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A new hybrid method for solving variational inequalities

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We introduce a new method for solving variational inequalities with monotone and Lipschitz-continuous operators acting in a Hilbert space. The iterative process based on the well-known projection method and the hybrid (or outer approximations) method. However, we do not use an extrapolation step in the projection method. The absence of one projection in our method is explained by a slightly different choice of sets in the hybrid method. We prove the strong convergence of the sequences generated by our method.

Introduction. Variational inequality theory is an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems arising in several branches of pure and applied sciences in a unified general framework. This field is dynamical and is experiencing an explosive growth in both theory and applications. Several numerical methods have been developed for solving variational inequalities and related optimization problems (see [1, 2] and references therein).

We consider the classical variational inequality problem, which is to find a point $x^* \in C$ such that

$$(Ax^*, x - x^*) \geqslant 0 \qquad \forall\, x \in C, \tag{1}$$

where C is a closed convex set in the Hilbert space H, $(\cdot,\cdot)$ denotes the inner product in H, and $A\colon H \to H$ is some mapping. We assume that the following conditions hold:

[(C1)] The solution set of (1), denoted by S, is nonempty.

[(C2)] The mapping A is monotone on C, i. e., $(Ax - Ay, x - y) \geqslant 0\ \forall\, x, y \in C$.

[(C3)] The mapping A is Lipschitz-continuous on C with constant $L > 0$, i. e., there exists $L > 0$ such that $\|Ax - Ay\| \leqslant L\|x - y\|\ \forall\, x, y \in C$.

In order to construct an algorithm which provides the strong convergence to a solution of (1), we propose the following method:

$$\begin{cases} x_0, z_0 \in C, \\ z_{n+1} = P_C(x_n - \lambda A z_n), \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{2}$$

Here, P_M denotes the metric projection on the set M, $\lambda \in (0, 1/L)$, and the sets C_n and Q_n are some half-spaces which will be defined in what follows.

The oldest algorithm that provides the convergence of a generated sequence under the above assumptions is the extragradient method proposed by G. M. Korpelevich in [3]. At present, there exist many efficient modifications of the extragradient method [4–9]. The natural question that arises in the case of an infinite-dimensional Hilbert space is how to construct a modified

Korpelevich's extragradient algorithm, which will provide the strong convergence. To answer this question, Nadezhkina and Takahashi [5] introduced the following method:

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = P_C(x_n - \lambda_n A y_n), \\ C_n = \{w \in C \colon \|z_n - w\| \leqslant \|x_n - w\|\}, \\ Q_n = \{w \in C \colon (x_n - w, x_0 - x_n) \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \end{cases} \tag{3}$$

where $\lambda_n \in [a, b] \subseteq (0, 1/L)$. Under the above assumptions (C1)–(C3), they proved that the sequence (x_n) generated by (3) converges strongly to $P_S x_0$. Their method is based on the extragradient method and on the hybrid method proposed in [10]. The computational complexity of (3) on every step is three computations of a metric projection and two computations of A. Inspired by this scheme, Censor, Gibali, and Reich [11, 12] presented the following algorithm:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda A x_n), \\ T_n = \{w \in H \colon (x_n - \lambda A x_n - y_n, w - y_n) \leqslant 0\}, \\ z_n = \alpha_n x_n + (1 - \alpha_n) P_{T_n}(x_n - \lambda A y_n), \\ C_n = \{w \in H \colon \|z_n - w\| \leqslant \|x_n - w\|\}, \\ Q_n = \{w \in H \colon (x_n - w, x_0 - x_n) \geqslant 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{4}$$

In contrast to (3), the sets C_n and Q_n are half-spaces. Hence, it is much more simplier to calculate $P_{C_n \bigcap Q_n} x_0$ than that on the general convex set C. Therefore, we will not take into consideration this projection in the next schemes. On the second step, only the projection onto the half-space T_n, rather than onto the set C like in (3), is calculated. However, on every step of (4), we need to calculate A at two points, as well as in (3).

In this work, we show that, with some other choice of sets C_n, it is possible to throw out the step of extrapolation in (3) or in (4), which consists in $y_n = P_C(x_n - \lambda A x_n)$. It is easy to see that our method (2) on every iteration needs only one computation of the projection (as in (4)) and only one computation of A.

Preliminaries. In order to prove our main result, we need the following statements (see [2]). At first, the following well-known properties of the projection mapping will be used throughout this paper.

Lemma 1. *Let M be a nonempty closed convex set in H, $x \in H$. Then*
i) $(P_M x - x, y - P_M x) \geqslant 0 \ \forall y \in M$;
ii) $\|P_M x - y\|^2 \leqslant \|x - y\|^2 - \|x - P_M x\|^2 \ \forall y \in M$.

