot\nabla\eta}{\left|\nabla\varphi\right|^2}=\frac{\left[\vec{B}\times\nabla\psi\right]}{\left|\vec{B}\right|^2}\cdot\nabla\eta=0 \tag{18}$$

and describes toroidal Alfvén modes. The relation (18) means that the toroidal amplitude $\eta$ does not depend on the longitudinal angle $\varphi$. This was taken into account when analysing the equation (16) in the article [8]. All spatial gradients in this equation are multiplied by the magnetic field and have the form $\vec{B}\cdot\nabla$, so it is convenient to switch to the field-aligned derivative. Since this is a one-dimensional problem, we can consider a single harmonic by replacing $\frac{\partial^2}{\partial t^2}$ with $-\omega^2$. Now we can rewrite the equation (16) in the form

$$\Omega^2\eta + \frac{1}{\cos^{13}\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\cos\theta}\frac{\partial\eta}{\partial\theta}\right) = 0, \qquad (19)$$

where $\Omega = \omega/\omega_A$ is the dimensionless frequency, $\omega_A = B_0/L\sqrt{\rho}$ is the Alfvén frequency, $B_0 = M/L^3$ is the equatorial magnetic field. We assume the plasma density $\rho$ to be constant along the field line.

Adding ideal conductive boundary conditions for toroidal modes (17), we can easily get a WKB solution to the Sturm-Liouville eigenvalue problem (19), (17):

$$\Omega_m = \frac{\pi m}{\int\limits_{-\theta_0}^{+\theta_0}\cos^7\theta \mathrm{d}\theta},\ m = 1,\,2,\,3,\ldots,$$

$$\eta_m = \frac{1}{\Omega_m\cos^3\theta}\sin\left[\Omega_m\int\limits_{-\theta_0}^{+\theta_0}\cos^7\theta \mathrm{d}\theta\right]. \qquad (20)$$

As one can see, $\Omega_m$ and thus $\omega$ are continuous functions of $\theta_0$ and thus of $L$. Let us follow the approach of [20]. Consider two neighbouring magnetic surfaces with slightly different equatorial distances $L - \Delta L$ and $L + \Delta L$. Assuming $\Delta L$ to be small, let us calculate the average perturbed toroidal magnetic field $\delta B_\varphi$ in this region:

$$\langle\delta B_\varphi(L,\theta,t)\rangle_L = \frac{1}{2\Delta L}\int\limits_{L-\Delta L}^{L+\Delta L}\delta B_\varphi(L,\theta,t)\,\mathrm{d}L = \frac{1}{2\Delta L}\int\limits_{L-\Delta L}^{L+\Delta L} a(L,\theta)\,e^{-i\omega(L)t}\mathrm{d}L. \qquad (21)$$

Here $\omega(L)$ is defined by equation (20) and $a(L,\theta)$ is defined by the initial conditions at $t = 0$. Switching the integration variable from $L$ to $\omega$, we obtain

$$\langle\delta B_\varphi(L,\theta,t)\rangle_\omega = \frac{1}{2\Delta L}\int\limits_{\omega_-}^{\omega_+} b(\omega,\theta)\,e^{-i\omega t}\mathrm{d}\omega, \qquad (22)$$

where $b(\omega,\theta) = \frac{a(L(\omega),\theta)}{\partial\omega/\partial L}$, $L(\omega)$ is a function, inverse to $\omega(L)$, $\omega_\pm = \omega(L \pm \Delta L)$. Note that the function $b(\omega,\theta)$ exists only when $\frac{\partial\Omega}{\partial L} \neq 0$.

Integrating the right-hand side of equation (22) by parts, we get

$$\langle\delta B_\varphi(L,\theta,t)\rangle_\omega = \frac{1}{2\Delta L}\left[\left.\frac{ib(\omega)e^{-i\omega t}}{t}\right|_{\omega_-}^{\omega_+} - \frac{i}{t}\int\limits_{\omega_-}^{\omega_+}\frac{\partial b(\omega,\theta)}{\partial\omega}e^{-i\omega t}\mathrm{d}\omega\right]. \qquad (23)$$

As one can easily see, this expression tends to zero at $t \to \infty$, i. e. the perturbations are damped.

