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On new expanders of unbounded degree for practical applications in informatics

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A method of construction of new examples of families of expander graphs of unbounded degree is presented. The property of being an expander seems significant in many of these mathematical, computational, and physical contexts. Even more, expanders are surprisingly applicable in other computational aspects: in the theory of error correcting codes, computer networking theory, the theory of pseudorandomness, etc. We present the new families of $(q+1)$-regular graphs with the second largest eigenvalue of at most $2\sqrt{q}$ for every prime power q (geometrical Ramanujan graphs). In particular, we construct a family of new $(q+1)$-regular Ramanujan graphs of girth 6 of order $2(1+q+q^2+q^3)$. They are not isospectric to the geometry of the simple Lie group $B_2(q)$.

Introduction. The reader can find missing definitions of graph theory in [1]. We assume that all graphs under consideration are simple. The girth of a graph is a minimal length of its cycle. The spectrum of a graph is formed by eigenvalues of its adjacency matrix.

We say that a family of regular graphs of bounded degree q and increasing order n *has an expansion constant* c, $c > 0$, if the inequality $|\partial A| \geqslant c|A|$ holds for each subset A of the vertex set X, $|X| = n$, with $|A| \leqslant n/2$. The expansion constant of the family of q-regular graphs can be estimated via the upper limit $q - \lambda_n$, $n \to \infty$, where λ_n is the second largest eigenvalue of a family representative of order n.

According to the known Alon–Boppana theorem, the large enough members of an infinite family of d-regular graphs with constant d satisfy the inequality $\lambda \geqslant 2\sqrt{d-1} - o(1)$, where λ is the second largest eigenvalue in absolute value.

This result motivates the concept of a Ramanujan graph, which is a graph of degree d with the second largest eigenvalue bounded from above by $2\sqrt{d-1}$ (see [2]).

N. Alon [3] demonstrated the importance of families of expanding graphs of unbounded degree. This class contains families of graphs of unbounded degree d_i with the second largest λ_i such that the upper bound of the sequence λ_i/d_i is bounded away from 1. We refer to such families of graphs as geometrical expanders.

We consider families of graphs G_i of increasing degree k_i with the second largest eigenvalue λ_i such that the upper limit α of $\lambda_i/2\sqrt{k_i-1}$ is at most 1. We refer to such families of graphs as geometrical Ramanujan graphs.

The natural examples of geometrical Ramanujan graphs are known regular generalized m-gons ($m = 3, 4, 6$) and their affine parts. For instance, the finite projective plane $PG_2(q)$ over a finite field $\mathbb{F}_q$ (generalized 3-gon) has degree $q+1$ and the second largest eigenvalue $\sqrt{q}$. So, we can take the infinite family of classical edge transitive graphs $PG_2(p_i)$, where p is a fixed prime, and the integer i is unbounded, and form the family of geometrical Ramanujan graphs with $\alpha = 1/2$. Our conjecture is that the minimal constant α for the family of geometrical Ramanujan graphs is $\alpha' = 1/2$.

In fact, the known families of Ramanujan graphs of unbounded degree play an important role in the theory of finite geometries and have many practical applications, for example, in networking theory and the construction of a class of error-correcting codes (the so-called LDPC codes).

Root system and generalized polygons. One of the classes of small world bipartite graphs with additional geometric properties important for many practical applications is a class of regular generalized m-gons, i. e., regular tactical configurations of diameter m and girth $2m$. For each parameter m, a regular generalized m-gon has degree $q+1$ and order $2(1+q+\cdots+q^{m-1})$ [4].

