doi: https://doi.org/10.15407/dopovidi2017.01.010
UDC 517.5

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Semilinear equations in a plane and quasiconformal mappings

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We consider generalizations of the Bieberbach equation with nonlinear right parts, which makes it possible to study many problems of mathematical physics in inhomogeneous and anisotropic media with smooth characteristics. We establish interconnections of these semilinear equations with quasiconformal mappings, obtain on this basis, a series of theorems on the existence of their solutions that blow-up on the boundary of a unit disk, as well as on punctured unit disks and rings, and give their explicit representations.

Keywords: *semilinear elliptic equations, Bieberbach equation, quasiconformal mappings, Beltrami equation, Keller–Osserman condition.*

1. Introduction. Let Ω be a domain in the complex plane $\mathbb{C}$, and let $A(z)$ be a symmetric 2×2 matrix such that $\det A = 1$ with real entries, which are smooth in $z \in \Omega$, maybe except isolated points, and

$$\frac{1}{K}|\xi|^2 \leqslant \langle A(z)\xi,\xi\rangle \leqslant K|\xi|^2, \quad 1 \leqslant K < \infty \quad \forall \xi \in \mathbb{R}^2. \tag{1}$$

We study the blow-up problem for the model semilinear equation

$$\operatorname{div}\,[A(z)\nabla u\,(z)] = e^{u(z)} \text{ in } \Omega \tag{2}$$

and show that the well-known Liouville—Bieberbach function solves this problem under an appropriate choice of the matrix $A(z)$. The proof is based on the fact that every regular solution u can be expressed as $u(z) = T\,(w(z))$, where $w : \Omega \to G$ stands for a quasiconformal homeomorphism generated by the matrix $A(z)$, and T is a solution of the semilinear weighted Bieberbach equation

$$\Delta T\,(w) = m(w) e^{T(w)} \text{ in } G. \tag{3}$$

Here, the weight $m\,(w)$ is the Jacobian of the inverse mapping $\omega^{-1}(w)$:

Recall that, given a bounded domain Ω in $\mathbb{C}$, solutions to a semilinear equation

$$\Delta u(z) = f(u(z)) \tag{4}$$

are called its *boundary blow-up solutions* or its *large solutions*, if

$$u(z) \to +\infty \text{ as } d(z) := \mathrm{dist}\,(z, \partial\Omega) \to 0\,. \tag{5}$$

In the last decades, semilinear equations became a central subject of studies in the theory of nonlinear partial differential equations. The study of such equations is of interest because of their numerous applications to actual problems of differential geometry, mathematical physics, logistic problems, etc.: see, e.g., [1, 2], and the extended bibliography therein. In this context, the consideration of matrix-valued functions instead of scalar functions in (2) makes it possible to study many physical processes not only in inhomogeneous but also in anisotropic media that is very actual at present.

The existence of a large solution to Eq. (4) is related to the existence of a maximal solution u of (4) in Ω, which, in turn, depends on the so-called Keller—Osserman condition, see [3, 4]. Namely, J.B. Keller and R. Osserman provided a sharp condition on the growth of f at infinity, which guarantees that the set of solutions to (4) is uniformly bounded from above in compact subsets of Ω. Qualitatively, the condition means that the superlinearity of f at infinity is sufficiently strong. They derived an a priori estimate for solutions to (4) in terms of $\rho(z) = \mathrm{dist}\,(z, \partial\Omega)$. This estimate implies that the Eq. (4) possesses a maximal solution in a bounded domain. Under some additional conditions on Ω, the maximal solution blows up everywhere on the boundary. Thus, we arrive at a large solution.

Recall that a function $f \in C(\mathbb{R}_+)$ satisfies the *Keller—Osserman condition* if there exists a positive non-decreasing function h such that

$$f(t) \geqslant h(t), \forall t \in \mathbb{R}_+ \text{ and } \int\limits_{t_0}^{\infty} \left\{ \int\limits_{0}^{t} h(s)\,ds \right\}^{-1/2} dt < \infty \text{ for all } t_0 > 0. \tag{6}$$