Two next lemmas are also well-known.

Lemma 2. *Assume that $A\colon C \to H$ is a continuous and monotone mapping. Then x^* is a solution of* (1) *iff x^* is a solution of the following problem:*

$$\text{find}\, x \in C, \qquad \textit{such that} \qquad (Ay, y - x) \geqslant 0 \qquad \forall\, y \in C.$$

Remark 1. The solution set S of the variational inequality (1) is closed and convex.

We write $x_n \rightharpoonup x$ to indicate that the sequence (x_n) converges weakly to x, and $x_n \to x$ implies that (x_n) converges strongly to x.

Lemma 3 (Kadec–Klee property of a Hilbert space). *Let (x_n) be a sequence in H. Then it follows from $\|x_n\| \to \|x\|$ and $x_n \rightharpoonup x$ that $x_n \to x$.*

At last, we need the following result.

Lemma 4. *Let (a_n), (b_n), (c_n) be nonnegative real sequences, α, $\beta \in \mathbb{R}$, and let, for all $n \in \mathbb{N}$, the inequality $a_n \leqslant b_n - \alpha c_{n+1} + \beta c_n$ hold. If $\sum\limits_{n=1}^{\infty} b_n < +\infty$ and $\alpha > \beta \geqslant 0$, then $\lim\limits_{n\to\infty} a_n = 0$.*

3. Algorithm and its convergence. We now formally state our algorithm.

Algorithm 1 (Hybrid algorithm without extrapolation step).

1. Choose x_0, $z_0 \in C$ and two parameters $k > 0$ and $\lambda > 0$.
2. Given the current iterate x_n and z_n, compute

$$z_{n+1} = P_C(x_n - \lambda A z_n). \tag{5}$$

If $z_{n+1} = x_n = z_n$, then stop. Otherwise, construct sets C_n and Q_n as

$$\begin{aligned}
&C_0 = H,\\
&C_n = \Big\{ w \in H\colon \|z_{n+1} - w\|^2 \leqslant \|x_n - w\|^2 + k\|x_n - x_{n-1}\|^2 -\\
&\qquad - \left(1 - \frac{1}{k} - \lambda L\right)\|z_{n+1} - z_n\|^2 + \lambda L\|z_n - z_{n-1}\|^2 \Big\}, \qquad n \geqslant 1,\\
&Q_0 = H,\\
&Q_n = \{ w \in H\colon (x_n - w, x_0 - x_n) \geqslant 0\}, \qquad n \geqslant 1,
\end{aligned} \tag{6}$$

and calculate

$$x_{n+1} = P_{C_n \cap Q_n} x_0.$$

3. Set $n \leftarrow n + 1$ and return to step 2.

We remark that the sets C_n look like slightly complicated in contrast to (4). However, it is only for superficial examination; for a computation, it does not matter. In (6) and in (3), both C_n are some half-spaces.

First, we note that the stopping criterion in Algorithm 1 is valid.

Lemma 5. *If $z_{n+1} = x_n = z_n$ in Algorithm 1, then $x_n \in S$.*

The next lemma is central to our proof of the convergence theorem.

Lemma 6. *Let (x_n) and (z_n) be two sequences generated by Algorithm 1, and let $z \in S$. Then*

$$\|z_{n+1} - z\|^2 \leqslant \|x_n - z\|^2 + k\|x_n - x_{n-1}\|^2 - \left(1 - \frac{1}{k} - \lambda L\right)\|z_{n+1} - z_n\|^2 + \lambda L\|z_n - z_{n-1}\|^2.$$

Proof. By Lemma 1 we have

$$\|z_{n+1}-z\|^2 \leqslant \|x_n-\lambda Az_n-z\|^2-\|x_n-\lambda Az_n-z_{n+1}\|^2 = \\ = \|x_n-z\|^2-\|x_n-z_{n+1}\|^2-2\lambda(Az_n,z_{n+1}-z). \tag{7}$$

Since A is monotone and $z\in S$, we see that $(Az_n,z_n-z)\geqslant 0$. Thus, adding $2\lambda(Az_n,z_n-z)$ to the right-hand side of (7), we get

$$\|z_{n+1}-z\|^2 \leqslant \|x_n-z\|^2-\|x_n-z_{n+1}\|^2-2\lambda(Az_n,z_{n+1}-z_n) = \\ = \|x_n-z\|^2-\|x_n-x_{n-1}\|^2-2(x_n-x_{n-1},x_{n-1}-z_{n+1})- \\ -\|x_{n-1}-z_{n+1}\|^2-2\lambda(Az_n,z_{n+1}-z_n)=\|x_n-z\|^2-\|x_n-x_{n-1}\|^2- \\ -2(x_n-x_{n-1},x_{n-1}-z_{n+1})-\|x_{n-1}-z_n\|^2-\|z_n-z_{n+1}\|^2- \\ -2\lambda(Az_n-Az_{n-1},z_{n+1}-z_n)+2(x_{n-1}-\lambda Az_{n-1}-z_n,z_{n+1}-z_n). \tag{8}$$

