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The Dirichlet problem for a Beltrami equation of the second type

Criteria on the existence of regular solutions of the Dirichlet problem for the degenerate Beltrami equation $\overline{\partial} f = \nu \overline{\partial f}$ in a Jordan domain of the complex plane $\mathbb{C}$ are given.

Let D be a domain in the complex plane $\mathbb{C}$. Throughout this paper, we use the notations $z = x + iy$, $B(z_0, r)\colon = \{z \in \mathbb{C}\colon |z - z_0| < r\}$ for $z_0 \in \mathbb{C}$ and $r > 0$, $\mathbb{B}(r)\colon = B(0, r)$, $\mathbb{B}\colon = \mathbb{B}(1)$, and $\overline{\mathbb{C}}\colon = \mathbb{C} \bigcup \infty$.

The purpose of this paper is to study the Dirichlet problem

$$\begin{cases} f_{\overline{z}} = \nu(z) \cdot \overline{f_z}, & z \in D, \\ \lim\limits_{z \to \zeta} \operatorname{Re} f(z) = \varphi(\zeta), & \forall\, \zeta \in \partial D, \end{cases} \tag{1}$$

in a Jordan domain D of the complex plane $\mathbb{C}$ with continuous boundary data $\varphi(\zeta)$. Here, $\nu(z)$ stands for a measurable coefficient satisfying the inequality $|\nu(z)| < 1$ a. e. in D. The degeneracy of the ellipticity for Beltrami equations of the second type $f_{\overline{z}} = \nu(z) \cdot \overline{f_z}$ is controlled by the dilatation coefficient

$$K_\nu(z) := \frac{1 + |\nu(z)|}{1 - |\nu(z)|} \in L^1_{\text{loc}}. \tag{2}$$

Note that the Beltrami equations of the second type take a key part in many problems of mathematical physics, see, e. g., [11].

We will look for a solution as a continuous, discrete, and open mapping $f\colon D \to \mathbb{C}$ of the Sobolev class $W^{1,1}_{\text{loc}}$ and such that the Jacobian $J_f(z) \neq 0$ a. e. in D. Such a solution will be called a *regular solution* of the Dirichlet problem (1) in a domain D. Recall that a mapping $f\colon D \to \mathbb{C}$ is called *discrete* if the preimage $f^{-1}(y)$ consists of isolated points for every $y \in \mathbb{C}$ and *open* if f maps every open set $U \subseteq D$ onto an open set in $\mathbb{C}$.

For the uniformly elliptic case, i. e. when $K_\nu(z) \leqslant K < \infty$ a. e. in D, the Dirichlet problem was studied in [1]. The solvability of the Dirichlet problem for the degenerate Beltrami equations of the first type

$$f_{\overline{z}} = \mu(z) \cdot f_z \tag{3}$$

was studied in [5, 8], and [12]. Recall that the problem of the existence of homeomorphic solutions for equation (3) was solved in the uniformly elliptic case where $\|\mu\|_\infty < 1$ long ago, see, e. g., [1].

The existence problem for the degenerate Beltrami equations (3), when $K_\mu \notin L^\infty$, is currently an active area of research, see, e. g., survey [7] and monograph [8] and references therein. A series of criteria on the existence of regular solutions for a Beltrami equation of the second type were given in our recent papers [3, 4]. There, a homeomorphism $f \in W^{1,1}_{\rm loc}(D)$ is called a *regular solution* of the equation if f satisfies the equation a. e. in D, and $J_f(z) = |f_z|^2 - |f_{\overline{z}}|^2 \neq 0$ a. e. in D.

To derive criteria of the existence of regular solutions for the Dirichlet problem (1) in a Jordan domain $D \subset \mathbb{C}$, we make use of the approximate procedure based on the existence theorems in the case $K_\nu \in L^\infty$ given in [1] and convergence theorems for the Beltrami equations of the second type when $K_\nu \in L^1_{\rm loc}$ established in [3]. The Schwarz formula

$$f(z) = i\,{\rm Im}\, f(0) + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} {\rm Re}\, f(\zeta) \cdot \frac{\zeta + z}{\zeta - z}\frac{d\zeta}{\zeta}, \tag{4}$$

that allows one to recover an analytic function f in the unit disk $\mathbb{B}$ by its real part $\varphi(\zeta) = {\rm Re} f(\zeta)$ on the boundary of $\mathbb{B}$ up to a purely imaginary additive constant $c = i{\rm Im} f(0)$, see, e. g., Section 8, Chapter III, Part 3 in [6], and the Arzela–Ascoli theorem combined with moduli techniques are also used.

