UDC 336.71

EVALUATING THE FINANCIAL FLOWS OF BESSEL PROCESSES BY USING SPECTRAL ANALYSIS

© 2017 BURTNYAK I. V., MALYTSKA H. P.

UDC 336.71

Burtnyak I. V., Malytska H. P. Evaluating the Financial Flows of Bessel Processes by Using Spectral Analysis

The article solves the two-parameter task of evaluating the intensity of diffuse Bessel processes by the methods of spectral theory. In particular, barriers for cost of options, where the derivative of financial flows turns into zero, have been considered, and a task for the two-barrier option has been solved, which corresponds to Bessel process. A Green's function has been built for the diffusion Bessel process of the two-barrier option, decomposed according to the first-type system of Bessel functions. The barriers are taken in such a way that the derivative of financial flow in terms of price is turned to zero, i.e. there are the points where flow can acquire extreme values. On the basis of Green's function, the value of securities has been calculated. It is handier to use similar barriers when monitoring a stock market. The Green's function for this task, which represents the probability of spreading the option price, is represented through the Fourier series. This provides an opportunity to evaluate the intensity of financial flows in stock markets.

Keywords: *spectral theory, barrier option, financial flows, Bessel functions, Green's function, singular parabolic operator, infinitesimal operator.*
Fig.: *1.* ***Formulae:*** *12.* ***Bibl.:*** *9.*

Burtnyak Ivan V. *– PhD (Economics), Associate Professor of the Department of Economic Cybernetics, Precarpathian National University named after V. Stefanyk (57 Shevchenka Str., Ivano-Frankivsk, 76018, Ukraine)*
E-mail: *bvanya@meta.ua*
Malytska Hanna P. *– PhD (Physics and Mathematics), Associate Professor of the Department of Mathematical and Functional Analysis, Precarpathian National University named after V. Stefanyk (57 Shevchenka Str., Ivano-Frankivsk, 76018, Ukraine)*

УДК 336.71

Буртняк І. В., Малицька Г. П. Оцінка фінансових потоків Беселівських процесів за допомогою спектрального аналізу

У статті розв'язано двопараметричну задачу оцінювання інтенсивності Беселівських дифузійних процесів методами спектральної теорії. Зокрема, розглянуто бар'єри для вартості опціонів, в яких похідна фінансових потоків перетворюється в нуль, розв'язано задачу для двобар'єрного опціону, що відповідає процесу Бесселя. Здійснено побудову функції Гріна для дифузійного процесу Бесселя двобар'єрного опціону, яка розкладена по системі функцій Бесселя першого роду. Бар'єри взяті таким чином, щоб у них похідна фінансового потоку за ціною перетворювалася в нуль, тобто це точки, де потік може набувати екстремальних значень. На основі функції Гріна проведено обчислення вартості похідних цінних паперів. Для проведення моніторингу фондового ринку зручно використовувати саме такі бар'єри. Функцію Гріна цієї задачі, яка репрезентує ймовірність поширення ціни опціону, представлено через ряди Фур'є. Це дає можливість оцінити інтенсивність фінансових потоків фондових ринків.

Ключові слова: *спектральна теорія, бар'єрний опціон, фінансові потоки, функції Бесселя, функція Гріна, сингулярний параболічний оператор, інфінітіземальний оператор.*
Рис.: *1.* ***Формул:*** *12.* ***Бібл.:*** *9.*

Буртняк Іван Володимирович *– кандидат економічних наук, доцент кафедри економічної кібернетики, Прикарпатський національний університет ім. В. Стефаника (вул. Шевченка, 57, Івано-Франківськ, 76018, Україна)*
E-mail: *bvanya@meta.ua*
Малицька Ганна Петрівна *– кандидат фізико-математичних наук, доцент кафедри математичного та функціонального аналізу, Прикарпатський національний університет ім. В. Стефаника (вул. Шевченка, 57, Івано-Франківськ, 76018, Україна)*

УДК 336.71

Буртняк И. В., Малицкая А. П. Оценка финансовых потоков процессов Бесселя с помощью спектрального анализа

