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Prime ends on the Riemann surfaces

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We prove criteria for the homeomorphic extension of mappings with finite distortion between the domains on Riemann surfaces to the boundary by prime Carathéodory ends.

Keywords: *Riemann surfaces, prime Carathéodory ends, homeomorphic extension, boundary behavior, mappings of finite distortion, Sobolev mappings.*

The theory of the boundary behavior in the prime ends for the mappings with finite distortion has been developed in [1] and [2] for the plane domains and in [3] for the spatial domains. The pointwise boundary behavior of the mappings with finite distortion in regular domains on Riemann surfaces was recently studied by us in [4]. Moreover, the problem was investigated in regular domains on the Riemann manifolds for $n \geqslant 3$ as well as in metric spaces, see, e.g., [5]. For basic definitions and notations, discussions, and historic comments in the mapping theory on Riemann surfaces, see our previous papers [4, 6, 7].

1. Definition of the prime ends and preliminary remarks. We act similarly to Carathéodory under the definition of the prime ends of domains on a Riemann surface $\mathbb{S}$, see Chapter 9 in [8]. First of all, recall that a continuous mapping $\sigma: I \to \mathbb{S}$, $I = (0,1)$, is called a *Jordan arc* in $\mathbb{S}$ if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations $\sigma, \overline{\sigma}$, and $\partial\sigma$ for $\sigma(I), \overline{\sigma(I)}$, and $\overline{\sigma(I)} \setminus \sigma(I)$, correspondingly. A Jordan arc σ in a domain $D \subset \mathbb{S}$ is called a *cross-cut* of the domain D if σ splits D, i.e. $D \setminus \sigma$ has more than one (connected) component, $\partial\sigma \subseteq \partial D$, and $\overline{\sigma}$ is a compact set in $\mathbb{S}$.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of D is called a *chain* in D if:

(i) $\overline{\sigma}_j \cap \overline{\sigma}_j = \varnothing$ for every $i \neq j, \quad i, j = 1, 2, \ldots;$

(ii) σ_m splits D into 2 domains, one of which contains σ_{m+1}, and another one contains σ_{m-1} for every $m > 1$;

(iii) $\delta(\sigma_m) \to 0$ as $m \to \infty$.

Here, $\delta(E) = \sup\limits_{p_1, p_2 \in S} \delta(p_1, p_2)$ denotes the diameter of a set E in $\mathbb{S}$ with respect to an arbitrary metric δ in $\mathbb{S}$ agreed with its topology, see [4, 5].

Correspondingly to the definition, a chain of cross-cuts σ_m generates a sequence of domains $\sigma_m \subset D$ such that $d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots$ and $D \cap \partial \quad d_m = \sigma_m$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma_k'\}$ are called *equivalent* if, for every $m = 1, 2, \ldots$, the domain d_m contains all domains d_k' except a finite number, and, for every $k = 1, 2, \ldots$, the domain d_k' contains all domains d_m except a finite number, too. A *prime end* P of the domain D is an equivalence class of chains of cross-cuts of D.

Here, E_D will denote the collection of all prime ends of a domain D and $\overline{D}_p = D \cup E_D$ is its completion by prime ends. A basis of neighborhoods of a prime end P of D can be defined in the following way. Let d be an arbitrary domain from a chain in P. Denote, by d^*, the union of d and all prime ends of D having some chains in d. Just all such d^* form a basis of open neighborhoods of the prime end P. The corresponding topology on $\overline{D}_P$ is called the *topology of prime ends.*

Later on, we everywhere apply the following *conditions A*:

Let $\mathbb{S}$ and $\mathbb{S}'$ be Riemann surfaces, D and D' be domains in $\overline{\mathbb{S}}$ and $\overline{\mathbb{S}}'$, correspondingly, $\partial D \subset \mathbb{S}$ and $\partial D' \subset \mathbb{S}'$ have finite collections of nondegenerate components, and let $f : D \to D'$ be a homeomorphism of finite distortion with $K_f \in L^1_{\mathrm{loc}^*}$.