The matrices $M_1 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $M_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, and $M_3 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ form a complete list of 2×2 Cartan matrices (see [5]). In [5], a lattice H with basis $\{\alpha_1,\alpha_2\}$, i. e., the set $\{\lambda_1\alpha_1+\lambda_2\alpha_2|\lambda_1,\lambda_2\in Z\}$, was considered. For an arbitrary two-dimensional matrix $A=(a_{i,j})$ from the list above, let us introduce two linear transformations r_1 and r_2 of the lattice H given by the formula

$$r_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i.$$

For M_k, $k=1,2,3$, we have $r_1^2=e$, $r_2^2=e$, and $(r_1r_2)^m=e$, where $m=3,4,6$ for $k=1,2,3$, accordingly. These conditions are generic relations for the Weyl group $W(M_k)$ corresponding to the 2×2 Cartan matrix M_k (see [6]). The set $\phi=\{g(\alpha_i)|g\in W(M_k), i=1,2\}$ is called a root system.

We use the concept of a *root system* ϕ, which is a configuration of vectors in a Euclidean space satisfying certain geometrical properties. Given a root system ϕ, we can always choose a set of positive roots $\phi^+=\phi\bigcap\{\lambda_1\alpha_1+\lambda_2\alpha_2|\lambda_i\geqslant 0, i=1,2\}$ ($|\phi^+|=|\phi^-|$, where ϕ^- a set of negative roots), in a fixed basis. For generalized regular m-gons, we have $\phi^+(M_1)=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2\}$, $\phi^+(M_2)=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2\}$, $\phi^+(M_2)=\{\alpha_1,\alpha_2,\alpha_1+\alpha_2,2\alpha_1+\alpha_2,3\alpha_1+\alpha_2,3\alpha_1+2\alpha_2\}$.

An element of ϕ^+ is called a *simple root* if it cannot be written as the sum of two elements of ϕ^+. α_1, α_2 are called simple roots. To determine remaining elements of the set ϕ^+, we use the linear operators r_1 and r_2. Originally in [7], α_1^*, α_2^* were linear functionals defined on H and given by the formula

$$\alpha_i^*(\alpha_j)=\begin{cases} 1, & i=j,\\ 0, & i\neq j,\end{cases}$$

$$\alpha_i^*(s\alpha_1+t\alpha_2)=s\alpha_i^*(\alpha_1)+t\alpha_i^*(\alpha_2),$$

where $s\alpha_1+t\alpha_2\in\phi^+$ is a positive root.

Let $\Delta=\varsigma\oplus L$ be the direct sum of ς and L, ς being the vector space of formal linear combinations $a\alpha_1^*+b\alpha_2^*$, where a and b are elements from $\mathbb{F}_q$. L is the set of all linear combinations of the form $\sum\limits_{\alpha\in\phi^+} t_\alpha\alpha$ and $t_\alpha\in\mathbb{F}_q$. λ, $\mu\in\phi^+$, and the bilinear product $\langle\cdot,\cdot\rangle\colon\Delta\times\Delta\to\Delta$ is defined as follows:

$$\langle a\lambda,b\mu\rangle=\begin{cases}(ab)(r+1)(\lambda+\mu), & \lambda+\mu\in\phi^+,\\ 0, & \lambda+\mu\notin\phi^+,\end{cases}$$

$$\langle\alpha_i^*,\alpha_j^*\rangle=0,\qquad \langle\alpha_i^*,\lambda\rangle=\alpha_i^*(\lambda)\lambda,\qquad \langle\lambda,\alpha_i^*\rangle=-\alpha_i^*(\lambda)\lambda.$$

Here, r is an integer uniquely determined by the condition $\mu-r\lambda\in\phi$, $\mu-(r+1)\lambda\notin\phi$.

The generalized polygon P_m ($m = 3, 4, 6$) can be embedded into Δ in a way such that $(p)I[l]$, $(p)I[l] \in P_m$ implies $\langle p, l \rangle = 0$ for sufficiently large char $\mathbb{F}_q$ (see [6]).

More general constructions [7, 8] allow one to get the interpretation of the geometries of Shevalley groups over finite fields of "sufficiently large characteristic" and some new incidence systems.

