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Invariant solutions of a system of Euler equations that satisfy the Rankine–Hugoniot conditions

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We consider equations of hydrodynamics with certain additional constraints. Group-theoretical methods are applied to find invariant solutions of a system of Euler equations that satisfy the Rankine–Hugoniot conditions.

Keywords: *group-theoretical methods, invariant solutions, Euler equations.*

The past century in mathematical physics was marked with a large number of research papers on particular solutions of nonlinear differential equations. Besides the fact that exact solutions are almost always interesting themselves, they also have a valuable practical application to the verification of various numerical methods of solution of nonlinear differential equations.

There are many examples of explicitly solved problems of fluid mechanics in the literature. All known solutions and multiparametric families of new particular solutions appear to be obtainable by means of group-theoretical methods [1–8]. Moreover, these methods are useful for finding the particular solutions of nonlinear differential equations that satisfy certain prescribed initial or boundary conditions.

In the present paper, we look for invariant solutions of a system of Euler equations that satisfy the Rankine–Hugoniot conditions.

1. Formulation of the problem. To describe the motion of a nonviscous compressible liquid, we use the system of equations

$$D_t u^k(t,x) + \rho^{-1}\nabla_k p(t,x) = 0, \quad D_t\rho(t,x) + \rho\nabla_k u^k(t,x) = 0, \tag{1}$$

where $t \in R^1$, $x \in R^n$ $(n = 1, \ldots, 3)$, $u^k(t,x)$ stands for the k-th component of medium's velocity $(k = 1, \ldots, n)$, p is the pressure, ρ is the liquid density, and $D_t = \frac{\partial}{\partial t} + u^k\nabla_k$ is the total derivative with respect to the time with $\nabla_k = \frac{\partial}{\partial x_k}$. Repeating indices mean the summation, unless otherwise stated.

The main thermodynamical characteristics of the medium ρ, p, and T are expected to be connected by an expression

$$p = \Phi(\rho, T), \tag{2}$$

where Φ is a smooth (piecewise smooth) function. We also assume that the process described by system (1), (2) is either isothermal ($T = \text{const}$) or homothermal ($v_k T = 0$, $k = 1, \ldots, n$). Therefore, T does not depend on spatial coordinates, and the equation of state (2) reads

$$p = F(\rho, T) \tag{3}$$

with another function F.

In order to represent system (1) in a form convenient for the following analysis, we introduce the notations

$$u_\mu^k = \frac{\partial u^k}{\partial x_\mu}, \quad \rho_\mu = \frac{\partial \rho}{\partial x_\mu}, \quad p_k = \frac{\partial p}{\partial x_k},$$

where $k = 1, \ldots, n$, $\mu = 0, \ldots, n$, and $x_0 = t$. Using these notations, we represent system (1) in the form

$$u_0^k + u_j^k u^j + \rho^{-1} p_k = 0, \quad \rho_0 + u^j \rho_j + \rho u_j^j = 0. \tag{4}$$

Substituting (3) into the first equation of (4), we obtain

$$u_0^k + u^j u_j^k + \rho^{-1} F_\rho \rho_k = 0, \tag{5}$$

$$\rho_0 + u^j \rho_j + \rho u_j^j = 0, \tag{6}$$

where $F_\rho = \frac{\partial F}{\partial \rho}$.

For the symmetry analysis of system (5), (6), we use the infinitesimal Sophus Lie method. Its brief description is the following. Let

$$F^\nu(x, u, u_{(1)}) = 0, \quad \nu = 1, \ldots, N, \tag{7}$$

be a system of first-order differential equations, where $x = (x_1, \ldots, x_n)$, $u = (u^1, \ldots, u^m)$, and $u_{(1)} = Du$. We consider a one-parameter local group G of transformations

$$x' = f(x, u; a): \left. f\right|_{a=0} = x, \; u' = g(x, u; a): \left. g\right|_{a=0} = u \tag{8}$$