В статье решена двухпараметрическая задача оценивания интенсивности диффузных процессов Бесселя методами спектральной теории. В частности, рассмотрены барьеры для стоимости опционов, где производная финансовых потоков превращается в ноль, решена задача для двухбарьерного опциона, что соответствует процессу Бесселя. Построена функция Грина для диффузионного процесса Бесселя двухбарьерного опциона, разложенная по системе функций Бесселя первого рода. Барьеры взяты таким образом, чтобы в них производная финансового потока по цене превращалась в ноль, то есть это точки, где поток может приобретать экстремальные значения. На основе функции Грина проведено вычисление стоимости ценных бумаг. Для проведения мониторинга фондового рынка удобно использовать именно такие барьеры. Функция Грина этой задачи, которая репрезентирует вероятность распространения цены опциона, представлена через ряды Фурье. Это дает возможность оценить интенсивность финансовых потоков фондовых рынков.

Ключевые слова: *спектральная теория, барьерный опцион, финансовые потоки, функции Бесселя, функция Грина, сингулярный параболический оператор, инфинитиземальный оператор.*
Рис.: *1.* ***Формул:*** *12.* ***Библ.:*** *9.*

Буртняк Иван Владимирович *– кандидат экономических наук, доцент кафедры экономической кибернетики, Прикарпатский национальный университет им. В. Стефаника (ул. Шевченко, 57, Ивано-Франковск, 76018, Украина)*
E-mail: *bvanya@meta.ua*
Малицкая Анна Петровна *– кандидат физико-математических наук, доцент кафедры математического и функционального анализа, Прикарпатский национальный университет им. В. Стефаника (ул. Шевченко, 57, Ивано-Франковск, 76018, Украина)*

Bessel processes play an important role in financial mathematics, since they are inherently closely related to models of geometric Brownian motion and Cox-Ingresol-Ross processes [1]. We are interested in considering those Bessel processes, which present generalization of the Ornstein-Uhlenbeck process for barrier options [2]. At certain characteristics, the diffusion process with the Bessel operator never hits zero, and a number of papers [3] are dedicated to these cases. We consider those cases when the derivative of the financial flow of the Bessel process can hit zero. These conditions are used to determine the excessive growth rate of the stock portfolio as well as explain how exceeding the growth rate of the market portfolio provides a measure of internal volatility in the market at any given time [4].

A significant part of the problems of financial mathematics are described by diffusion processes or stochastic differential equations. In 1998, Kaufman showed that the Bessel diffusion $\{Z_t,\ t \geq 0\}$ with constant negative drift and an infinitesimal generator has the form [5]:

$$(Gf)(z) = \frac{1}{2}\sigma^2 f''(z) + \left(\frac{d}{z} + c\right) f'(z).$$

Problem statement. The spectral method is applied to derivative financial instruments, in particular there presented the price for the derivative $u(t, x)$ through a function that is neutral to the risk of expecting the future value of the real-valued process X, that is as

$$u(t,\ x) = \tilde{E}_x[H(X_t)] = \int H(y)p(t,\ x,\ y)\,dy,$$

where $p(t, x, y)$ – transition density of X with the probability p. If the infinitesimal generator L of the real-valued process is self-ajoint in the Hilbert space with the increment of the measure $m(x)\,dx$, and the L – spectrum is discrete, then the transition density of X is developing with respect to its own functions [6]:

$$p(t,\ x,\ y) = m(y)\sum\nolimits_n e^{-\lambda nt}\varphi_n(y)\varphi_n(x),$$

where $\{\lambda_n\}$ – eigenvalues, $L\{\varphi_n\}$ – eigenfunctions: therefore, $L\{\varphi_n\} = \lambda_n \varphi_n$.

Let us consider the process for which the operator L has the form:

$$L = \partial^2_{xx} + x^{-1}\partial_x - x^{-2}p^2, \qquad (1)$$

where p – constant value called an index, $x > 0$.