New constructions. Ramanujan graphs and expanders. To create graphs, which have interesting properties, we can use the root system and a special binary operation. We choose simple roots α_1, α_2 and then the remaining $n-2$ elements, by making up the n-element set ϕ_n^+. The third element is the sum of simple roots $\alpha_1 + \alpha_2$. The following elements can be obtained by a simple addition of the element α_1 or α_2 to the one of the already chosen non-simple roots. Obviously, there is only one 3-element set $\phi_3^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. The sets ϕ_4^+ consisting of four elements can be chosen by two ways: $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ and $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, but they are symmetric and give the same results. There are three ways to choose non-symmetric sets ϕ_5^+: $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 2\alpha_1 + 2\alpha_2\}$, $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\}$. We have $\phi_3^+ = \phi^+(M_1)$, $\phi_4^+ = \phi^+(M_2)$, but ϕ_5^+ cannot be obtained, by using linear transformations r_1, r_2 and any Cartan matrix. In this article, we only consider cases for $n = 3, 4, 5$. In our construction, we simplify the concept and define α_1^*, α_2^* as follows:

$$\alpha_i^*(\alpha_j) = \begin{cases} 1, & i{=}j, \\ 0, & i \neq j, \end{cases}$$

$$\alpha_i^*(s\alpha_1 + t\alpha_2) = 1\alpha_i^*(\alpha_1) + 1\alpha_i^*(\alpha_2).$$

Here, $s\alpha_1 + t\alpha_2 \in \phi^+$ is a positive root.

Let $\Delta = \varsigma \oplus L$ be the direct sum of ς and L as above, λ, $\mu \in \phi^+$, let a and b be elements from $\mathbb{F}_q$, and let us redefine the binary operation $\langle \cdot, \cdot \rangle \colon \Delta \times \Delta \to \Delta$:

$$\langle a\lambda, b\mu \rangle = \begin{cases} (ab)(\lambda + \mu), & \lambda + \mu \in \phi^+, \\ 0, & \lambda + \mu \notin \phi^+, \end{cases}$$

$$\langle \alpha_i^*, \alpha_j^* \rangle = 0, \qquad \langle \alpha_i^*, \lambda \rangle = \alpha_i^*(\lambda)\lambda, \qquad \langle \lambda, \alpha_i^* \rangle = -\alpha_i^*(\lambda)\lambda,$$

where $i, j = 1, 2$. We use the operators defined above to simplify our concept. Before the determination of incidence relations, we describe the set of vertices. Let $\Gamma(n, \phi_n^+, \mathbb{F}_q)$ denote a bipartite graph obtained, by using the n-element set ϕ_n^+, scalars from $\mathbb{F}_q$, and the binary operator $\langle \cdot, \cdot \rangle$. Traditionally in geometrical bipartite graphs, one set of vertices is called a set of points P, and another set of vertices is called a set of lines L.

First, let us consider an ordinary n-gon as a bipartite graph with the vertex set $V = P \bigcup L = \{(1), (2), \ldots, (n)\} \bigcup \{[1, 2], [2, 3], \ldots, [n-1, n], [n, 1]\}$. We can write the incidence relation I in an n-gon as follows:

$$(m)I[s, t] \Longleftrightarrow m = s \vee m = t.$$

The line is incident with a point if this point belongs to the line.

Let a vertex of the type t_i be defined as a vertex corresponding to the i-element subset of ϕ_n^+, $i = 0, 1, 2, \ldots, n-1$, and let A_i, B_i denote i-element closed subsets of ϕ_n^+. We create two ascending sequences of closed subsets of ϕ_n^+. The second element of the first sequence is $\{\alpha_1\}$, and the second element for the second sequence is $\{\alpha_2\}$:

$$A_0 = \{\varnothing\} \subset A_1 = \{\alpha_2\} \subset A_2 \subset A_3 \subset \cdots \subset A_{n-1} = \phi_n^+ \setminus \{\alpha_1\},$$

$$B_0 = \{\varnothing\} \subset B_1 = \{\alpha_1\} \subset B_2 \subset B_3 \subset \cdots \subset B_{n-1} = \phi_n^+ \setminus \{\alpha_2\}.$$

