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The damped sloshing in an upright circular tank due to an orbital forcing

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The nonlinear Narimanov—Moiseev-type modal system with linear damping terms is employed to study the damped steady-state resonant sloshing in an upright circular tank due to a prescribed horizontal orbital (elliptic) tank motion with the forcing frequency close to the lowest natural sloshing frequency. Whereas the undamped sloshing implies coexisting the co-directed (with forcing) and counter-directed angular progressive waves (swirling), the damping makes the counter-directed swirling impossible as the forcing orbit tends to a circle.

Keywords: *sloshing, damping, steady-state waves.*

An upright circular cylindrical rigid tank performs a small-magnitude prescribed periodic horizontal motion, which is described by the two generalized coordinates $r_0\eta_1(t)$ and $r_0\eta_2(t)$ (r_0 is the tank radius) as shown in fig. 1. Those tank motions are relevant for bioreactors [1]. In contrast to industrial containers whose dimensions are relatively large, the bioreactors have $r_0 \approx 5-10$ [cm] that requires accounting for the damping associated with a laminar boundary layer and the bulk viscosity.

The problem is studied in the nondimensional statement provided by the characteristic size r_0 and time $1/\sigma$, where σ is the forcing frequency close to the lowest natural sloshing frequency σ_{11}. The nondimensional forcing magnitude is small, i.e. $\eta_i(t) = O(\varepsilon), i = 1, 2$. Fig. 1 illustrates the adopted nomenclature. The unknowns, ς and Φ (the velocity potential), are defined in the tank-fixed coordinate system and can be found from either the corresponding free-surface problem or its equivalent variational formulation. Using the Fourier-type representation (in the cylindrical coordinates)

$$\varsigma(r,\theta,t) = \sum_{M,i}^{\infty} J_M(k_{Mi}r)\cos(M\theta)\, p_{Mi}(t) + \sum_{m,i}^{\infty} J_m(k_{mi}r)\sin(m\theta)\, r_{mi}(t) \qquad (1)$$

makes it possible to derive an approximate system of ordinary differential equations (nonlinear modal equations [2]) with respect to the free-surface generalized coordinates $p_{Mi}(t)$

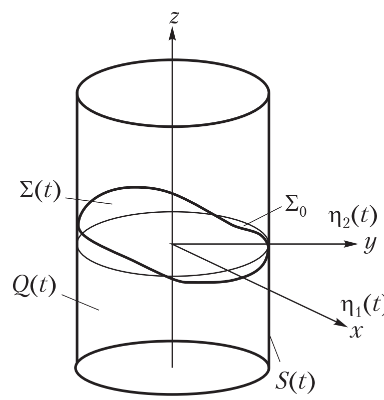

Fig. 1. The domain $Q(t)$ is confined by the free surface $\Sigma(t)$ ($z = \varsigma(r, \theta, t)$) and the wetted tank surface $S(t)$. Sloshing is considered in the tank-fixed coordinate system $Oxyz$ whose coordinate plane Oxy coincides with the mean (hydrostatic) free surface Σ_0; Oz is the symmetry axis. Small-magnitude periodic tank excitations are governed by generalized coordinates $\eta_1(t)$ (surge) and $\eta_2(t)$ (sway)

and $r_{mi}(t)$; here, $J_M(\cdot)$ is the Bessel functions of the first kind, k_{Mi} are the radial wave numbers ($J'_M(k_{Mi}) = 0$), and $\sigma_{Mi} = \sqrt{k_{Mi}\tanh(k_{Mi}h)g / r_0}$ are the dimensional natural sloshing frequencies (g is the gravity acceleration).