It is known that if f is non-decreasing and satisfies the Keller—Osserman condition, then a *large solution exists in every bounded smooth domain*. Uniqueness in smooth domains was established under some additional conditions on f, see, e.g., [5], Section 5.3. It is easy to check that the functions $f(t) = e^t$ and $f(t) = t^p$, $p > 1$, satisfy (6). The semilinear equation

$$\Delta u(z) = e^{u(z)}, \tag{7}$$

as far as we know, was first investigated by Bieberbach in his pioneering work [6] related to the study of the Riemannian geometry and automorphic functions in a plane. More precisely, if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant Gaussian curvature $-g^2$; then $\Delta u = g^2 e^{2u}$. It is this work that has stimulated numerous studies in the field of semilinear differential equations in R^n, $n \geq 1$, and Eq. (7) continues to play the role of one of the fundamental model equations of the theory. It is important to note that, in simply connected planar domains Ω, the large solutions for Eq. (7) are expressed explicitly by means of the Liouville—Bieberbach formula

$$u(z) = \log \frac{8|f'(z)|^2}{(1-|f(z)|^2)^2}, \tag{8}$$

where f stands for a conformal map $f : \Omega \to \mathbb{D} := \{z : |z| < 1\}$.

For the model case of the equation

$$\Delta u = e^{au}, \quad a > 0, \tag{9}$$

the following result holds, see [5], Theorem 5.3.7.

Theorem A. *Let Ω be a bounded domain in $\mathbb{C}$ such that $\partial\Omega = \partial\overline{\Omega}^c$. Then there exists one and only one blow-up solution to (9).*

2. The main lemma. Let us associate the complex-valued function

$$\mu(z) = \frac{1}{\det(I+A)}(a_{22} - a_{11} - 2ia_{12}) \tag{10}$$

with the above matrix-valued function $A(z)$, where I stands for the unit matrix. Then the condition of ellipticity (1) is written as

$$|\mu(z)| \leqslant \frac{K-1}{K+1}. \tag{11}$$

Thus, we associate the Beltrami equation

$$\omega_{\overline{z}}(z) = \mu(z)\omega_z(z) \tag{12}$$

with A, where $\omega_{\overline{z}} = \overline{\partial}\omega = (\omega_x + i\omega_y)/2$, $\omega_z = \partial\omega = (\omega_x - i\omega_y)/2$, $z = x + iy$, and ω_x and ω_y are partial derivatives of ω with respect to x and y, correspondingly.

In turn, the Beltrami equation (12) generates a quasiconformal homeomorphism $\omega : \Omega \to G$, see, e.g., [7; 8, p. 67], where one can take, as the domain G any plane domain, which is conformally equivalent to Ω. In what follows, we will say that A, μ, and ω are agreed.

Lemma 1. Let $\omega : \Omega \to G$ *be a homeomorphic solution to the Beltrami equation (12) agreed with the matrix-valued function A; let T be a real-valued function in $C^2(G)$, and $u = T \circ \omega$. Then*

$$\operatorname{div}[A(z)\nabla u(z)] = J_\omega(z)\Delta T(w), \ w = \omega(z), \tag{13}$$

where $J_\omega(z)$ stands for the Jacobian of the mapping $\omega(z)$.

3. On semilinear equations in a unit disk. Below, we confine ourselves to a few examples of the application of Lemma 1 to the study of some properties of boundary blow-up solutions of a classical model semilinear elliptic equation $\operatorname{div}[A(z)\nabla u] = p(z)e^u$.

Note first of all that, given a complex—valued function μ, satisfying (11), one can invert the algebraic system (10) to obtain the corresponding matrix-valued function

$$A(z) = \begin{pmatrix} \dfrac{|1-\mu|^2}{1-|\mu|^2} & \dfrac{-2\operatorname{Im}\mu}{1-|\mu|^2} \\ \dfrac{-2\operatorname{Im}\mu}{1-|\mu|^2} & \dfrac{|1+\mu|^2}{1-|\mu|^2} \end{pmatrix}. \tag{14}$$

Theorem 1. *Let $k(t)$, $0 < t < 1$; be a complex-valued smooth function, $|k(t)| \leqslant q < 1$, and let $A(z)$ be given in the unit disk $\mathbb{D}$ by (14) with $\mu(z) = k(|z|)z/\overline{z}$. Then the boundary blow-up solution to the equation*

$$\operatorname{div}[A(z)\nabla u(z)] = p(|z|)\, e^{u(z)}, \tag{15}$$

where

$$p(t) = \frac{1-|k(t)|^2}{|1-k(t)|^2}\exp\left\{\int_1^t \mathrm{Re}\frac{4k(\tau)}{1-k(\tau)}\frac{d\tau}{\tau}\right\},$$

has the explicit representation

$$u(z) = \log\frac{8}{(1-|\omega|^2)^2}$$

with

$$|\omega| = \exp\left\{\int_1^{|z|}\mathrm{Re}\frac{1+k(t)}{1-k(t)}\frac{dt}{t}\right\}.$$