We should note that L is a singular parabolic operator, an infinitesimal one, to which a number of operators where $\sigma^2 = 2x^2$ is reduced, L is called the Bessel operator.

Let us study L for eigenvalues and eigenfunctions using the Sturm – Liouville theory, thus $Lv = -\lambda^2 v$, we obtain the equation $v'' + \frac{v'}{x} - \frac{p^2}{x^2}v = -\lambda^2 v$, after multiplying by we have

$$x^2v'' + xv' + (\lambda^2x^2 - p^2)v = 0. \qquad (2)$$

Equation (2) is a Bessel equation with the parameter λ.

The solution of equation (2), except for the partial values of p, is not expressed in terms of elementary functions (in the finite form), these non-elementary functions are called Bessel functions, they are widely used in economics, technology and physics. Since the Euler-Bessel equation is a linear one, its total integral can be put in the form

$$v = C_1v_1 + C_2v_2,$$

where v_1, v_2 are any two linearly independent partial solutions of the Euler – Bessel equation, and C_1, C_2 are arbitrary constants.

In the case of $p \geq 0$ we make a substitution $v = x^p w$ and obtain for the function w the following equation:

$$w'' + \frac{2p+1}{x}w' + w = 0.$$

The solution of the resulting equation is a power series that is absolutely convergent for all $x \in (-\infty;\ \infty)$ and has the form:

$$v = x^p w = \frac{(x/2)}{\Gamma(p+1)} + \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{p+2m}}{2\ldots m(p+1)\ldots(p+m)\Gamma(p+1)}, \qquad (3)$$

where Γ – gamma function. By transforming (3) on the basis of properties of the gamma function, we obtain a Bessel function of the first kind of the p-th order:

$$J_p(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{p+2m}}{\Gamma(m+1)\Gamma(p+m+1)}.$$

Note. Since equation (2) contains p^2, the substitution of p for $(-p)$ does not influence the solution of the equation, thus there exists a solution for any value of p.

If p is not an integer, then the Bessel functions cannot be linearly dependent and the general integral of equation (2) has the form:

$$J = C_1J_p(x) + C_2J_{-p}(x).$$

With an integer p we find one more partial solution:

$$Y_p(x) = \frac{J_p(x)\cos p\pi + J_{-p}(x)}{\sin p\pi},$$

which is expressed by a Bessel function of the second kind that is undefined at $x = 0$. Using L'Hôpital's rule we find the boundary for $x \to 0$ and by this value define the function at zero:

$$Y_0 = \frac{2}{\pi}J_0(x)\left(\ln\frac{x}{2} + C\right) - \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^m (x/2)^{2m}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m}\right).$$

For any value of p the following formulas can be used:

$$\frac{d}{dx}(x^p J_p(x)) = x^p J_{p-1}(x),$$

$$\frac{d}{dx}(x^{-p} J_p(x)) = x^{-p} J_{p+1}(x).$$

The Bessel functions $J_p(\lambda x)$, $J_p(\mu x)$ where λ and μ are the roots of the equation $J_p(x) = 0$ are orthogonal on the interval [0, 1] with the weight x, thus

$$\int_0^1 xJ_p(\lambda x)J_p(\mu x)dx = 0,\ \lambda \neq \mu,$$

and if $\lambda = \mu$, the two cases are possible:

$$\int_0^1 xJ_p^2(\lambda x)dx = \begin{cases} \frac{1}{2}J_p'^2(\lambda),\ J_p(\lambda) = 0, \\ \frac{1}{2}\left(1 - \frac{p^2}{\lambda^2}\right)J_p^2(\lambda),\ J_p'(\lambda) = 0. \end{cases}$$

For all $\alpha, \beta \geq 0$, $\alpha + \beta > 0$ there exists a countable set of positive roots

$$\alpha v_k'(\mu) + \beta\mu v_k(\mu) = 0,$$

whose boundary point is at infinity.