For bigger n, the set ϕ_n^+ has more roots, and the above sequences can be chosen in many ways. Now, choosing elements from these two sequences alternately, we create a set of points and a set of lines. For lines, we choose the sets $B_0 = \{\varnothing\}$, $A_1 = \{\alpha_2\}$, B_2, $A_3, \ldots, \phi_n^+ \setminus \{\alpha_j\}$. For points, we choose $A_0 = \{\varnothing\}$, $B_1 = \{\alpha_1\}$, A_2, $B_3, \ldots, \phi_n^+ \setminus \{\alpha_i\}$, where $i = 1$ and $j = 2$ if n is odd and $i = 2$ and $j = 1$ if n is even. The number of vertices in the obtained graph $\Gamma(n, \phi_n^+, \mathbb{F}_q)$ is $|V| = 2(1 + q + q^2 + \cdots + q^{n-1})$. The graph is bipartite $V = P \bigcup L$, and the set V consists of

2 elements of type t_0–$((1), \alpha_1^*)$ and $[[1,2], \alpha_2^*]$,

$2q$ elements of type t_1–$((2), \alpha_1^* + p_1\alpha_1)$ and $[[1,2], \alpha_2^* + l_1\alpha_2]$,

$2q^2$ elements of type t_2–$\Big((n), \alpha_2^* + \sum\limits_{\alpha \in A_2} p_\alpha \alpha\Big)$ and $\Big[[2,3], \alpha_1^* + \sum\limits_{\alpha \in B_2} l_\alpha \alpha\Big]$,

$\vdots$

$2q^{n-1}$ elements of type t_{m-1}–$\Big((\lceil (n+2)/2 \rceil) + \alpha_j^* + \sum\limits_{\alpha \in \phi_n^+ \setminus \{\alpha_i\}} p_\alpha \alpha\Big)$ and $\Big[[\lfloor (n+2)/2 \rfloor, \lfloor (n+$ $+ 4)/2 \rfloor]] + \alpha_i^* + \sum\limits_{\alpha \in \phi_n^+ \setminus \{\alpha_j\}} l_\alpha \alpha\Big]$, where $i = 1$ and $j = 2$ if n is odd, and $i = 2$ and $j = 1$ if n is even and p_1, l_1, p_α, $l_\alpha \in \mathbb{F}_q$.

Brackets and parentheses will allow the reader to distinguish points $(\cdot)$ and lines $[\cdot]$. The set of edges consists of all pairs $\{(p), [l]\}$, for which $(p) I_\Gamma [l]$. The incidence relation I_Γ in graphs $\Gamma(n, \phi_n^+, \mathbb{F}_q)$ is determined by using the operator $\langle \cdot, \cdot \rangle$.

The incidence relation for the graph $\Gamma(n, \phi^+, \mathbb{F}_q)$ is defined as follows. Let ψ_1 and ψ_2 be a closed subset of the set of positive roots ϕ^+, and let Σ_p and Σ_l be a linear combination of elements of the sets ψ_1 and ψ_2, accordingly, with scalars from $\mathbb{F}_q$. Point $(p) = ((m), \alpha_i^* + \Sigma_p)$ is incident to line $[l] = [[s,t], \alpha_j^* + \Sigma_l]$ (we denote it by $(p)I[l]$) if and only if

$$(m = s \vee m = t) \wedge (\langle \alpha_i^* + \Sigma_p, \alpha_j^* + \Sigma_l \rangle_{|\psi_1 \cap \psi_2} = 0).$$

This construction allows us to obtain new structures similar in some aspects to generalized regular polygons, but with different properties, in general. It is easy to see that this is a symmetric incidence relation (the graphs are simple). For $n = 3$, this construction yields a projective plane, which is commonly known. For $n = 4$, the set of roots is the same as for a generalized quadrangle, but we obtain two structures with different properties. For $n = 5$, the set ϕ_5^+ cannot be derived from the Cartan matrix, and we obtain over a dozen of new structures with different properties.