In particular, among the quasiconformal automorphisms of the unit disc $\mathbb{D}$,

$$\omega(z) = \frac{z}{|z|}\exp\left\{\int_1^{|z|}\frac{1+k(t)}{1-k(t)}\frac{dt}{t}\right\}, \tag{16}$$

agreed with $\mu(z) = k(|z|)z/\bar{z}$ from Theorem 1, there is a variety of volume-preserving maps, for which $J_\omega(z) \equiv 1$, $z \in \mathbb{D}$. Hence, Theorem 1 implies the following statement that may have of independent interest.

Corollary 1. *Let a matrix-valued function* $A(z)$ *be given in* $\mathbb{D}$ *by* (14) *with*

$$\mu(z) = k(|z|)\frac{z}{\bar{z}},\ k(t) = \nu^2(t) \pm i\nu(t)\sqrt{1-\nu^2(t)}, \tag{17}$$

where $\nu(t)$, $0 < t < 1$, *is a real-valued smooth function*. If $|\nu(t)| \leqslant q < 1$, *then there exists one and only one boundary blow-up solution to the generalized Bieberbach equation*

$$\mathrm{div}\,[A(z)\nabla u] = e^u,\ z \in \mathbb{D}, \tag{18}$$

which is written explicitly by the same Liouville—Bieberbach formula

$$u(z) = \log\frac{8}{(1-|z|^2)^2}. \tag{19}$$

For example, the well-known spiral mapping

$$s(z) = ze^{2i\log|z|}$$

is just volume-preserving. Indeed, in this case, the Beltrami coefficient

$$\mu(z) = \frac{\omega_{\bar{z}}}{\omega_z} = \frac{1}{2}(1+i)\frac{z}{\bar{z}},$$

and we see that it corresponds to (17) with $\nu(t) \equiv 1/\sqrt{2}$. If $z = \rho e^{i\varphi}$, the matrix A generated by such $\mu(z)$ will depend on φ only and has the form

$$A = \begin{pmatrix} 3-2(\cos 2\varphi - \sin 2\varphi) & -2(\cos 2\varphi + \sin 2\varphi) \\ -2(\cos 2\varphi + \sin 2\varphi) & 3+2(\cos 2\varphi - \sin 2\varphi) \end{pmatrix}. \tag{20}$$

The spiral mappings play an important role in applications. F. Gehring employed $s(z)$ in [9] to solve the well-known Bers' problem on the structure of the universal Teichmüller space. F. John [10] used the mapping $s(z)$ to study the uniqueness of a non-linear elastic equilibrium for prescribed boundary displacements. The spiral mappings also play an important role in the theory of smooth planar mappings with constant principal stretches, see [11, 12]. The problem of conformal differentiation as well as many regularity, distortion, and rotation problems for quasiconformal mappings in the plane were also investigated by means of the spiral mapping. We recall a typical such problem that goes back to F. John and is closely related to the non-linear elasticity theory. In [13], he showed that if $f:\mathbb{C}\to\mathbb{C}$ is a $(1+\varepsilon)$ — bi-Lipschitz mapping and if, for some $0<a<b$, we have $f(z)=z$ for $|z|>b$; $f(z)=z\,e^{i\theta}$ for $|z|<a$, then

$$|\theta|\leqslant C\,(1+\log(b/a))\,\varepsilon\,. \tag{21}$$

The angle estimate (21) follows from the basic stability theorems in [13] for $(1+\varepsilon)$—bi-Lipschitz mappings in a plane. The BMO technique [14] also plays an important role for (21). Quasiconformal methods lead to the sharp solution of John's problem, see [8, 15], Chapter 13. It is shown that the logorithmic spiral gives the extremum to F. John's angle distortion problem for plane bi-Lipschitz mappings.