Furthermore, the nonlinear Narimanov—Moiseev-type modal system [2] (the infinite-dimensional system of ordinary differential equations with respect to $p_{Mi}(t)$ and $r_{mi}(t)$) is equipped with the linear damping terms $2\xi_{Mi}\bar{\sigma}_{Mi}\dot{p}_{Mi}$ and $2\xi_{Mi}\bar{\sigma}_{Mi}\dot{r}_{Mi}$, where the damping coefficients ξ_{Mi} are taken according to the formula by Miles [3], which provides a rather accurate theoretical prediction of the logarithmic decrements of the natural sloshing modes due to the boundary layer and the bulk viscosity. The 2π-periodic solutions of the modified modal system describe the resonant steady-state sloshing. To find the asymptotic steady-state solutions, we use the Bubnov—Galerkin procedure [2, 4] by posing the lowest-order components of the primary resonantly excited modes as

$$p_{11}(t) = a\cos t + \bar{a}\sin t + O(\varepsilon),\; r_{11}(t) = \bar{b}\cos t + b\sin t + O(\varepsilon), \tag{2}$$

where the nondimensional amplitudes a, $\bar{a}$, $\bar{b}$, and b are of $O(\varepsilon^{1/3})$. Having known these amplitudes approximates the steady-state free-surface elevations as the superposition of the two out-of-phase angular modes

$$\varsigma(r,\theta,t) = J_1(k_{11}r)[(a\cos\theta + \bar{b}\sin\theta)\cos t + (\bar{a}\cos\theta + b\sin\theta)\sin t] + O(\varepsilon^{1/3}), \tag{3}$$

which implies the so-called swirling (angular progressive wave) unless $(a\cos\theta + \bar{b}\sin\theta)$ and $(\bar{a}\cos\theta + b\sin\theta)$ are congruent patterns ($\Leftrightarrow ab = \overline{ab}$). The latter means that (3) determines a standing wave. Occurrence of swirling and standing waves was in many details discussed in [2, 4—6].

The Bubnov—Galerkin procedure leads to a necessary solvability condition with respect of a, $\bar{a}$, $\bar{b}$, and b appearing as a system of nonlinear algebraic equations [2, 4, 5]. To describe the steady-state sloshing, we should solve the system for any $\bar{\sigma}_{11} = \sigma_{11} / \sigma$ close to 1. The first Lyapunov method can be used to study the stability. The algebraic system is rederived in terms of the integral amplitudes A, B (the main wave elevation components in the Ox and Oy directions, respectively) and the phase-lags ψ, φ:

$$A = \sqrt{a^2 + \bar{a}^2} \text{ and } B = \sqrt{b^2 + \bar{b}^2} \tag{4a}$$

$$a = A\cos\psi,\; \bar{a} = A\sin\psi,\; \bar{b} = B\cos\varphi,\; \bar{b} = B\sin\varphi, \tag{4b}$$