If $v(x)$ is the solution of (2), then the function $v(\lambda x)$ will also be the solution of the equation of the following form:

$$x^2v'' + xv' + (\lambda^2x^2 - p^2)v = 0. \qquad (4)$$

Equation (4) is a Bessel equation with the parameter λ. Any solution of equation (2) expressed by a Bessel function has an infinite set of positive roots that are close to the roots of the function sin ($x + \omega$), which has the form: $k_n = n\pi - \omega$, ω = const, n – an integer (it is similar for negative roots, because they are symmetrical relative to the origin of coordinates), if $k_n \neq 0$ they are simple roots and form a countable set [7].

Since Bessel functions are alternating series, the calculation of values can be performed using the Leibniz lemma, which makes it possible to determine the accuracy of the approximation.

To find the eigenfunctions and eigenvalues, let us consider the following boundary value problem:

$$x^2 v_k'' + x v_k' + (\lambda^2{}_k x^2 - p^2) v_k = 0, \quad (5)$$

$$\left| v_k \right|_{x=0} < +\infty, \quad (6)$$

$$\alpha v_k'(x_0) + \beta v_k(x_0) = 0. \quad (7)$$

Thus, we are considering a Sturm – Liouville problem. The given problem has a unique solution. We impose condition (6) because $x = 0$ is a special point of equation (5) and the operator L. x_0 is a regular point of equation (5). The values of λ_k with which the boundary value problem (5) – (7) has a non-trivial solution v_k are called eigenvalues, and v_k – eigenfunctions of the problem. It is known that under conditions (6) the operator L has a countable number of eigenvalues, they are simple and not negative [8]. The multiplication of L by x^2 does not change either the eigenvalues, the eigenfunctions, or their quantity.

Let us consider the following problem:

$$\begin{cases} x^2 v_k'' + x v_k' + (\lambda_k^2 x^2 - p^2) v_k = 0, \\ \left| v_k \right|_{x=0} < +\infty, \\ v_k'(x_0) = 0. \end{cases} \quad (8)$$

From (6) it follows that $p \geq 0$. Let us consider the case with $p > 0$, since (2) has as its integral $v = C_1 J_p(x) + C_2 Y_p(x)$, then, based on the properties of the Bessel function, problem (8) has the following solution:

$$v = C_1 J_p(\lambda x) + C_2 Y_p(\lambda x).$$

Taking into account the boundary conditions we will have $C_2 = 0$ and $C_1 J_p(\lambda_k x_0) = 0$, thus $J_p(\lambda_k x_0) = 0$, therefore $\lambda_k x_0 = \mu_k$, where $\lambda_k = \dfrac{\mu_k}{x_0}$, $0 < \mu_1 < \mu_2 < \ldots < \mu_k < \ldots$, where μ_k are the roots of $J_p'(\mu_k) = 0$. The norm of $v_k(x)$ will have the form:

$$\left\| v_k(x) \right\|^2 = \frac{1}{2}\left(x_0^2 - \frac{v^2}{\mu_k^2} \right)(J_p(\mu_k))^2, \; k = 1, 2, \ldots$$

Let us consider the case with $p = 0$. Thus we have the problem:

$$\begin{cases} (xv')' + \lambda^2 x v = 0, \\ \left| v \right|_{x=0} < +\infty, \\ v'(x_0) = 0. \end{cases} \quad (9)$$

With $\lambda = 0$ the solution is $v(x) \equiv 1$, then it follows that $\lambda = 0$ is the eigenvalue and $v_0(x) \equiv 1$ is the eigenfunction.

Let us consider $\lambda > 0$. As in the previous case we will find $v_k(x) = J_0\left(\dfrac{\mu_k}{x_0} x \right)$, $k = 1, 2, \ldots$, $\lambda_k = \dfrac{\mu_k}{x_0}$, where μ_k are positive roots of the equation $J_p'(\mu) = 0$, in this case the norm $v_k(x)$ will be equal to:

$$\left\| v_k(x) \right\|^2 = \frac{x_0^2}{2}(J_0(\mu_k))^2, \quad k = 1, 2, \ldots$$

Results of the research. Let us consider the Bessel process described by the equation:

$$\frac{\partial v(t,x)}{\partial t} = \frac{\partial^2 v(t,x)}{\partial x^2} + x^{-1}\frac{\partial v(t,x)}{\partial x} - -p^2 x^{-2} v(t,x), \; 0 < x < x_0, \quad (10)$$

and the boundary condition

$$v(0, x) = K(e^x - 1)^+, \; v_x'(t, x_0) = 0, \quad (11)$$

where K is a strike value. The process is homogeneous, therefore, $v(t, x) = \varphi(t)v(x)$.