2. On BMO, VMO, and FMO functions. Recall that a real-valued function u in a domain D in $\mathbb{C}$ is said to be a *bounded mean oscillation* in D, abbr. $u \in {\rm BMO}(D)$, if $u \in L^1_{\rm loc}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int\limits_B |u(z) - u_B|\, dxdy < \infty, \tag{5}$$

where the supremum is taken over all disks B in D and

$$u_B = \frac{1}{|B|} \int\limits_B u(z)\, dxdy.$$

We write $u \in {\rm BMO}_{\rm loc}(D)$ if $u \in {\rm BMO}(U)$ for every relatively compact subdomain U of D (we also write BMO or ${\rm BMO}_{\rm loc}$ if it is clear from the context what D is).

The class BMO was introduced by John and Nirenberg (1961) in [10] and soon became an important concept in harmonic analysis, the theory of partial differential equations, and related areas.

A function u in BMO is said to have a *vanishing mean oscillation*, abbr. $u \in$ VMO, if the supremum in (5) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [15]. There exist a number of papers devoted to the study of partial differential equations with coefficients of the class VMO.

Following [9], we say that a function $u\colon D \to \mathbb{R}$ has a *finite mean oscillation* at a point $z_0 \in D$ if

$$\overline{\lim_{\varepsilon \to 0}} \;\; \fint\limits_{B(z_0,\varepsilon)} |u(z) - \widetilde{u}_\varepsilon(z_0)|\, dxdy < \infty, \tag{6}$$

where $\widetilde{u}_\varepsilon(z_0)$ $\tilde{u}_\varepsilon(z_0) = \fint\limits_{B(z_0,\varepsilon)} u(z)\, dxdy$ is the mean value of the function $u(z)$ over the disk $B(z_0, \varepsilon)$ with small $\varepsilon > 0$. We also say that a function $u\colon D \to \mathbb{R}$ has a *finite mean oscillation* in D, abbr. $u \in$ FMO(D) or simply $u \in$ *FMO*, if relation (6) holds at every point $z_0 \in D$.

Remark 1. Clearly, BMO $\subset$ FMO. There exist the examples showing that FMO is not ${\rm BMO}_{\rm loc}$, see, e. g., [8]. By definition, FMO $\subset L^1_{\rm loc}$, but FMO is not a subset of $L^p_{\rm loc}$ for any $p > 1$ in comparison with ${\rm BMO}_{\rm loc} \subset L^p_{\rm loc}$ for all $p \in [1, \infty)$.

Proposition 1. *If, for some collection of numbers* $u_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} |u(z) - u_\varepsilon| \, dxdy < \infty, \tag{7}$$

then u *has a finite mean oscillation at* z_0.

Corollary 1. *If, for a point* $z_0 \in D$,

$$\overline{\lim_{\varepsilon\to 0}} \ \fint_{B(z_0,\varepsilon)} |u(z)| \, dxdy < \infty \tag{8}$$

then u *has a finite mean oscillation at* z_0.

Remark 2. Note that the function $u(z) = \log(1/|z|)$ belongs to BMO in the unit disk $\mathbb{B}$ and hence also to FMO. However, $\widetilde{u}_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, showing that condition (8) is only sufficient, but not necessary for a function u to have a finite mean oscillation at z_0.

Lemma 1. *Let* $u\colon D \to \mathbb{R}$ *be a nonnegative function with finite mean oscillation at* $0 \in D$, *and let* u *be integrable in* $B(0, e^{-1}) \subset D$. *Then*

$$\int_{A(\varepsilon, e^{-1})} \frac{u(z)\, dxdy}{\left(|z| \log \frac{1}{|z|}\right)^2} \leqslant C \cdot \log\log \frac{1}{\varepsilon}, \qquad \forall\, \varepsilon \in (0, e^{-e}). \tag{9}$$

Here, we use the notation $A(\varepsilon, \varepsilon_0) = \{z \in \mathbb{C}\colon \varepsilon < |z| < \varepsilon_0\}$.

3. The main lemma. The following lemma is the main tool for deriving criteria on the existence of regular solutions for the Dirichlt problem with degenerate Beltrami equations of the second type in a Jordan domain $D \subset \mathbb{C}$.