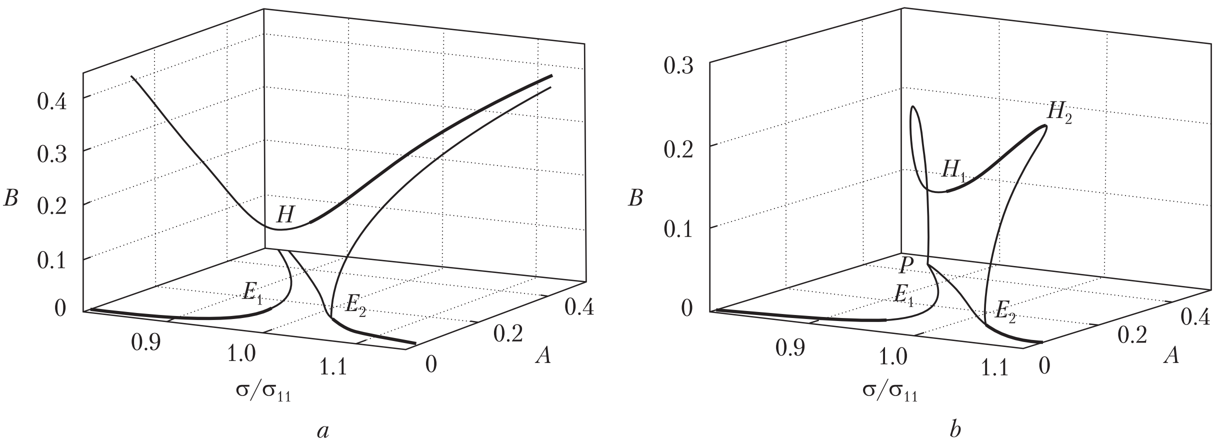

Fig. 2. Response curves in the $(\sigma/\sigma_{11}, A, B)$ -space for the longitudinal ($\varepsilon = 0$) harmonic forcing in the *Oxz*-plane, $h/r_0 = 1.5$, the nondimensional forcing amplitude $\eta_{1a} = 0.01$ ($\eta_{2a} = 0$). The undamped sloshing ($\xi = 0$) is presented in (a) and the damped case ($\xi = 0.02$) is shown in (b). There is no stable steady-state sloshing between E_1 and E_2, where irregular (chaotic) waves are expected. Curves on (close to) the $(\sigma/\sigma_{11}, A)$-plane correspond to the (almost) planar wave regime

$$\begin{cases} A[\bar{\sigma}_{11}^2 - 1 + m_1 A^2 + (m_3 - F)B^2] = \varepsilon_x \cos\psi; \quad A[DB^2 + \xi] = \varepsilon_x \sin\psi; \\ B[\bar{\sigma}_{11}^2 - 1 + m_1 B^2 + (m_3 - F)A^2] = \varepsilon_y \sin\varphi; \quad B[DA^2 - \xi] = \varepsilon_y \cos\varphi; \end{cases} \tag{5a}$$

$$\begin{cases} F = (m_3 - m_1)\cos^2(\alpha) = (m_3 - m_1)/(1 + C^2), \\ D = (m_3 - m_1)\sin(\alpha)\cos(\alpha) = (m_3 - m_1)C/(1 + C^2), \end{cases} \tag{5b}$$

where $\alpha = \varphi - \psi$, $C = \tan\alpha$, $0 \leqslant \varepsilon_y \leqslant \varepsilon_x \neq 0$, $F(\alpha)$ and $D(\alpha)$ are π-periodic functions of the phase-lags difference α, and $\varepsilon_x, \varepsilon_y$ are linear functions of the forcing amplitudes η_{1a}, η_{2a} The coefficients m_1 and m_2 are known functions of the liquid depth (see, [2, 4]) but $\xi = 2\xi_{11}$ (damping rate of the two lowest natural sloshing modes). A special numerical scheme [7] was developed to solve (5), i.e. to describe how the main wave amplitude components A and B change versus σ/σ_{11}.

The undamped resonant steady-state sloshing due to longitudinal excitations along the Ox axis $(\varepsilon_x > 0, \varepsilon_y = 0, \xi = 0)$ was analyzed in [2, 4]. A planar standing wave and the swirling are identified. In terms of (4) and (5) with $\xi = 0$ these imply $B = 0, \sin\psi = 0, C = 0,$ and $AB \neq 0, \sin\psi = \cos\varphi = 0,$ $(C = \pm\infty)$, respectively. The swirling consists of two identical angular progressive waves occurring in either counter- or clockwise directions, they correspond to $C = +\infty$ and $-\infty$ respectively. Fig. 2, *a* presents the corresponding response curves. Case (*b*) shows the linear damping effect with $\xi = 0.02$ The branches belonging (close) to the plane $\sigma/\sigma_{11}, A$ are responsible for the (almost) planar standing wave regime. The regime is stable to the left of E_1 and to the right of E_2. It becomes unstable in a neighborhood of the primary resonance $\sigma/\sigma_{11} = 1$, where the stable swirling (to the right of $H(H_1)$) and irregular waves (the steady-state sloshing is unstable) between E_1 and $H(H_1)$ are predicted. The damping removes infinite points on the response curves of (*a*), so that the steady-state swirling branching in (*b*) constitutes an arc pinned