in the space R^{n+m} of the variables (x, u). Transformations (8) induce a one-parameter group of transformations in the space of the variables $u_{(1)}$,

$$u_{(1)}{}' = \Psi(x, u, u_{(1)}; a): \left. \Psi\right|_{a=0} = u_{(1)}, \tag{9}$$

where $\Psi(x, u, u_{(1)}; a)$ is a function which can be determined, if we know f and g. As a result, we have a one-parameter group $G_{(1)}$ of transformations in the space R^{n+m+nm} of the variables $(x, u, u_{(1)})$. Transformations (9) are referred to as the *prolongation* of transformations (8), and the group $G_{(1)}$ is the *first prolongation* of G [1, Chapter 2.3].

Definition 1. System of equations (7) is said to be *invariant with respect to the group G of point transformations* (8), if the manifold determined by Eqs. (7) in the space R^{n+m+nm} is invariant with respect to the first prolongation $G_{(1)}$ of the group G.

Let

$$X = \xi^j(x, u)\frac{\partial}{\partial x_j} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{10}$$

where

$$\xi^j(x, u) = \left.\frac{\partial f(x, u, a)}{\partial a}\right|_{a=0}, \tag{11}$$

$$\eta^\alpha(x,u) = \left.\frac{\partial g(x,u,a)}{\partial a}\right|_{a=0}. \tag{12}$$

The operator X is said to be the *infinitesimal operator* of the one-parameter group G^1 of transformations, and the functions ξ^j and η^α are its *coordinates*. The first prolongation of the group G^1 corresponds to an infinitesimal operator of the form

$$X_{(1)} = X + \zeta_i^\alpha \frac{\partial}{\partial u_j^\alpha}, \tag{13}$$

where

$$\zeta_i^\alpha = \frac{\partial \eta^\alpha}{\partial x_i} + u_i^\mu \frac{\partial \eta^\alpha}{\partial u^\mu} - u_j^\alpha \left(\frac{\partial \xi^j}{\partial x_i} + u_i^\beta \frac{\partial \xi^j}{\partial u^\beta} \right). \tag{14}$$

One of the prominent results in the group theory of continuous transformations is the fact that the invariance criterion for a differential equation with respect to the group G^1 is stated in terms of the correspondent infinitesimal symmetry operator, cf. [2].

Proposition 1. *System of equations* (7) *is invariant with respect to the group* G^1 *if and only if*

$$X_{(1)} F^\nu(x,u,u_{(1)})\Big|_{F=0} = 0, \; \nu = 1, \ldots, N. \tag{15}$$

Condition (15) is equivalent to a system of first-order linear differential equations in x, u, and $u_{(1)}$ called the *system of determining equations*.

Thus, the problem of finding the maximal local group of point transformations that are admissible for system (7) is to determine the coordinates of the infinitesimal operators that generate its one-parameter subgroups.

In the case of system (5), (6), the infinitesimal symmetry operator is expected to be of the form

$$Z = \xi^\mu(x,u,\rho)\frac{\partial}{\partial x_\mu} + \eta^k(x,u,\rho)\frac{\partial}{\partial u^k} + \Lambda(x,u,\rho)\frac{\partial}{\partial \rho}, \tag{16}$$

where $\mu = 1, \ldots, n$, $k = 1, \ldots, n$.