Lemma 2. *Let* D *be a Jordan domain in* $\mathbb{C}$ *with* $0 \in D$ *and let* $\nu\colon D \to \mathbb{C}$ *be a measurable function with* $K_\nu \in L^1(D)$. *Suppose that, for every* $z_0 \in \overline{D}$, *there exist* $\varepsilon_0 = \varepsilon(z_0) > 0$ *and a family of measurable functions* $\psi_{z_0,\varepsilon}\colon (0, \infty) \to (0, \infty)$, $\varepsilon \in (0, \varepsilon_0)$, *such that*

$$0 < I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\, dt < \infty, \tag{10}$$

and such that

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0} K_\nu(z) \cdot \psi^2_{z_0,\varepsilon}(|z - z_0|)\, dxdy = o(I^2_{z_0}(\varepsilon)) \tag{11}$$

as $\varepsilon \to 0$. *Then the Dirichlet problem* (1) *has a regular solution* f *with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi\colon \partial D \to \mathbb{R}$.

Here, we assume that ν is extended by zero outside the domain D.

Corollary 2. *Let* D *be a Jordan domain in* $\mathbb{C}$ *with* $0 \in D$ *and let* $\nu\colon \mathbb{B} \to \mathbb{C}$ *be a measurable function with* $K_\nu \in L^1(\mathbb{B})$. *Suppose that, for every* $z_0 \in \overline{\mathbb{B}}$ *and some* $\varepsilon_0 > 0$,

$$\int_{\varepsilon<|z-z_0|<\varepsilon_0} K_\nu(z) \cdot \psi^2(|z - z_0|)\, dxdy \leqslant O\left(\int_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt\right) \tag{12}$$

as $\varepsilon \to 0$, where $\psi\colon (0,\infty) \to (0,\infty)$ is a measurable function such that

$$\int\limits_0^{\varepsilon_0} \psi(t)\,dt = \infty, \qquad 0 < \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\,dt < \infty, \qquad \forall\,\varepsilon \in (0,\varepsilon_0). \tag{13}$$

Then the Dirichlet problem (1) *has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function $\varphi\colon \partial D \to \mathbb{R}$.*

4. Existence theorems. Everywhere further, we assume that the function $\nu\colon D \to \mathbb{C}$ is extended by zero outside the domain D. In particular, by Lemmas 1 and 2 with $\psi_{z_0,\varepsilon}(t) \equiv \equiv 1/(t\log 1/t)$, we have the following result.

Theorem 1. *Let D be a Jordan domain in $\mathbb{C}$ with $0 \in D$ and let $\nu\colon D \to \mathbb{B}$ be a measurable function such that $K_\nu(z) \leqslant Q(z) \in FMO$. Then the Dirichlet problem* (1) *has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function $\varphi\colon \partial D \to \mathbb{R}$.*

Now, Theorem 1 and Corollary 1 yield the following conclusion.

Corollary 3. *In particular, if*

$$\overline{\lim_{\varepsilon\to 0}} \ \ -\!\!\!\!\!\!\int\limits_{B(z_0,\varepsilon)} \frac{1+|\nu(z)|}{1-|\nu(z)|}\,dxdy < \infty, \qquad \forall\, z_0 \in \overline{D}, \tag{14}$$

then the Dirichlet problem (1) *in a Jordan domain D, $0 \in D$, has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function $\varphi\colon \partial D \to \mathbb{R}$.*

Similarly, choosing the function $\psi_{z_0,\varepsilon}(t) \equiv 1/t$, in Lemma 2, we come to the following statement.

Theorem 2. *Let D be a Jordan domain in $\mathbb{C}$ with $0 \in D$ and let $\nu\colon D \to \mathbb{B}$ be a measurable function such that $K_\nu \in L^1_{\rm loc}(D)$. Suppose that*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\nu(z)\frac{dxdy}{|z-z_0|^2} = o\biggl(\biggl[\log\frac{1}{\varepsilon}\biggr]^2\biggr), \qquad \forall\, z_0 \in \overline{D}, \tag{15}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \delta(z_0)$. Then the Dirichlet problem (1) *has a regular solution f with* $\operatorname{Im} f(0) = 0$ *for each nonconstant continuous function $\varphi\colon \partial D \to \mathbb{R}$.*