From the Sturm – Liouville theory we have:

$$v(t, x) = \sum_{n=1}^{\infty} c_{np} e^{-\frac{\mu_n^2}{x_0^2}t} J_p\left(\frac{\mu_n}{x_0} x \right) + c_0, \; p > 0, \; \mu_n > 0,$$

where μ_n – positive roots of the equation are calculated by the formula with $p > 0$, $c_{0p} = 0$ if $p = 0$ then

$$c_{0p} = \frac{K_1 \int_0^{x_0} x(e^x - 1)dx}{\int_0^{x_0} x dx} = \frac{K_1\left(x_0 e^x - e^{x_0} + 1 - \frac{x_0}{2} \right)}{\frac{x_0}{2}} =$$

$$= K_1(2x_0^{-1} e^{x_0} - 2x_0^{-2} e^{x_0} + 2x_0^{-2} - 1), \; K_1 = e^{-\lambda^2 T},$$

$$c_{np} = \frac{K \int_0^{x_0} x(e^x - 1) J_p\left(\frac{\mu_n}{x_0} x \right) dx}{\int_0^{x_0} x J_p^2\left(\frac{\mu_n}{x_0} x \right) dx}.$$

The financial flows have the following form:

$$u(t, x) = \sum_{n=1}^{\infty} K c_{np} e^{-\left(\frac{\mu_n}{\ln K} \right)^2 (T-t)} J_p\left(\mu_n \ln \frac{x}{K} \right).$$

In the case when the process is completed at time T, when $X_T = K$:

$$u(t, x) = \sum_{n=0}^{\infty} K c_n e^{-\left(\frac{\mu_n}{\ln \frac{R}{L}} \right)^2 (T-t)} J_p\left(\frac{\mu_n\left(\ln \frac{x}{L} \right)}{\ln \frac{R}{L}} \right),$$

where $L < x < R$, L, R – barriers, K – strike value, and c_{np} are calculated as follows:

$$c_{np} = 2K \frac{\int_0^1 t(e^{Kt} - 1) J_p(\mu_n t) dt}{J_{p+1}^2(\mu_n)}.$$

We have calculated the expansion of the financial flow in terms of the system of Bessel functions J_p of the first kind, while the distribution of the flows is set by the Green function of the corresponding problem. Therefore, for the calculations it is convenient to expand the Green function in terms of the system of Bessel functions. The process that we consider can be expressed by a correspondent inhomogeneous boundary value problem:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+x^{-1}\frac{\partial u}{\partial x}-\frac{p^2 u(t,x)}{x^2}+f(t,x),\ \ x>0, \quad \mathbf{(12)}$$

where $f(t, x)$ is twice continuously differentiable in x and continuously differentiable in t, absolutely integrable with the derivatives, $(t, x) \in [0, +\infty)$, and is expressed as follows:

$$f(t,\ x)=\sum_{n=0}^{\infty} f_n(t)\,J_p\left(\frac{\mu_n x}{x_0}\right),$$

$$0<x<x_0<+\infty,\ \ 0<t<T,$$

where μ_n are the roots of the equation $J_p(\mu_n) = 0$.