Acting by operator (16) on Eqs. (5), (6), we obtain a rather cumbersome system of first-order linear differential equations. Eliminating the variables u_0^k and ρ_0 by virtue of their expressions from (5) and (6), we transform it to another system of equations, where the quantities x_α, u^k, u_j^k, and ρ_j will be treated as independent variables. As the coordinates of the infinitesimal operator do not depend on u_j^k and ρ_j, the two equations obtained from (5) and (6) by means of criterion (15) can be split with respect to these variables. As a result, we have the system of differential equations

$$\eta^k u^l + \xi_k^l = 0, \quad \eta^k u^l + \xi_l^k = 0, \quad k \neq l,$$

$$\eta^j + u^j \xi_0^0 - \xi_0^j - \sum_{i=1}^n \xi_i^j u^i = 0, \quad \Lambda_\rho + \rho^{-1}\Lambda + \xi_k^k - \xi_0^0 - \eta^k u^k = 0,$$

$$\Lambda_0 + \sum_{i=1}^n (u^l \Lambda_l + \rho u_l^l) = 0, \tag{17}$$

$$2F_\rho(\xi_0^0 - \xi_k^k) + F_{\rho\rho} A + F_{0\rho}\xi^0 = 0, \tag{18}$$

where $\xi^0 = \xi^0(x_0)$, $\xi^k = \xi^k(x)$, $\eta^k = \eta^k(x,u)$, $\Lambda = \Lambda(x,u,\rho)$. In all formulae (17), (18), there is no summation over repeating indices.

Note that the arbitrary function F appears only in (18). This equation is referred to as a *classifying condition.*

2. Symmetry of system (5), (6). It is easy to check by direct calculations that system (17) has the solution

$$\xi^0 = \theta x_0^2 + \lambda x_0 + \alpha, \quad \Lambda = \left(c - \frac{n}{2}\dot{\xi}^0(x_0)\right)\rho,$$

$$\xi^k = \left(\frac{1}{2}\dot{\xi}^0(x_0) + \delta\right)x_k + \mu^k x_0 + \sum_{l=1}^{n} a_l^k x_l + \nu^k,$$

$$\eta^k = \theta x_k + \mu^k + \sum_{l=1}^{n} a_l^k u^l + \left(\delta - \frac{1}{2}\dot{\xi}^0(x_0)\right)u^k, \tag{19}$$

where $\dot{\xi}^0(x_0) = d\xi^0(x_0)/dx_0$, $a_l^k = -a_k^l$, and c, α, δ, θ, λ, μ^k, and ν^k are arbitrary parameters. Substituting this solution into (18), we have

$$\left(\frac{n}{2}\dot{\xi}^0(x_0) - c\right)\varphi_\rho - \xi^0\varphi_0 = (\dot{\xi}^0(x_0) - 2\delta)\varphi, \tag{20}$$

where $\varphi(\rho,t) = F_\rho(\rho,t)$. Note that the parameters a_l^k, μ^k, and ν^k (and α in the case where $\varphi_0 = 0$) are not involved in system (20). Therefore, for an arbitrary function $F(\rho,t)$, system (17), (18) admits the solution

$$\xi^0 = 0, \ \xi^k = \sum_{j=1}^{n} a_j^k x_j + \mu^k x_0 + \nu^k, \ \eta^k = \sum_{j=1}^{n} a_j^k u^j + \mu^k, \ a_j^k = -a_k^j. \tag{21}$$

In the case $F_\rho = \varphi(\rho)$ the same solution with $\xi^0 = \alpha = \mathrm{const}$ is also possible.

The functions ξ^0, ξ^k, and η^k ($k = 1, \ldots, n$) defined by (21) correspond to the differential operators

$$P_k = \frac{\partial}{\partial x_k}, \ G_k = x_0\frac{\partial}{\partial x_k} + \frac{\partial}{\partial u^k}, \ J_{kr} = x_k\frac{\partial}{\partial x_r} - x_r\frac{\partial}{\partial x_k} + u^k\frac{\partial}{\partial u^r} - u^r\frac{\partial}{\partial u^k}. \tag{22}$$

It is easy to check that the vector space $\langle P_k, G_k, J_{kr}\rangle$, $k = 1, \ldots, n$, $r = 1, \ldots, n$, is closed under the Lie bracket

$$X, Y \to [X,Y] = XY - YX, \tag{23}$$

and, therefore, the space of these operators possesses the structure of a Lie algebra. This is a general property [2], namely, the set of infinitesimal operators that generate the one-parameter groups of transformations admissible for a differential equation (or a system) necessarily form a Lie algebra. Operators (22) with $P_0 = \frac{\partial}{\partial x_0}$ (the case $\alpha = \xi^0 \neq 0$) form the Lie algebra of the Galilean group. Therefore, the following statement holds.