In Table 1, we present the incidence relations for graph $\Gamma(4, \phi_4^+, \mathbb{F}_q)$ when the sequences of closed sets are: $\{\alpha_1\} \subset \{\alpha_1, 2\alpha_1 + \alpha_2\} \subset \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2\}$, $\{\alpha_2\} \subset \{\alpha_2, \alpha_1 + \alpha_2\} \subset$ $\subset \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. For these sequences, we obtain better results than for the second possibility, $\lambda_1 = \sqrt{3q}$.

We checked every possibility to create the ascending sequences of closed subsets of ϕ_5^+. For some of them, we obtained $\lambda_1 = 2\sqrt{q}$. But, for $\Gamma(5, \{\alpha_1\}, \{\alpha_1 + \alpha_2\}, \{2\alpha_1 + \alpha_2\}, \{\alpha_1 + 2\alpha_2\}, \mathbb{F}_q)$ and

Table 1. Incidence relations for graph $\Gamma(4, \phi_4^+, \mathbb{F}_q)$

	$((1), \varnothing)$	$((2), p_1)$	$((4), p_1, p_2)$	$((3), p_1, p_2, p_3)$
$[[1,2], \varnothing]$	+	+	—	—
$[[4,1], l_1]$	+	—	$+: l_1 = p_1$	—
$[[2,3], l_1, l_2]$	—	+ $p_1 = l_1$	—	$+: p_1 = l_1$ $p_2 - l_2 = p_3 l_1$
$[[3,4], l_1, l_2, l_3]$	—	—	$+: p_1 = l_1$, $p_2 = l_2$	$+: l_2 - p_2 = p_1 l_1$, $l_3 - p_3 = p_1 l_2$

Table 2. Incidence relations for graph $\Gamma(5, \phi_5^+, \mathbb{F}_q)$

	$((1), \varnothing)$	$((2), p_1)$	$((5), p_1, p_2)$	$((3), p_1, p_2, p_3)$	$((4), p_1, p_2, p_3, p_4)$
$[[1,2], \varnothing]$	+	+	—	—	—
$[[1,5], l_1]$	+	—	$+: p_1 = l_1$	—	—
$[[2,3], l_1, l_2]$	—	+ $p_1 = l_1$	—	$+: p_1 = l_1$ $p_2 - l_2 = p_3 l_1$	—
$[[4,5], l_1, l_2, l_3]$	—	—	$+: p_1 = l_1$, $p_2 - l_2 = p_1 l_3$	—	$+: p_1 = l_1$, $p_2 - l_2 = p_1 l_3 + p_3 l_1$ $p_3 = l_3$
$[[3,4], l_1, l_2, l_3, l_4]$	—	—	—	$+: p_1 = l_1$, $p_2 - l_2 =$ $= p_1 l_3 + p_3 l_1$ $p_3 = l_3$	$+: p_2 - l_4 = l_3 p_1$, $p_3 - l_3 = l_1 p_1$ $p_4 - l_2 = l_1 p_3$

Table 3. Expanding properties of graphs $\Gamma(4, \phi^+, \mathbb{F}_q)$ and $\Gamma(4, \phi^+, \mathbb{Z}_{2r})$