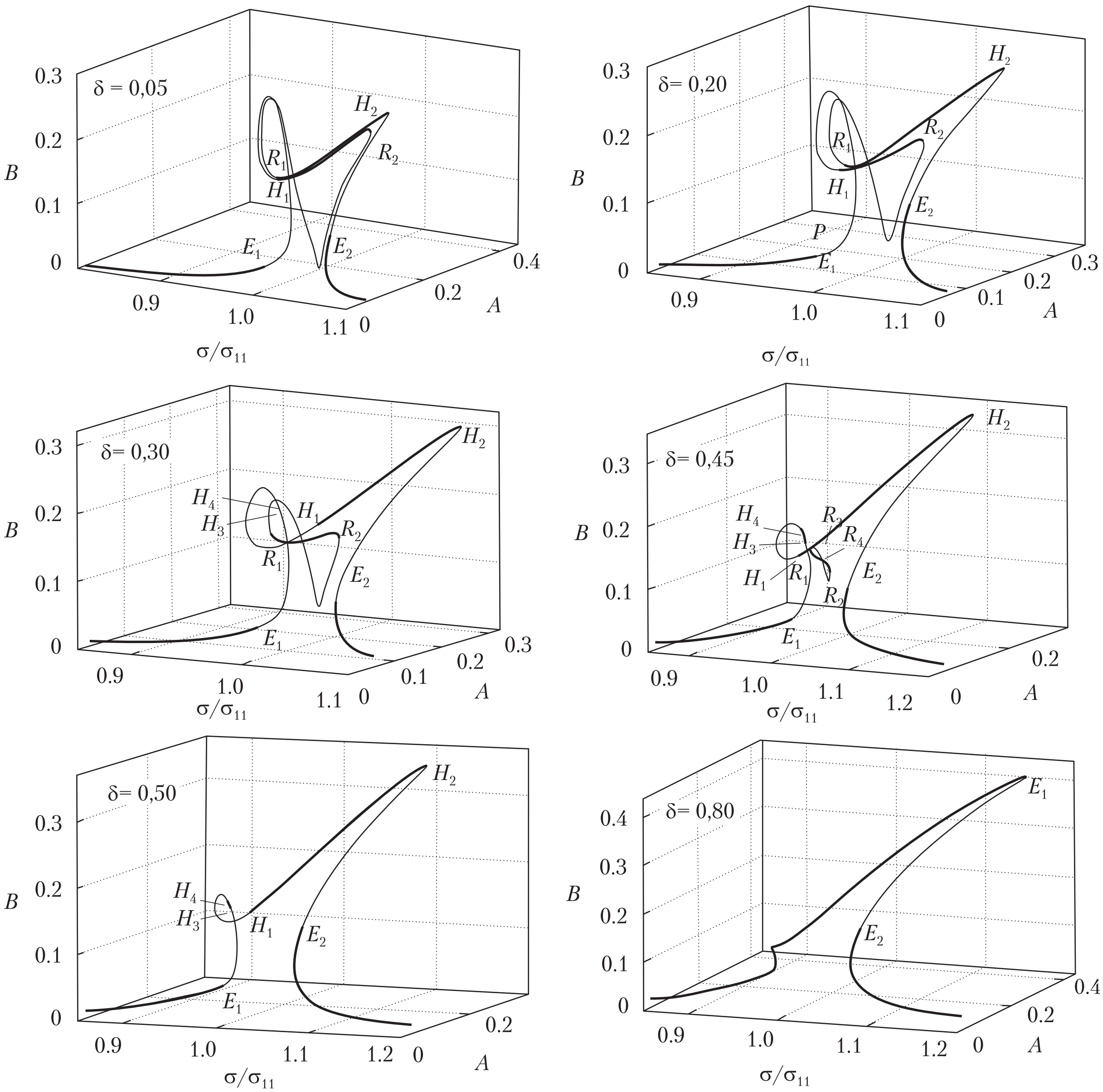

Fig. 3. Response curves for $\delta = \varepsilon_y/\varepsilon_x > 0$ in the $(\sigma/\sigma_{11}, A, B)$-space. The steady-state resonant sloshing is due to an elliptic counterclockwise forcing with $\eta_{1a} = 0.01$, $\eta_{2a} = \delta\eta_{1a}$; $\xi = 0.02$. All the points on the response curves correspond to the swirling. The bold lines mark the stability

at E_2 and P, which can be treated as bifurcation points, where the swirling emerges from the (almost) planar steady-state wave regime.