Theorem 1. *For an arbitrary function* $F_\rho = \varphi(\rho,t)$, *the system of equations* (5), (6) *admits an* $\frac{n(n+3)}{2}$*-parameter group of transformations with the Lie algebra generated by operators* (22). *In the case where* F_ρ *does not depend on* x_0 *explicitly, system* (5), (6) *admits the Galilean group* $G(n)$.

Thereby, we have found the symmetry of system (5), (6) under any functional relationship $p = F(\rho,t)$. However, for some values of F, the symmetry of this system appears to be essentially wider. In order to list all the cases of symmetry extensions, it is necessary to get the set of solutions of Eq. (20) under various constraints on the parameters involved in this equation.

As a result of solving Eq. (20), we have found 12 cases of symmetry extension for the system in question. The corresponding functions φ_ν and the set of infinitesimal symmetry operators admitted by system (5), (6) are presented in Table.

Observe that, for all state equations that admit an extension of the symmetry (except the first one, where $\varphi = \varphi_1 = M\rho^{2/n}$), an arbitrary one-parameter invariance group of Euler equations is generated by an operator of the form

$$Z = (\alpha + \lambda x_0)\frac{\partial}{\partial x_0} + Ax_k \frac{\partial}{\partial x_k} + Bu^k \frac{\partial}{\partial u^k} + L\rho\frac{\partial}{\partial \rho}. \qquad (24)$$

Operator (24) with the constraint α = 0 is referred to as the *generator of scale transformations*. The solutions of system (5–6) that are invariant with respect to this operator are called *self-similar solutions*.

List of inequivalent cases for the equations of state and the corresponding operators

$\varphi = F_\rho$	Z_ν	Notes
$\varphi_1 = M\rho^{2/n}$	$Z_1 = \alpha P_0 + \lambda L_1 + \delta L_2 + n(\delta - \frac{\lambda}{2})L_3 + \theta L_0$, $L_0 = x_0^2\frac{\partial}{\partial x_0} + x_0 x_k \frac{\partial}{\partial x_k} + (x_k - x_0 u^k)\frac{\partial}{\partial u^k} + (-1)^n x_0 \rho \frac{\partial}{\partial \rho}$, $P_0 = \frac{\partial}{\partial x_0}$, $L_1 = x_0 \frac{\partial}{\partial x_0} + \frac{1}{2}x_k \frac{\partial}{\partial x_k} - \frac{1}{2}u^k \frac{\partial}{\partial u^k}$, $L_2 = x_k \frac{\partial}{\partial x_k} + u^k \frac{\partial}{\partial u^k}$, $L_3 = \rho\frac{\partial}{\partial \rho}$	
$\varphi_2 = M\rho^\theta$	$Z_2 = \alpha P_0 + \lambda L_1 + \delta L_2 + \frac{2}{\theta}(\delta - \frac{\lambda}{2})L_3$,	$\theta \neq 0$
$\varphi_3 = Mx_0^\sigma \rho^\theta$	$Z_3 = \lambda L_1 + \delta L_2 + \left[\frac{2}{\theta}(\delta - \frac{\lambda}{2}) - \frac{\sigma}{\theta}\lambda\right]L_3$	$\theta \neq 0$
$\varphi_4 = \rho^{2/n}G(\gamma)$, $\gamma = \rho^{2/n}x_0^\sigma$	$Z_4 = \lambda L_1 + \frac{\kappa}{n}\lambda L_2 + n\lambda(\frac{\kappa}{n} - \frac{1}{2})L_3$	$\sigma = 1 - \frac{2\kappa}{n}$
$\varphi_5 = Mx_0^\sigma$	$Z_5 = \lambda L_1 + \frac{\sigma+1}{2}\lambda L_2 + (\mu - \frac{n}{2}\lambda)L_3$	
$\varphi_6 = x_0^{-1}G(\rho)$	$Z_6 = \lambda L_1$	
$\varphi_7 = \Phi(\rho^{2/n}x_0^\sigma)$	$Z_7 = \lambda L_1 + \frac{\lambda}{2}L_2 + n\frac{\sigma}{2}\lambda L_3$	$\sigma = 1 - \frac{2\kappa}{n}$
$\varphi_8 = \Phi(\rho^{2/n}e^{-\sigma x_0})$	$Z_8 = \alpha P_0 + n\frac{\sigma}{2}\alpha L_3$	
$\varphi_9 = e^{\sigma x_0}\Phi(\rho)$	$Z_9 = \alpha P_0 + \frac{\sigma}{2}\alpha L_2$	
$\varphi_{10} = x_0^\sigma \Phi(\rho)$	$Z_{10} = \lambda\left[L_1 + \frac{\sigma+1}{2}L_2\right]$	
$\varphi_{11} = \Phi(\rho)$	$Z_{11} = \alpha P_0 + \lambda\left[L_1 + \frac{1}{2}L_2\right]$	
$\varphi_{12} = \rho^\kappa \Phi(x_0)$	$Z_{12} = \delta(L_3 + \frac{\kappa}{2}L_2)$	