The problem can be solved using the following equation:

$$u(t,\ x)=\sum_{n=0}^{\infty} T_n(t)\,J_p\left(\frac{\mu_n x}{x_0}\right).$$

By substituting (12) we obtain:

$$\sum_{n=0}^{\infty} T_n'(t)J_p\left(\frac{\mu_n x}{x_0}\right)=\sum_{n=0}^{\infty}\left\{\left[\frac{\left(J_p\left(\frac{\mu_n x}{x_0}\right)\right)''_{x^2}}{}+\frac{\left(J_p\left(\frac{\mu_n x}{x_0}\right)\right)'_x}{x}-\frac{p^2 J_p\left(\frac{\mu_n x}{x_0}\right)}{x^2}+\lambda_n^2 J_p\left(\frac{\mu_n x}{x_0}\right)\right]-\right.$$

$$\left.-\lambda_n^2 J_p\left(\frac{\mu_n x}{x_0}\right)\right\}T_n(t)+\sum_{n=0}^{\infty} f_n(t)\,J_p\left(\frac{\mu_n x}{x_0}\right),$$

then

$$\sum_{n=0}^{\infty}\left[T_n'(t)+\lambda_n^2 T_n(t)-f_n(t)\right]J_p\left(\frac{\mu_n x}{x_0}\right)\equiv 0,$$

therefore, $T_n'(t)+\lambda_n^2 T_n(t)-f_n(t)=0,\ \lambda_n=\frac{\mu_n}{x_0},\ n\in N$

with the initial condition $T_n(0) = 0$.

The inhomogeneous differential equation of the first order is solved by the method of constant variation. Since $T_n'(t)+\lambda_n^2 T_n(t)=0$ has the first integral $T_n(t)=Ce^{-\lambda_n^2 t}$ (the solution of a inhomogeneous equation), then $T_n(t)=C(t)e^{-\lambda_n^2 t}$, thus

$$C'^{(t)}=f_n(t)e^{\lambda_n^2 t},\ C(t)=\int_0^t e^{\lambda_n^2\beta} f_n(\beta)\,d\beta+C_1.$$

$$T_n(t)=\int_0^t e^{\lambda_n^2\beta} f_n(\beta)\,d\beta e^{-\lambda_n^2 t}+C_1 e^{-\lambda_n^2 t} \text{ with } t=0,\ C_1=0,$$

$$T_n(t)=\int_0^t e^{-\lambda_n^2(t-\beta)} f_n(\beta)\,d\beta,$$

therefore, $u(t,\ x)=\sum_{n-0}^{\infty}\int_0^t e^{-\lambda_n^2(t-\beta)} f_n(\beta)\,d\beta\, J_p\left(\frac{\mu_n x}{x_0}\right).$

Taking into account that

$$f_n(t)=\int_0^{x_0}\xi f_n(\xi,\ t)\ J_p\left(\frac{\mu_n \xi}{x_0}\right)d\xi\left(\int_0^{x_0} x J_p^2\left(\frac{\mu_n x}{x_0}\right)dx\right)^{-1},$$

we have:

$$u(t,\ x)=\sum_{n=0}^{\infty}\int_0^t e^{-\lambda_n^2(t-\beta)}\int_0^{x_0}\xi f(\xi,\ t)\,J_p\left(\frac{\mu_n \xi}{x_0}\right)\times$$

$$\times d\beta d\xi\, J_p\left(\frac{\mu_n x}{x_0}\right)\left(\int_0^{x_0} y J_p^2\left(\frac{\mu_n x}{x_0}\right)dy\right)^{-1}=$$

$$=\int_0^{x_0}\int_0^t\sum_{n-0}^{\infty}\left(y J_p^2\left(\frac{\mu_n y}{x_0}\right)dy\right)^{-2} e^{-\lambda_n^2(t-\beta)}\times$$

$$\times\xi J_p\left(\frac{\mu_n \xi}{x_0}\right)J_p\left(\frac{\mu_n x}{x_0}\right)f(\xi,\ t)\,d\xi d\beta,$$

therefore,

$$G(t-\beta,\ x,\ \xi)=\sum_{n=0}^{\infty}\xi J_p\left(\frac{\mu_n \xi}{x_0}\right)J_p\left(\frac{\mu_n x}{x_0}\right)e^{-\frac{\mu_n^2}{x_0}(t-\beta)}\times$$

$$\times\left(\frac{1}{2}\left(x_0^2-\frac{p^2}{\mu_k^2}\right)(J_p(\mu_k))^2\right)^{-1},$$