In [5], we showed that any orbital small-magnitude periodic tank motions are equivalent, to within the higher-order terms, to an artificial elliptic-type horizontal excitation with $\varepsilon_y = \delta\varepsilon_x$, $0 < \delta \leqslant 1$. How the response curves of the damped steady-state sloshing change with increasing δ is shown in Fig. 3. When $\delta \neq 0$, all the steady-state sloshing regimes are of the swirling type. Specifically, there are no identical swirling waves with opposite directions, as it has been in the longitudinal case (each point on $PH_1H_2E_2$ in Fig. 2, *b* implies the pair of these waves). The connected branching in Fig. 2, *b* splits into the response curve $E_1H_1H_2E_2$ existing for any σ/σ_{11} and $0<\delta\leqslant 1$ and corresponding to the co-directed (with the counterclockwise elliptic forcing) angular progressive waves and the loop-like branch with R_1 and R_2 whose points imply the counter-directed swirling. Fig. 3 shows that the latter branch disappears, as δ increases. This is a very interesting fact, which contradicts the steady-state analysis of the undamped sloshing in [2], where both the co- and counter-directed angular progressive waves exist and can be stable in certain frequency ranges for any $0<\delta\leqslant 1$.

In summary, the linear viscous damping matters for the orbitally-excited sloshing in bioreactors of an upright circular cylindrical shape. It affects qualitatively and quantitatively the steady-state sloshing and the corresponding response curves. The most interesting fact is that the damping, even being relatively small, makes the counter-directed angular progressive waves (swirling) impossible, as the forcing orbit tends to a circle. This fact contradicts the the undamped steady-state analysis, but it is qualitatively consistent with model tests by M. Reclari in [1].

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ХЛЮПАННЯ ІЗ ДЕМПФУВАННЯМ
У ВЕРТИКАЛЬНОМУ ЦИЛІНДРИЧНОМУ БАКУ
ПРИ ОРБІТАЛЬНИХ ЗБУРЕННЯХ

З використанням нелінійної модальної системи Наріманова—Мойсеєва з лінійним демпфуванням вивчається затухаюче усталене хлюпання рідини у вертикальному круговому баку при заданому горизонтальному орбітальному (еліптичному) русі посудини з вимушеною частотою, близькою до власної частоти коливань. Тоді як випадок без демпфування включає як співнапрямлені (із напрямком орбітального руху), так і протилежно напрямлені кутові прогресивні хвилі, демпфування робить неможливим існування протилежно направленої хвилі при збуреннях, близьких до кругових.

Ключові слова: *хлюпання рідини, демпфування, усталені хвилі.*

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ПЛЕСКАНИЕ С ДЕМПФИРОВАНИЕМ
В ВЕРТИКАЛЬНОМ ЦИЛИНДРИЧЕСКОМ БАКЕ
ПРИ ОРБИТАЛЬНЫХ ВОЗБУЖДЕНИЯХ

С использованием нелинейной модальной системы Нариманова—Моисеева с линейным демпфированием изучается затухающее установившееся плескание жидкости в вертикальном круговом баке при заданном горизонтальном орбитальном (эллиптическом) движении сосуда с вынужденной частотой, близкой к собственной частоте колебаний жидкости. В то время как случай без демпфирования включает как сонаправленные (с направлением орбитального движения), так и противоположно направленные угловые прогрессивные волны, демпфирование делает невозможным существование противоположно направленной волны при возбуждениях, близких к круговым.

Ключевые слова: *плескание жидкости, демпфирование, установившиеся волны.*