Theorem 2. *The symmetry extension of system* (5), (6) *is possible in 12 cases presented in Table. The maximal invariance group for this system is the*

$\left[\frac{n(n+3)}{2}+4\right]$*-parameter projective group*

This group is admissible for system (5), (6) *if and only if* $F_\rho = c\rho^{2/n}$.

Remark 1. Observe that the one-dimensional case is special. Namely, the two first equations in system (17) appear only when $n > 1$. As was demonstrated in [9], for the equation of state of the form $p = \frac{M}{3}\rho^3$, which describes an ideal polytropic gas, system (5), (6) under $n = 1$, admits an infinite group. Due to this fact, the general solution was obtained for system (5), (6) in this case [9].

3. Invariant solutions of system (5), (6) and Rankine–Hugoniot conditions. In this section, we find solutions of system (5), (6) in the case $n = 1$ that are compatible with the Rankine–Hugoniot conditions.

In this case, each operator that generates a one-parameter group of admissible transformations for system (5), (6) can be presented as

$$Z = (\alpha + \lambda t + \theta t^2)\frac{\partial}{\partial t} + (\mu t + \nu + Ax + \theta xt)\frac{\partial}{\partial x} + (\theta x + \mu + Bu - \theta tu)\frac{\partial}{\partial u} + (L - \theta t)\rho\frac{\partial}{\partial \rho}, \qquad (25)$$

where $t = x_0$, $x = x_1$; α, δ, θ, λ, μ, and ν are arbitrary constant parameters, $B = \delta - \lambda/2$, $A = B + \lambda$, and L is a function of these parameters.

Following the well-known technique [1, 2], we find the solutions of (5), (6) that are invariant with respect to a one-parameter group of transformations with infinitesimal symmetry operator of the form (25) by means of the transition to invariant variables, which can be expressed via solutions of the equation

$$ZJ(t, x, u, \rho) = 0. \qquad (26)$$

In order to list the cases where invariant solutions are applicable to the description of a point explosion in the medium with the state equation $p = F(\rho, t)$, it is necessary to analyze the invariance of the manifold determined by the boundary conditions with respect to transformations generated by operator (25).