As $z_n=P_C(x_{n-1}-\lambda Az_{n-1})$ and $z_{n+1}\in C$, we have

$$(x_{n-1}-\lambda Az_{n-1}-z_n,z_{n+1}-z_n)\leqslant 0. \tag{9}$$

Using the triangle, Cauchy–Schwarz, and the Cauchy inequalities, we obtain

$$2(x_n-x_{n-1},x_{n-1}-z_{n+1})\leqslant \\ \leqslant \|x_n-x_{n-1}\|^2+\|x_{n-1}-z_n\|^2+k\|x_n-x_{n-1}\|^2+\frac{1}{k}\|z_{n+1}-z_n\|^2. \tag{10}$$

Since A is Lipschitz-continuous, we get

$$2\lambda(Az_n-Az_{n-1},z_{n+1}-z_n)\leqslant 2\lambda L\|z_n-z_{n-1}\|\|z_{n+1}-z_n\|\leqslant \\ \leqslant \lambda L(\|z_{n+1}-z_n\|^2+\|z_n-z_{n-1}\|^2). \tag{11}$$

Combining inequalities (8)–(11), we see that

$$\|z_{n+1}-z\|^2\leqslant\|x_n-z\|^2+k\|x_n-x_{n-1}\|^2-\left(1-\frac{1}{k}-\lambda L\right)\|z_{n+1}-z_n\|^2+\lambda L\|z_n-z_{n-1}\|^2,$$

which completes the proof.

We now can state and prove our main convergence result.

Theorem 1. *Assume that (C1)–(C3) hold, and let $\lambda\in(0,1/(2L))$, $k>1/(1-2\lambda L)$. Then the sequences (x_n) and (z_n) generated by Algorithm 1 converge strongly to P_Sx_0.*

Proof. It is evident that the sets C_n and Q_n are closed and convex. By Lemma 6, we have that $S\subseteq C_n$ for all $n\in\mathbb{Z}^+$. Let us show by induction that $S\subseteq Q_n$ for all $n\in\mathbb{Z}^+$. For $n=0$, we have $Q_0=H$. Suppose that $S\subseteq Q_n$. It is sufficient to show that $S\subseteq Q_{n+1}$. Since $x_{n+1}=P_{C_n\cap Q_n}x_0$ and $S\subseteq C_n\bigcap Q_n$, it follows that $(x_{n+1}-z,x_0-x_{n+1})\geqslant 0\ \forall z\in S$. From this by the definition of Q_n, we conclude that $z\in Q_{n+1}\ \forall z\in S$. Thus, $S\subseteq Q_{n+1}$ and, hence, $S\subseteq C_n\bigcap Q_n$ for all $n\in\mathbb{Z}^+$. For this reason, the sequence (x_n) is defined correctly. Let $\bar{x}=P_Sx_0$.

Since $x_{n+1} \in C_n \bigcap Q_n$ and $\bar{x} \in S \subseteq C_n \bigcap Q_n$, we have $\|x_{n+1} - x_0\| \leqslant \|\bar{x} - x_0\|$. Therefore, (x_n) is bounded. From $x_{n+1} \in C_n \bigcap Q_n \subseteq Q_n$ and $x_n = P_{Q_n} x_0$, we obtain

$$\|x_n - x_0\| \leqslant \|x_{n+1} - x_0\|. \tag{12}$$

Hence, there exists $\lim\limits_{n\to\infty} \|x_n - x_0\|$. In addition, since $x_n = P_{Q_n} x$ and $x_{n+1} \in Q_n$, Lemma 1 yields

$$\|x_{n+1} - x_n\|^2 \leqslant \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \tag{13}$$

From this, it may be concluded that the series $\sum\limits_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent. In fact, relations (13) and (12) yield

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 \leqslant \|\bar{x} - x_0\|^2 - \|x_1 - x_0\|^2 < +\infty.$$

Since $x_{n+1} \in C_n$, we see that

$$\|z_{n+1} - x_{n+1}\|^2 \leqslant \|x_{n+1} - x_n\|^2 + k\|x_n - x_{n-1}\|^2 - \left(1 - \frac{1}{k} - \lambda L\right)\|z_{n+1} - z_n\|^2 + \\ + \lambda L \|z_n - z_{n-1}\|^2.$$