Number field	$\lambda_0 = q + 1$	λ_1	$2\sqrt{q}$	Ring	$\lambda_0 = 2r + 1$	λ_1	$2\sqrt{2r}$
$\mathbb{F}_2$	3	2.2882	2.8284	$\mathbb{Z}_4$	5	4	4
$\mathbb{F}_3$	4	3	3.4641	$\mathbb{Z}_6$	7	6	4.899
$\mathbb{F}_4$	5	3.4641	4	$\mathbb{Z}_8$	9	8	5.6569
$\mathbb{F}_5$	6	3.8730	4.4721	$\mathbb{Z}_{10}$	11	10	6.3246
$\mathbb{F}_7$	8	4.5826	5.2915	$\mathbb{Z}_{12}$	13	12	6.9282
$\mathbb{F}_8$	9	4.899	5.6568	$\mathbb{Z}_{14}$	15	14	7.4833
$\mathbb{F}_9$	10	5.1962	6	$\mathbb{Z}_{16}$	17	16	8
$\mathbb{F}_{11}$	12	5.7446	6.6332	$\mathbb{Z}_{18}$	19	18	8.4853
$\mathbb{F}_{13}$	14	6.2450	7.2111	$\mathbb{Z}_{20}$	21	20	8.9443
$\mathbb{F}_{17}$	18	7.1414	8.2462	$\mathbb{Z}_{22}$	23	22	9.3808
$\mathbb{F}_{19}$	20	7.5498	8.7178	$\mathbb{Z}_{24}$	25	24	9.798
$\mathbb{F}_{23}$	24	8.3066	9.59	$\mathbb{Z}_{26}$	27	26	10.198

the sequences $\{\alpha_1\} \subset \{\alpha_1, 2\alpha_1+\alpha_2\} \subset \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2\} \subset \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$ and $\{\alpha_2\} \subset \{\alpha_2, \alpha_1 + 2\alpha_2\} \subset \{\alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2\} \subset \{\alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + 2\alpha_2\}$, the results are the best with $\lambda_1 < 2\sqrt{q}$. We present the incidence relations for these choices of sequences in Table 2. Table 3 and Table 4 contain results for the second largest eigenvalue λ_1 for the graphs described in Tables 1 and 2, accordingly. In addition, we consider the case where we use a ring $\mathbb{Z}_{2r}$ ($r \in \mathbb{Z}_+$) and modulo operations instead of the number field $\mathbb{F}_q$. Note that the vertex $((m), \alpha_i^* + p_1\alpha_i + p_2\alpha_3 + p_3\alpha_4)$, where α_i is a simple root and α_3, α_4 are nonsimple roots, can be identified with $((m), p_1, p_2, p_3)$ to simplify the notation of incidence relations.

We announce the following statement.

Theorem 1. *Graphs $\Gamma(4, \phi^+, \mathbb{F}_q)$ for an arbitrary prime power q have girth 6. The incidence graph of the geometry of a simple group of the Lie type $B_2(q)$ is not isospectric to $\Gamma(4, \phi^+, \mathbb{F}_q)$.*

Table 4. Expanding properties of graphs $\Gamma(5, \phi^+, \mathbb{F}_q)$

Number field	regularity $q+1$ first eigenvalue	second eigenvalue	$2\sqrt{q}$
$\mathbb{F}_2$	3	2.4495	2.8284
$\mathbb{F}_3$	4	3.2004	3.4641
$\mathbb{F}_4$	5	4	4
$\mathbb{F}_5$	6	4.1317	4.4721
$\mathbb{F}_7$	8	4.8887	5.2915
$\mathbb{F}_8$	9	5.6569	5.6569
$\mathbb{F}_9$	10	5.5433	6
$\mathbb{F}_{11}$	12	6.1283	6.6332

Theorem 2. *If $\phi_5^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$ and the sequences of closed sets are $\{\alpha_1\} \subset \{\alpha_1, 2\alpha_1+\alpha_2\} \subset \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2\} \subset \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$, $\{\alpha_2\} \subset \{\alpha_2, \alpha_1+2\alpha_2\} \subset \{\alpha_2, \alpha_1+2\alpha_2, \alpha_1+\alpha_2\} \subset \{\alpha_2, \alpha_1+2\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+2\alpha_2\}$, then graphs $\Gamma(5, \phi^+, \mathbb{F}_q)$ have girth 8 for an arbitrary prime power q.*

Based on the results contained in Tables 3 and 4, we formulate the following conjectures.