The role of "boundary conditions" in the case of point explosion is played by the Rankine–Hugoniot conditions [10]

$$\rho_2(u_2 - D) + \rho_1 D = 0, \quad \rho_2(u_2 - D)^2 + p_2 = \rho_1 D^2 + p_1, \qquad (27)$$

which represent the discontinuity of main characteristics of shock waves in a material medium. In formula (27), the quantities with index 2 describe the values of these functions behind the shock wave front, and those with the index 1 before it. The medium is expected to be motionless, $u_1 = 0$, D is the velocity of the shock wave front, and p_1, ρ_1 are constants, $\rho_1 > 0$.

It is obvious that, in the one-dimensional case, the motion of the shock wave front in the point explosion problem can be determined by a relation $x_{\text{front}} = g(t)$ with a certain function g. Therefore, the manifold M defined by the boundary conditions (27) is determined by the system

$$x - g(t) = 0, \qquad (28)$$

$$\rho[u - \dot{g}(t)] + \rho_1\dot{g}(t) = 0, \qquad (29)$$

$$\rho[u - \dot{g}(t)]^2 + p(\rho, t) - \rho_1\dot{g}^2(t) - p_1 = 0, \qquad (30)$$

where ρ_1, p_1 are constants that are equal to initial values of the density and the pressure in the medium, correspondingly, $g^{(t)}$ is the unknown function, and $\dot{g}(t)=dg(t)/dt$.

Note that infinitesimal operator of the form (25) with coefficients involving quadratic terms is admissible if and only if $p=\frac{M}{3}\rho^3$. In this case system (5), (6) has a general solution. Therefore, we can set $\theta = 0$ in (25).

Applying the infinitesimal invariance criterion (15) to the manifold M, we obtain the system

$$\mu - L\dot{g} = 0, \tag{31}$$

$$\nu + \mu t + Ag(t) - (\alpha + \lambda t)\dot{g}(t) = 0, \tag{32}$$

$$\frac{\rho_1^2}{\rho}\dot{g}^2(t)(L+2B) + \rho L p_\rho + (\alpha+\lambda t)p_t - 2\rho_1(\mu + B\dot{g})\dot{g} = 0. \tag{33}$$

To satisfy condition (31) in the case $L \neq 0$, it is necessary that $g(t) = St + R$, where S and R are some constants, $S \neq 0$. Formula (32) implies that $L = -B$. Analyzing the functions $\varphi = F_\rho$ and the corresponding operators Z_ν (see Table 1), we conclude that the case $L \neq 0$ is possible only for a state equation of the form

$$p = c - \frac{M}{\rho},\ M = (S\rho_1)^2, \tag{34}$$

which corresponds to the function $\varphi = M\rho^\sigma$ with $\sigma = -2$.

Since $L = 0$ for the other cases, conditions (31–33) can be represented as

$$\mu = L = 0, \tag{35}$$

$$\nu + Ag - (\alpha+\lambda t)\dot{g} = 0, \tag{36}$$

$$2B\frac{\rho_1^2}{\rho}\dot{g}^2 - 2\rho_1\dot{g}^2 B + (\alpha+\lambda t)p_t = 0. \tag{37}$$

It is necessary to analyze condition (37) now. Note that the operators listed in Table 1 can be partitioned into two groups according to the criterion whether L is a multiple of B. Thereby, the first group consists of Z_1, Z_2, Z_4, Z_{11}, and Z_{12}. For the operator Z_4, the restrictions (35) imply that the corresponding function φ_4 does not depend on t and, therefore, coincides with φ_{11}. For the operator Z_{12}, the restriction $L = 0$ makes the operator to vanish.

By virtue of (35), the functions φ_1, φ_2, and φ_{11} correspond to the same infinitesimal symmetry operator

$$Z_{II} = (\alpha+\lambda t)\frac{\partial}{\partial t} + (\nu + \lambda x)\frac{\partial}{\partial x}. \tag{38}$$

Therefore, we can consider these three cases together. Denote the function that corresponds to operator (38) by $\Phi_{II}(\rho) = \varphi_1(\rho) = \varphi_2(\rho) = \varphi_{11}(\rho)$. It is clear that $p_{II} = \Phi_{II}(\rho)H(t)$ with a certain function $H(t)$. For operator (38), Eq. (37) is equivalent to the condition

$$(\alpha+\lambda t)(p_{II})_t = 0, \tag{39}$$

which leads to $H = c = \text{const}$.