The base for our research is the following, see Lemma 7.1 in [9] or Theorem 2 in [10].

Lemma A. *Under conditions A, suppose that*

$$\int\limits_{R(p_0,\varepsilon,\varepsilon_0)} K_f(p)\cdot \psi^2_{p_0,\varepsilon,\varepsilon_0}(h(p,p_0))\, dh(p) = o(I^2_{p_0,\varepsilon_0}(\varepsilon)), \quad \forall\, p_0 \in \partial D \tag{1}$$

as $\varepsilon \to 0$ *for all* $\varepsilon_0 < \delta(p_0)$, *where* $R(p_0,\varepsilon,\varepsilon_0) = \{p \in S : \varepsilon < h(p,p_0) < \varepsilon_0\}$ *and* $\psi_{p_0,\varepsilon,\varepsilon_0}(t) : (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, *is a family of measurable functions such that*

$$0 < I_{p_0,\varepsilon_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{p_0,\varepsilon,\varepsilon_0}(t)\, dt < \infty, \quad \forall \varepsilon \in (0,\varepsilon_0).$$

Then f *can be extended to a homeomorphism* $\tilde{f}$ *of* $\overline{D}_P$ *onto* $\overline{D}'_P$.

2. On the extension to the boundary by prime ends. We assume in this section that the function K_f is extended by zero outside of D.

Theorem 1. *Under conditions A, suppose in addition that*

$$\int\limits_0^{\varepsilon_0} \frac{dr}{\|K_f\|(p_0,r)} = \infty, \quad \forall_0 \in \partial D, \quad \varepsilon_0 < \delta(p_0) \tag{2}$$

where

$$\|K_f\|(p_0,r) := \int\limits_{S(p_0,r)} K_f(p)\, ds_h(p). \tag{3}$$

Then f *can be extended to a homeomorphism* $\tilde{f}$ *of* $\overline{D}_P$ *onto* $\overline{D}'_P$.

Here, $S(p_0,r)$ denotes the circle $\{p \in \mathbb{S} : h(p,p_0) = r\}$.

Proof. Indeed, for the functions

$$\psi_{p_0,\varepsilon_0}(t) := \begin{cases} 1/\|K_f\|(p_0,t), & t \in (0,\varepsilon_0), \\ 0, & t \in [\varepsilon_0,\infty), \end{cases} \tag{4}$$

we have, by the Fubini theorem, that

$$\int\limits_{R(p_0,\varepsilon,\varepsilon_0)} K_f(p)\psi^2_{p_0,\varepsilon_0}(h(p,p_0))\,dh(p)=\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dr}{\|K_f\|(p_0,r)}, \tag{5}$$

where $R(p_0,\varepsilon,\varepsilon_0)$ denotes the ring $\{p\in S:\varepsilon<h(p,p_0)<\varepsilon_0\}$ and, consequently, condition (1) holds by (2) for all $p_0\in\partial D$ and $\varepsilon_0\in(0,\varepsilon(p_0))$.

Here, we have used the standard conventions in the integral theory that $a/\infty=0$ for $a\neq\infty$ and $0\cdot\infty=0$.

Thus, Theorem 1 follows immediately from Lemma A.

Corollary 1. *In particular, the conclusion of Theorem 1 holds if*

$$k_{p_0}(r)=O\left(\log\frac{1}{r}\right),\quad \forall p_0\in\partial D, \tag{6}$$

as $r\to 0$, *where* $k_{p_0}(r)$ *is the average of* K_f *over the infinitesimal circle* $S(p_0,r)$.

Choosing $\psi(t):=\dfrac{1}{t\log 1/t}$ in (1), we obtain, by Lemma A, the next result, see also Lemma 4.1 in [11] or Lemma 13.2 in [12].