Conjecture 1. *Graphs $\Gamma(4, \phi^+, \mathbb{F}_q)$ for an arbitrary prime power $q > 2$ are $(q+1)$-regular Ramanujan graphs with $\lambda_1 = \sqrt{3q}$.*

Conjecture 2. *Graphs $\Gamma(4, \phi^+, \mathbb{Z}_{2r})$ for arbitrary $r \in \mathbb{Z}_+$ are $(2r+1)$-regular expander graphs with constant spectral gap $|2r+1-\lambda_1| = 1$.*

Conjecture 3. *If $\phi_5^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$, and the sequences of closed sets are $\{\alpha_1\} \subset \{\alpha_1, 2\alpha_1+\alpha_2\} \subset \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2\} \subset \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$, $\{\alpha_2\} \subset \{\alpha_2, \alpha_1+2\alpha_2\} \subset \{\alpha_2, \alpha_1+2\alpha_2, \alpha_1+\alpha_2\} \subset \{\alpha_2, \alpha_1+2\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+2\alpha_2\}$, then, for an arbitrary prime power q, graphs $\Gamma(5, \phi^+, \mathbb{F}_q)$ are $(q+1)$-regular Ramanujan graphs, and $\lambda_1 \leqslant 2\sqrt{q}$.*

4. On the comparison with Moore graphs of girth 8. Let $v(2m, k+1)$ be the minimal order of a $(k+1)$-regular graph of girth $2m$. Then $v(2m, k+1) \geqslant 2(1+k+k^2+\cdots+k^{m-1})$ (see [9]).

In fact, the graphs $\Gamma(4, \phi^+, F_q)$ have some similarity with Moore cages of girth 8 and degree $q+1$. Really, $|V(\Gamma(4, \phi^+, F_q))| = 2(1+q+q^2+q^3) = v(q+1, 8)$.

As is known, the cages with such parameters are regular generalized 4-gons of degree $q+$ $+1$. They are Ramanujan graphs. We produce another family of Ramanujan graphs of order $v(q+1, 8)$. Girth 6 indicates that our graphs are not isomorphic to Moore graphs. The tables demonstrate that the graphs from these two families of the same order and degree are not isospectric.

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Про нові експандери необмеженого степеня для практичного застосування в інформатиці

Розглянуто метод побудови нових прикладів родин графів-експандерів необмеженого степеня. Графи з властивістю експансії пов'язані з багатьма концепціями чистої математики, теорії обчислень та фізики. Крім того, експандери застосовуються в різних напрямках інформатики: теорії кодування, теорії мереж, теорії псевдовипадкових процесів і т. д. Наведено приклади сімейств $(q+1)$*-регулярних графів таких, що їх друге власне число не перевищує подвоєного кореня квадратного з* q *(родин геометричних графів Рамануджана). Зокрема, побудовано родину нових* $(q+1)$*-регулярних графів Рамануджана обхвату 6 порядку* $2(1+q+q^2+q^3)$*, але вони не є ізоспектральними до геометрій простих груп типу Лі* $B_2(q)$*.*

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О новых экспандерах неограниченной степени для практического применения в информатике

Представлен метод построения новых примеров семейств графов-экспандеров неограниченной степени. Графы со свойством экспансии связаны с многими концепциями в чистой математике, теории вычислений и физики. Кроме того, экспандеры применяются в различных направлениях информатики: теории кодирования, теории сетей, теории псевдослучайных процессов и т. д. Приведены примеры семейств $(q+1)$*-регулярных графов с вторым собственным значением, не превышающим удвоенного корня квадратного из* q *(семейств геометрических графов Рамануджана). В частности, построено семейство новых* $(q+1)$*-регулярных графов Рамануджана обхвата 6 порядка* $2(1+q+q^2+q^3)$*, но они не изоспектральны геометриям простых групп типа Ли* $B_2(q)$*.*