Draw our attention to other cases. If $L = 0$, then the functions φ_7 and φ_8 coincide with φ_{11}, and the functions φ_3, φ_5, φ_6, and φ_{10} can be represented as

$$\varphi_{III} = t^\sigma \dot{\Phi}(\rho) \tag{40}$$

due to the fact that the infinitesimal symmetry operator for all these cases is the same, namely

$$Z_{III} = \lambda t \frac{\partial}{\partial t} + (\nu + \lambda x)\frac{\partial}{\partial x} + Bu\frac{\partial}{\partial u}, \tag{41}$$

where $A = \frac{\lambda}{2}(\sigma + 2)$, $B = \frac{\lambda\sigma}{2}$, $\sigma \neq 0$.

Formula (37) enables one to recover p_{III},

$$p_{III} = t^{\sigma}\Phi(\rho) + H(t). \tag{42}$$

Observe that the derivative of p_{III} with respect to t can be expressed as

$$(p_{III})_t = \frac{\sigma}{t}(p_{III} - H) + \frac{dH}{dt}. \tag{43}$$

So, condition (37) is equivalent to the equation

$$\sigma p_1 - \sigma H + t\frac{dH}{dt} = 0. \tag{44}$$

Hence, $H(t) = c_1 t^{\sigma} + c_2$ and

$$p_{III} = t^{\sigma}\Phi(\rho) + c, \; c = p_1. \tag{45}$$

The last case to be considered is $F_{\rho} = \varphi_9$. Then

$$p_{IV} = e^{2\kappa t}\Phi(\rho) + H(t), \; Z_{IV} = \alpha\left(\frac{\partial}{\partial t} + \kappa x\frac{\partial}{\partial x} + \kappa u\frac{\partial}{\partial u}\right), \tag{46}$$

and, hence, $B = \kappa\alpha$. Expressing $(p_{IV})_t$ in terms of p, H, and $\frac{dH}{dt}$ and using formulae (29) and (30), we find that

$$p_{IV} = e^{2\kappa t}\Phi(\rho) + p_1. \tag{47}$$

Hereby, all the functional relationships $p = F(\rho,t)$, for which the corresponding invariance solutions are compatible with the Rankine–Hugoniot conditions, are listed. In what follows, we determine the function $g(t)$ for each of these cases and verify that g=const, i.e. that a shock wave really propagates in a medium. Solving Eq. (36), we obtain

$$g_{II}(t) = \begin{cases} c_2 + \frac{\nu}{\alpha}t, & if \quad \lambda = 0, \\ c_2(\alpha + \lambda t) - \frac{\nu}{\alpha}, & if \quad \lambda \neq 0, \end{cases} \tag{48}$$

in the case $Z = Z_{\mathrm{II}}$, $p = \Phi(\rho) + p_1$;

$$g_{III}(t) = \begin{cases} \frac{\nu}{\lambda}\ln t + c_3, & if \quad \theta = \frac{\sigma}{2} + 1 = 0, \\ c_3 t^{\theta} - \frac{\nu}{\lambda\theta}, & if \quad \theta \neq 0, \end{cases} \tag{49}$$

in the case $Z = Z_{\mathrm{III}}$, $p = t^{\sigma}\Phi(\rho) + p_1$; and

$$g_{IV}(t) = c_4 e^{\kappa t} - \frac{\nu}{\kappa\alpha} \tag{50}$$

for $Z = Z_{\mathrm{IV}}$, $p = e^{2\kappa t}\Phi(\rho) + p_1$. In expressions (48–50), c_2, c_3, and c_4 are arbitrary constants.