Theorem 2. *Under conditions A, let* K_f *have a dominant* Q_{p_0} *in a neighborhood of each point* $p_0\in\partial D$ *with finite mean oscillation at* p_0. *Then* f *can be extended to a homeomorphism* $\tilde f:\overline{D}_p\to\overline{D}'_p$.

By Corollary 4.1 in [11] or Corollary 13.3 in [12] we obtain the following.

Corollary 2. *In particular, the conclusion of Theorem 2 holds if*

$$\overline{\lim_{\varepsilon\to 0}}\int\limits_{D(p_0,\varepsilon)} K_f(p)\,dh(p)<\infty,\quad \forall p_0\in\partial D, \tag{7}$$

where $D(p_0,\varepsilon)$ *is the infinitesimal disk* $\{p\in S:h(p,p_0)<\varepsilon_0\}$.

Corollary 3. *The conslusion of Theorem 2 holds if every point* $p_0\in\partial D$ *is a Lebesgue point of the function* K_f *or its dominant* Q_{p_0}.

The next statement also follows from Lemma A under the choice $\psi(t)=1/t$.

Theorem 3. *Under conditions A, let, for some* $\varepsilon_0>0$,

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0} K_f(p)\frac{dh(p)}{h^2(p,p_0)}=o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right)\quad as\quad \varepsilon\to 0\ \forall\ p_0\in\partial D. \tag{8}$$

Then f *can be extended to a homeomorphism of* $\overline{D}_p$ *onto* $\overline{D}'_p$.

Remark 1. Choosing the function $\psi(t)=1/(t\log 1/t)$ in Lemma A instead of $\psi(t)=1/t$, (8) can be replaced by the weaker condition

$$\int\limits_{\varepsilon<h(p,p_0)<\varepsilon_0}\frac{K_f(p)\,dh(p)}{\left(h(p,p_0)\log\dfrac{1}{h(p,p_0)}\right)^2}=o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right) \tag{9}$$

and (6) by the condition

$$k_{p_0}(r)=o\left(\log\frac{1}{\varepsilon}\log\log\frac{1}{\varepsilon}\right). \tag{10}$$

Of course, we could give here the whole scale of corresponding conditions of the logarithmic type, by using suitable functions $\psi(t)$.

3. The final remarks. Theorem 1 has a number of other consequences thanking to Theorem 8.1 in [9].

Theorem 4. *Under conditions A, let*

$$\int\limits_{D(p_0,\varepsilon_0)} \Phi_{p_0}(K_f(p))\,dh(p)<\infty, \quad \forall p_0\in\partial D \tag{11}$$

for $\varepsilon_0=\varepsilon(p_0)$ *and a nondecreasing convex function* $\Phi_{p_0}:[0,\infty)\to[0,\infty)$ *with*

$$\int\limits_{\delta(p_0)}^{\infty} \frac{d\tau}{\tau\Phi_{p_0}^{-1}(\tau)}=\infty \tag{12}$$

for $\delta(p_0)>\Phi_{p_0}(0)$. *Then* f *is extended to a homeomorphism of* $\overline{D}_p$ *onto* $\overline{D}'_p$.

Proof. Indeed, in the case of hyperbolic Riemann surfaces, (11) and (12) imply (2) by Theorem 8.1 in [9]. After this, Theorem 4 becomes a direct consequence of Theorem 1. In the simpler case of elliptic and parabolic Riemann surfaces, we can apply similarly Theorem 3.1 in [13] for the Euclidean plane instead of Theorem 8.1 in [9].

Corollary 4. *In particular, the conclusion of Theorem 4 holds if*

$$\int\limits_{D(p_0,\varepsilon_0)} e^{\alpha_0 K_f(p)}\,dh(p)<\infty \quad \forall p_0\in\partial D \tag{13}$$

for some $\varepsilon_0=\varepsilon(p_0)>0$ and $\alpha_0=\alpha(p_0)>0$.