Remark 3. Choosing the function $\psi(t) = 1/(t\log 1/t)$ instead of $\psi(t) = 1/t$ in Lemma 2, we are able to replace (15) by

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K_\nu(z)\,dxdy}{\left(|z-z_0|\log\dfrac{1}{|z-z_0|}\right)^2} = o\biggl(\biggl[\log\log\frac{1}{\varepsilon}\biggr]^2\biggr). \tag{16}$$

In general, we are able to give here the whole scale of the corresponding conditions in log, using functions $\psi(t)$ of the form $1/(t\log 1/t \cdot \log\log 1/t \cdot \cdots \cdot \log\ldots\log 1/t)$.

Theorem 3. *Let D be a Jordan domain in $\mathbb{C}$ with $0 \in D$, let $\nu\colon D \to \mathbb{B}$ be a measurable function, $K_\nu \in L^1(D)$, and let $k_{z_0}(r)$ be the mean value of $K_\nu(z)$ over the circle $|z-z_0| = r$. Suppose that*

$$\int\limits_0^{\delta(z_0)} \frac{dr}{r k_{z_0}(r)} = \infty, \qquad \forall\, z_0 \in \overline{D}. \tag{17}$$

Then the Dirichlet problem (1) *has a regular solution* f *with* $\mathrm{Im} f(0) = 0$ *for each nonconstant continuous function* $\varphi\colon \partial D \to \mathbb{R}$.

Theorem 3 also follows from Lemma 2 by a special choice of the functional parameter $\psi_{z_0,\varepsilon}(t) \equiv 1/[tk_{z_0}(t)]$.

Corollary 4. *In particular, the conclusion of Theorem 3 holds if*

$$k_{z_0}(r) = O\left(\log\frac{1}{r}\right) \qquad as \qquad r \to 0, \qquad \forall\, z_0 \in \overline{D}. \tag{18}$$

In fact, it is clear that condition (17) yields the whole scale of conditions in terms of log with the use of functions of the form $\log 1/r \cdot \log\log 1/r \cdots \log \cdots \log 1/r$ on the right-hand side in (18).

By Theorem 3.1 in [13], see also Theorem 3.17 in [14], it follows that conditions (19) and (20) below yield condition (17). Thus, by Theorem 3, we obtain the following significant result.

Theorem 4. *Let* D *be a Jordan domain in* $\mathbb{C}$ *with* $0 \in D$, *and let* μ *and* $\nu\colon D \to \mathbb{B}$ *be measurable functions such that*

$$\int\limits_D \Phi(K_\nu(z))\, dxdy < \infty, \tag{19}$$

where $\Phi\colon [0,\infty] \to [0,\infty]$ *is a non-decreasing convex function satisfying the condition*

$$\int\limits_\delta^\infty \frac{d\tau}{\tau\Phi^{-1}(\tau)} = \infty \tag{20}$$

for some $\delta > \Phi(0)$. *Then the Dirichlet problem* (1) *has a regular solution* f *with* $\mathrm{Im}\, f(0) = 0$ *for each nonconstant continuous function* $\varphi\colon \partial D \to \mathbb{R}$.

Remark 4. Note also that we may assume in Theorem 4 that the function $\Phi(t)$ is not convex and non-decreasing on the whole segment $[0,\infty]$, but only on a segment $[T,\infty]$ for some $T \in \in (1,\infty)$.

Corollary 5. *In particular, the conclusion of Theorem 4 holds if, for some* $\alpha > 0$,

$$\int\limits_D e^{\alpha K_\nu(z)} dxdy < \infty. \tag{21}$$

Finally we would like to note that our approach makes it possible to study the Dirichlet problem for the degenerate Beltrami equations also in finitely connected domains of the complex plane.

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Задача Діріхле для рівняння Бельтрамі другого роду

Для виродженого рівняння Бельтрамі $\overline{\partial}f = \nu\overline{\partial f}$ доведено критерії існування регулярного розв'язку задачи Діріхле у довільній жордановій області комплексної площини $\mathbb{C}$.

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Задача Дирихле для уравнения Бельтрами второго рода

Для вырожденного уравнения Бельтрами $\overline{\partial}f = \nu\overline{\partial f}$ доказаны критерии существования регулярного решения задачи Дирихле в произвольной жордановой области комплексной плоскости $\mathbb{C}$.