Therefore, if we restrict the consideration to the symmetry operators that do not contain any quadratic terms in their coefficients, the following theorem holds.

Theorem 3. *The four classes of invariant solutions to system* (5), (6) *compatible with the Rankine–Hugoniot conditions under* $n = 1$ *are:*

a) *solutions that are invariant with respect to the one-parameter subgroup generated by the operator*

$$Z_I = (\alpha + \lambda t)\frac{\partial}{\partial t} + (\nu - BSt + Ax)\frac{\partial}{\partial x} + Bu\frac{\partial}{\partial u} + L\rho\frac{\partial}{\partial \rho},$$

where $L = -B$, α, λ, ν, A, B, *and* S *are constants*, $B \neq 0$ *and* $S \neq 0$, *if*

$$p = c - \frac{(S\rho_1)^2}{\rho};$$

b) *solutions invariant with respect to the one-parameter subgroup generated by the operator*

$$Z_{II} = (\alpha + \lambda t)\frac{\partial}{\partial t} + (\nu + \lambda x)\frac{\partial}{\partial x},$$

if $p = \Phi(\rho) + p_1$;

c) *solutions invariant with respect to the one-parameter subgroup generated by the operator*

$$Z_{III} = \lambda t\frac{\partial}{\partial t} + (\nu + Ax)\frac{\partial}{\partial x} + Bu\frac{\partial}{\partial u},$$

where $A = \lambda(\frac{\sigma}{2} + 1)$, $B = \frac{\lambda\sigma}{2}$, if $p = t^{\sigma}\Phi(\rho)$;

d) *solutions invariant with respect to the one-parameter subgroup generated by the operator*

$$Z_{IV} = \alpha\frac{\partial}{\partial t} + (\nu + \kappa\alpha x)\frac{\partial}{\partial x} + \kappa\alpha u\frac{\partial}{\partial u},$$

if $p = e^{2\kappa t}\Phi(\rho) + p_1$.

Hereby, the cases where the boundary-value problem (5), (6), (27) admits invariant solutions are exhaustively described. Some special cases are considered in [11].

Conclusion. In this paper, the group analysis of a system of Euler equations with the equation of state of a medium is carried out, and the cases where the point explosion problem has invariant solutions are listed. The group classification provided for the state equations is of great practical importance, because there is no unified analytical expression that satisfactorily describes the relationship of thermodynamic parameters of a liquid throughout the domain, where these parameters vary. In many cases, the state equations listed in Table coincide with functional relationships known as the equations of state for a liquid in the limited ranges of thermodynamic parameters.

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ІНВАРІАНТНІ РОЗВ'ЯЗКИ СИСТЕМИ РІВНЯНЬ ЕЙЛЕРА,
ЩО ЗАДОВОЛЬНЯЮТЬ УМОВИ РЕНКІНА—ГЮГОНІО

Розглядаються рівняння гідродинаміки з певними додатковими обмеженнями. Для пошуку інваріантних розв'язків системи рівнянь Ейлера, що задовольняють умови Ренкіна—Гюгоніо, застосовуються теоретико-групові методи.

***Ключові слова**: теоретико-групові методи, інваріантні розв'язки, рівняння Ейлера.*

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ИНВАРИАНТНЫЕ РЕШЕНИЯ СИСТЕМЫ УРАВНЕНИЙ ЭЙЛЕРА,
УДОВЛЕТВОРЯЮЩИЕ УСЛОВИЯМ РЕНКИНА—ГЮГОНИО

Рассматриваются уравнения гидродинамики с некоторыми дополнительными ограничениями. Для нахождения инвариантных решений системы уравнений Эйлера, удовлетворяющих условиям Ренкина—Гюгонио, применяются теоретико-групповые методы.

***Ключевые слова:** теоретико-групповые методы, инвариантные решения, уравнения Эйлера.*