Remark 2. Note that, by Theorem 5.1 and Remark 5.1 in [14], condition (12) is not only sufficient but also necessary for a continuous extendability of all mappings f with the integral restriction (11) to the boundary.

Note also that, by Theorem 2.1 in [13], see also Proposition 2.3 in [15], (12) is equivalent to every of the conditions from the following series:

$$\int\limits_{\delta(p_0)}^{\infty} H'_{p_0}(t)\frac{dt}{t}=\infty, \quad \delta(p_0)>0, \tag{14}$$

$$\int\limits_{\delta(p_0)}^{\infty} \frac{dH_{p_0}(t)}{t}=\infty, \quad \delta(p_0)>0, \tag{15}$$

$$\int\limits_{\delta(p_0)}^{\infty} H_{p_0}(t)\frac{dt}{t^2}=\infty, \quad \delta(p_0)>0, \tag{16}$$

$$\int_{0}^{\Delta(p_0)} H_{p_0}\left(\frac{1}{t}\right)dt = \infty, \quad \Delta(p_0) > 0, \tag{17}$$

$$\int_{\delta_*(p_0)}^{\infty} \frac{d\eta}{H_{p_0}^{-1}(\eta)} = \infty, \quad \delta_*(p_0) > H_{p_0}(0), \tag{18}$$

where

$$H_{p_0}(t) = \log \Phi_{p_0}(t). \tag{19}$$

Here, the integral in (15) is understood as the Lebesgue—Stieltjes integral, and the integrals in (14) and (16)—(18) as the ordinary Lebesgue integrals.

It is necessary to give one more explanation. From the right hand sides in conditions (15, 16), we have in mind $+\infty$. If $\Phi_{p_0}(t) = 0$ for $t \in [0, t_*(p_0)]$, then $\mathrm{H}_{p_0}(t) = -\infty$ for $t \in [0, t_*(p_0)]$, and we complete the definition $\mathrm{H}'_{p_0}(t) = 0$ for $t \in [0, t_*(p_0)]$. Note that conditions (15) and (16) exclude that $t_*(p_0)$ belongs to the interval of integrability, because, in the contrary case, the left-hand sides in (15) and (16) are either equal to $-\infty$ or indeterminate. Hence we may assume in (14—17) that $\delta(p_0) > t_0$, correspondingly, $\Delta(p_0) < 1/t(p_0)$, where $t(p_0) := \sup\limits_{\Phi_{p_0}(t)=0} t$, set $t(p_0) = 0$ if $\Phi_{p_0}(0) > 0$.

The most interesting among the above conditions is (16), i.e. the condition:

$$\int_{\delta(p_0)}^{\infty} \log \Phi_{p_0}(t) \frac{dt}{t^2} = +\infty \ \textit{for some} \ \delta(p_0) > 0 \tag{20}$$

Finally, we note that the restriction on the nondegeneracy of boundary components of domains in conditions A is not essential, because this simplest case is included in our previous paper [4].

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ПРОСТІ КІНЦІ НА РІМАНОВИХ ПОВЕРХНЯХ

Доводяться критерії для гомеоморфного продовження на границю відображень зі скінченним спотворенням між областями на ріманових поверхнях по простих кінцях Каратеодорі.

Ключові слова: *ріманові поверхні, прості кінці за Каратеодорі, гомеоморфне продовження, гранична поведінка, відображення скінченного спотворення, відображення Соболєва.*

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ПРОСТЫЕ КОНЦЫ НА РИМАНОВЫХ ПОВЕРХНОСТЯХ

Доказываются критерии для гомеоморфного продолжения на границу отображений с конечным искажением между областями на римановых поверхностях по простым концам Каратеодори.

Ключевые слова: *римановы поверхности, простые концы по Каратеодори, гомеоморфное продолжение, граничное поведение, отображения конечного искажения, отображения Соболева.*