## SUMMARY

In the case of dipolar magnetic field configuration, it is not necessary to apply the ballooning approximation and the equations of small perturbations can be derived exactly (in the linearised sense, of course). These equations describe all three MHD modes – shear Alfvén mode, compressional Alfvén mode, which splits into a shear Alfvén mode and a slow magnetoacoustic mode in the limit of zero curvature, and a fast magnetoacoustic mode, which we manually cancelled from the equations. These equations contain a term $\mathrm{div}\vec{\xi}$, which has the physical meaning of plasma compressibility. This term, by definition of a divergence, contains a derivative in the $\nabla\psi$ direction, transverse to the magnetic surface. This derivative thus describes the transverse structure of the perturbations, i. e. the interaction between the perturbations localised at the neighbouring magnetic surfaces.

Due to this interaction and a continuous dependence of eigenmode frequencies on the label of the magnetic surface, the phase difference between the perturbations on the neighbouring magnetic surfaces will build up with time and their amplitude will decrease with time as $1/t$.

## ACKNOWLEDGEMENT

This research was supported by the State Fund for Fundamental Research of Ukraine, project No. F53.6/036.

## REFERENCES

[1] Cheremnykh S. O. & Agapitov O. 2011, Advances in Astronomy and Space Physics, 2, 103
[2] Agapitov A. V. & Cheremnykh O. K. 2011, Kinematika i Fizika Nebesnykh Tel, 27, 3, 17
[3] Agapitov A. V., Cheremnykh O. K. & Parnowski A. S. 2008, Adv. Space Res., 41, 1682
[4] Bateman G. 1978, *MHD Instabilities*, Cambridge, Mass., MIT Press
[5] Cheng C. Z. & Chance M. S. 1986, Physics of Fluids, 29, 3695
[6] Cheremnykh O. K. & Danilova V. V. 2011, Kinematics and Physics of Celestial Bodies, 27, 2, 63

[7] Cheremnykh O. K. & Parnowski A. S. 2006, Adv. Space Res., 37, 599
[8] Cheremnykh O. K., Parnowski A. S. & Burdo O. S. 2004, Planet. Space Sci., 52, 1217
[9] Coult N., Pilipenko V. & Engebretson M. 2007, Planet. Space Sci., 55, 694
[10] Fedorov E., Pilipenko V. & Engebretson M. J. 2001, J. Geophys. Res., 106, 6203
[11] Hameiri E. 1999, Physics of Plasmas, 6, 674
[12] Hameiri E. & Kivelson M. G. 1991, J. Geophys. Res., 96, 21125
[13] Klimushkin D. Y. 1998, Ann. Geophys., 16, 303
[14] Klimushkin D. Yu. 2007, Planet. Space Sci., 55, 722
[15] Kruskal M. D. & Kulsrud R. M. 1958, Physics of Fluids, 1, 265
[16] McIlwain C. E. 1961, J. Geophys. Res., 66, 3681
[17] Parnowski A. S. 2007, Ann. Geophys., 25, 1391
[18] Pilipenko V., Fedorov E. & Engebretson M. J. 2008, Earth, Planets and Space, 60, 949
[19] Pilipenko V., Fedorov E., Engebretson M. J. & Yumoto K. 2004, J. Geophys. Res., 109, A10204
[20] Polovin R. V. & Demutsky V. P. 1987, *Basic magnetic hydrodynamics*, Moscow, Energoatomizdat
[21] Shevchenko V. V. 1971, *Continuous transitions in open waveguides*, Golem Press
[22] Weinstein L. A. 1966, *Open resonators and open waveguides*, Moscow, Sov. Radio