Set $a_n = \|z_{n+1} - x_{n+1}\|^2$, $b_n = \|x_{n+1} - x_n\|^2 + k\|x_n - x_{n-1}\|^2$, $c_n = \|z_n - z_{n-1}\|^2$, $\alpha = (1 - (1/k) - L)$, $\beta = \lambda L$. By Lemma 6, since $\sum\limits_{n=1}^{\infty} b_n < +\infty$ and $\alpha > \beta$,

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

For this reason, (z_n) is bounded, and

$$\|z_{n+1} - z_n\| \leqslant \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\| \to 0.$$

As (x_n) is bounded, there exists a subsequence (x_{n_i}) of (x_n) such that (x_{n_i}) converges weakly to some $x^* \in H$. We will show that $x^* \in S$. It follows from (5) by Lemma 1 that

$$(z_{n_i+1} - x_{n_i} + \lambda A z_{n_i}, y - z_{n_i+1}) \geqslant 0 \qquad \forall\, y \in C.$$

This is equivalent to

$$0 \leqslant (z_{n_i+1} - z_{n_i} + z_{n_i} - x_{n_i}, y - z_{n_i+1}) + \lambda(Az_{n_i}, y - z_{n_i}) + \lambda(Az_{n_i}, z_{n_i} - z_{n_i+1}) \leqslant \\ \leqslant (z_{n_i+1} - z_{n_i}, y - z_{n_i+1}) + (z_{n_i} - x_{n_i}, y - z_{n_i+1})\lambda(Ay, y - z_{n_i}) + \\ + \lambda(Az_{n_i}, z_{n_i} - z_{n_i+1}) \qquad \forall\, y \in C. \tag{14}$$

In the last inequality, we used the monotonicity of A. Taking the limit in (14) as $i \to \infty$ and using $z_{n_i} \rightharpoonup x^* \in C$, we obtain $0 \leqslant (Ay, y - x^*)$ $\forall\, y \in C$. In view of Lemma 2, this implies that $x^* \in S$. Let us show $x_{n_i} \to x^*$. From $\bar{x} = P_S x_0$ and $x^* \in S$, it follows that $\|\bar{x} - x_0\| \leqslant \|x^* - x_0\| \leqslant \liminf\limits_{i\to\infty} \|x_{n_i} - x_0\| \leqslant \|\bar{x} - x_0\|$. Thus, $\lim\limits_{i\to\infty} \|x_{n_i} - x_0\| = \|x^* - x_0\|$. From this and $x_{n_i} - x_0 \rightharpoonup x^* - x_0$ by Lemma 3, we can conclude that $x_{n_i} - x_0 \to x^* - x_0$. Therefore, $x_{n_i} \to x^*$. Next, we have

$$\|x_{n_i} - \bar{x}\|^2 = (x_{n_i} - x_0, x_{n_i} - \bar{x}) + (x_0 - \bar{x}, x_{n_i} - \bar{x}) \leqslant (x_0 - \bar{x}, x_{n_i} - \bar{x}).$$

As $i \to \infty$, we obtain $\|x^* - \bar{x}\|^2 \leqslant (x_0 - \bar{x}, x^* - \bar{x}) \leqslant 0$. Hence, we have $x^* = \bar{x}$. Since the subsequence (x_{n_i}) was arbitrary, we see that $x_n \to \bar{x}$. It is clear that $z_n \to \bar{x}$.

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Новий гібридний метод для розв'язання варіаційних нерівностей

Запропоновано новий гібридний метод для розв'язання варіаційних нерівностей з монотонними і ліпшицевими операторами, що діють у гільбертовому просторі. Ітераційний процес базується на двох добре відомих методах: проективному та гібридному (або зовнішніх апроксимацій). Причому не використовується екстраполяційний крок у проективному методі. Відсутність однієї проекції досягається шляхом іншого вибору наборів множин у гібридному методі. Доведено сильну збіжність породжених методом послідовностей.

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Новый гибридный метод для решения вариационных неравенств

Предложен новый гибридный метод для решения вариационных неравенств с монотонными и липшицевыми операторами, действующими в гильбертовом пространстве. Итерационный процесс основан на двух хорошо известных методах: проективном и гибридном (или внешних аппроксимаций). Причем не используется экстраполяционный шаг в проективном методе. Отсутствие одной проекции достигается путем иного выбора наборов множеств в гибридном методе. Доказана сильная сходимость порожденных методом последовательностей.