$$u(t,\ x)=\int_0^t G(t-\tau,\ x,\ \xi)\,f(\tau,\ \xi)\,d\xi.$$

Since the problem of evaluating and studying two-dimensional barrier options is reduced to considering and solving boundary value problem [9]

$$\frac{\partial u(t,\ x)}{\partial t}=\frac{\partial^2 u(t,\ x)}{\partial x^2}+x^{-1}\frac{\partial u(t,\ x)}{\partial x}-\frac{p^2 u(t,\ x)}{x^2},$$

$$x\in[L,H],\ t\in[0,\ T],$$

$$u_x'(t,L)=0,\ u_x'(t,\ H)=0,$$

$$u_x'(T,x)=\max\left(\pm(x(T)-K),0\right)[\![_{(L<x(t)<H;\ t\in[0,T])}\,.$$

This problem is reduced to solving the boundary value problem for the singular parabolic equation:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial y^2}+y^{-1}\frac{\partial u}{\partial y}-\frac{p^2 u(t,\ x)}{y^2},$$

$$y=\ln x,\ y\in[A,\ B],\ t\in[0,\ T],\ A=\ln L,\ B=\ln H,$$

$$u_x'(t,\ A)=0,\ u_x'(t,\ B)=0,$$

$$u_x'(0,\ y)=\psi(e^{y(T)})=$$

$$=\max\left(\pm(x(T)-K),0\right)[\![_{(L<x(t)<H;\ t\in[0,T]}\,.$$

Taking into account all the considerations as to the solution of classical boundary value problems for the singular parabolic operator L, we have:

$$u(T,\ x)=\int_0^{\ln\frac{H}{L}}(e^{\xi}L-K)[\![_{(L<x(t)<H;\ t\in[0,T]}G(x,\xi)\,d\xi=$$

$$=\int_0^{\ln\frac{H}{L}}(e^{\xi}L-K)[\![_{(L<x(t)<H;\ t\in[0,T]),}$$

$$2\sum_{n=0}^{\infty}e^{-\left(\frac{\mu_n}{\ln\frac{H}{L}}\right)^2 t}J_p\left(\frac{\mu_n\xi}{\ln\frac{H}{L}}\right)J_p\left(\frac{\mu_n\ln\frac{x}{L}}{\ln\frac{H}{L}}\right)\times$$

$$\times\left(\left(\left(\ln\frac{H}{L}\right)^2-\frac{p^2}{\mu_k^2}\right)(J_p(\mu_k))^2\right)^{-1},$$

where is the $[\![_{(L<x(t)<H;\ t\in[0,T])}$ Heaviside step function.

Note. Since the roots of the Bessel functions of the first kind are simple, then between two neighboring roots of the Bessel functions there is a derivative root and we can assume that the roots of the derivative are distributed similarly to the roots of the functions. Thus for the Green's function and its first derivative the correct evaluation is

$$C\sum_{n=1}^{\infty} c_0 \ln^2 t < +\infty,\ \forall x \in [L, H],\ 0 < t < T,\ C > 0,\ c_0 > 0.$$

Approximate calculations do not require a large number of coefficients in a row because of the rapid convergence.

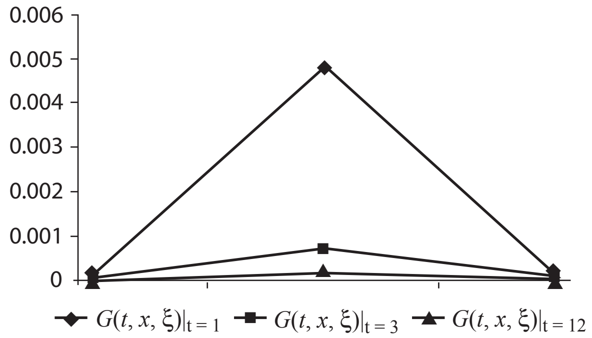

Fig. 1. Graph of the Green's function as the distribution density at *L* = 90, *H* = 120, ξ = 0,5

It should be noted that the Bessel diffusion is widely used in financial mathematics. Thus, the study considers the problem for the two-barrier option, which corresponds to the Bessel process.