4. On semilinear equations in a ring.

Theorem 2. *Let $A(z)$ be given in the annulus* $\{z\in\mathbb{C}: r<|z|<1\}$ by (14) with

$$\mu(z)=k(|z|)\frac{z}{\overline{z}},\;\; k(t)=\nu^2(t)\pm i\nu(t)\sqrt{1-\nu^2(t)}\,, \tag{22}$$

where $\nu(t)$ stands for a real-valued smooth function with $|\nu(t)|\leqslant q<1$, $0<t<1$.

Then there exists one and only one boundary blow-up solution to the equation

$$\operatorname{div}[A(z)\nabla u(z)]=e^{u(z)}, \tag{23}$$

which is given explicitly by the formula

$$u(z)=\log\frac{2\pi^2}{|z|^2\log^2 r\cdot\sin^2\left(\dfrac{\pi}{\log r}\log|z|\right)} \tag{24}$$

in the annulus $r<|z|<1$.

Making use of the limit in (24) as $r\to 0$, we get the following result, which may be of independent interest.

Corollary 2. *For each matrix-valued function $A(z)$ from Theorem 2, the equation*

$$\operatorname{div}[A(z)\nabla u(z)]=e^{u(z)} \tag{25}$$

and the Bieberbach equation

$$\nabla u(z)=e^{u(z)} \tag{26}$$

admit the boundary blow-up solution

$$u(z)=\log\frac{2}{|z|^2\log^2|z|} \tag{27}$$

in the punctured unit disk $0<|z|<1$.

5. The final remarks. The approach given in the last section to the construction of a boundary blow-up solution to the Bieberbach equation in the unit disk $\mathbb{D}$ with a singularity at the origin can be extended to the case of a finite number of singular points z_k, $|z_k|<1$, $k=1,2,\ldots,n$. Indeed, let

ISSN 1025-6415. Dopov. Nac. acad. nauk Ukr. 2017. № 1

$r > 0$ be such that all the circles $d_k = \{z : |z - z_k| \leqslant r\}$ belong to $\mathbb{D}$ and do not intersect each other. Denote, by $F_r(z)$, a conformal mapping of the circular multiconnected domain $\mathbb{D} \setminus \bigcup_{k=1}^{n} d_k$ onto the unit disc $\mathbb{D}$. Then the required solution with prescribed singularities at the points z_k is given by

$$u(z) = \lim_{r \to 0} \log \frac{8|F_r'(z)|^2}{(1 - |F_r(z)|^2)^2}. \tag{28}$$

Remark 1. Perhaps, a reader has pointed out that Lemma 1 was formulated for matrix-valued functions $A(z)$ with smooth entries. It was done only for the sake of simplicity in the exposition. We plan to publish the relevant results for the case of measurable entries satisfying the uniform ellipticity condition, as well as to study the case of degeneration.

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Received 20.09.2016

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ПОЛУЛИНЕЙНЫЕ УРАВНЕНИЯ НА ПЛОСКОСТИ
И КВАЗИКОНФОРМНЫЕ ОТОБРАЖЕНИЯ

Рассмотрены обобщения уравнения Бибербаха с нелинейными правыми частями, которые позволяют изучать многие проблемы математической физики в неоднородных и анизотропных средах с гладкими характеристиками. Установлены взаимосвязи этих полулинейных уравнений с квазиконформными отображениями и на этой основе получен ряд теорем существования их решений, взрывающихся на границе единичного круга, проколотых единичных кругах и кольцах, а также приведены их явные представления.

Ключевые слова*: полулинейные эллиптические уравнения, уравнение Бибербаха, квазиконформные отображения, уравнение Бельтрами, условие Келлера—Оссермана.*

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НАПІВЛІНІЙНІ РІВНЯННЯ НА ПЛОЩИНІ
ТА КВАЗІКОНФОРМНІ ВІДОБРАЖЕННЯ

Розглянуто узагальнення рівняння Бібербаха з нелінійними правими частинами, які дають можливість вивчати багато проблем математичної фізики в неоднорідних та анізотропних середовищах з гладкими характеристиками. Встановлено взаємозв'язки цих напівлінійних рівнянь з квазіконформними відображеннями і на цій основі отримано ряд теорем існування їх розв’язків, що вибухають на границі одиничного круга, проколотих одиничних кіл та кільцях, а також наведено їх явні зображення.

Ключові слова: *напівлінійні еліптичні рівняння, рівняння Бібербаха, квазіконформні відображення, рівняння Бельтрамі, умова Келлера-Оссермана.*