CONCLUSIONS

Thus, as a result of this study, a Green's function for the Bessel diffusion process of a two-barrier option expanded in terms of Bessel functions of the first kind is built. As barriers there chosen the points at which the derivative of the financial flow by price is equal to zero, that is, the points where the flow can take extreme values. By means of the Green's function the value of derivative prices are calculated. Barriers of this kind are convenient for stock market monitoring. ■

LITERATURE

1. Cox, J. C., Ingersoll, J. E. and Ross, S. A. A theory of the term structure of interest rates. *Econometrica*. 1985. Vol. 53. P. 385–407.

2. Буртняк І. В., Малицька Г. П. Дослідження процесу Орнштейна – Уленбека методами спектрального аналізу. *Проблеми економіки*. 2014. № 2. С. 349–356.

3. Yor, M. On square-root boundaries for Bessel processes, and pole-seeking Brownian motion. *Stochastic Analysis and Applications* (Lecture Notes Math. 1095), eds. A. Truman and D. Williams. Berlin: Springer, 1984. P. 100–107.

4. Linetsky, V. The spectral representation of Bessel processes with constant drift: applications in queueing and finance. *Journal of Applied Probability*. 2004. Vol. 41. No. 2. P. 327–¬344.

5. Coffman, E. G., Puhalskii, A. A. and Reiman, M. I. Polling Systems in Heavy Traffic: A Bessel Process Limit. *Mathematics Operations Research*. 1998. Vol. 23. Issue 2. P. 257–304.

6. Буртняк І. В., Малицька Г. П. Системний підхід до оцінювання опціонів на базі моделі CEV. *Бізнес Інформ*. 2016. № 10. С. 155–159.

7. Lebedev, N. N. Special Functions and Their Applications. New York: Dover, 1972.

8. Владимиров В. С. Уравнения математической физики. М.: Наука, 1981. -512 с.

9. Mendoza-Arriaga, R., Carr, P. and V. Linetsky. Time-changed markov processes in unified credit-equity modeling. *Mathematical Finance*. 2010. Vol. 20. Issue 4. P. 527–569.

REFERENCES

Burtniak, I. V., and Malytska, H. P. "Doslidzhennia protsesu Ornshteina – Ulenbeka metodamy spektralnoho analizu" [Investigation of the Ornshtein-Uhlenbeck process by spectral analysis methods]. *Problemy ekonomiky*, no. 2 (2014): 349-356.

Burtniak, I. V., and Malytska, H. P. "Systemnyi pidkhid do otsiniuvannia optsioniv na bazi modeli CEV" [System approach to options assessment based on the CEV model]. *Biznes Inform*, no. 10 (2016): 155-159.

Coffman, E. G., Puhalskii, A. A., and Reiman, M. I. "Polling Systems in Heavy Traffic: A Bessel Process Limit". *Mathematics Operations Research*. Vol. 23, no. 2 (1998): 257-304.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. "A theory of the term structure of interest rates". *Econometrica*. Vol. 53 (1985): 385-407.

Linetsky, V. "The spectral representation of Bessel processes with constant drift: applications in queueing and finance". *Journal of Applied Probability*. Vol. 41, no. 2 (2004): 327-344.

Lebedev, N. N. *Special Functions and Their Applications*. New York: Dover, 1972.

Mendoza-Arriaga, R., Carr, P., and Linetsky, V. "Time-changed markov processes in unified credit-equity modeling". *Mathematical Finance*. Vol. 20, no. 4 (2010): 527-569.

Vladimirov, V. S. *Uravneniya matematicheskoy fiziki* [Equations of mathematical physics]. Moscow: Nauka, 1981.

Yor, M. "On square-root boundaries for Bessel processes, and pole-seeking Brownian motion". In *Stochastic Analysis and Applications (Lecture Notes Math. 1095)*, 100-107. Berlin: Springer, 1984.