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On solvability of inhomogeneous boundary-value problems in Sobolev—Slobodetskiy spaces

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We investigate the most general class of Fredholm one-dimensional boundary-value problems in the Sobolev—Slobodetskiy spaces. Boundary conditions of these problems may contain a derivative of the whole or fractional order. It is established that each of these boundary-value problems corresponds to a certain rectangular numerical characteristic matrix with kernel and cokernel having the same dimension as the kernel and cokernel of the boundary-value problem. The sufficient conditions for the sequence of the characteristic matrices of a specified boundary-value problems to converge are found.

Keywords: inhomogeneous boundary-value problem, Sobolev—Slobodetskiy space, Fredholm operator, index of operator.

Introduction. Boundary-value problems for systems of ordinary differential equations arise in many problems of analysis and its applications. Unlike Cauchy problems, the solutions to such problems may not exist or may not be unique. Thus, it is interesting to investigate the nature of the solvability of inhomogeneous boundary-value problems in the functional Sobolev and Sobolev—Slobodetskiy spaces and the dependence of their solutions on the parameter. For Fredholm boundary-value problems, similar issues have been investigated in papers [1-5]. The case of underdefined or overdefined boundary-value problems in Sobolev spaces was investigated in paper [6].

Statement of the problem. Let a finite interval $(a, b) \subset \mathbb{R}$ and parameters

$$\{m, l\} \subset \mathbb{N}, \quad s \in (0, \infty) \setminus \mathbb{N}, \quad 1 \leqslant p < \infty,$$

be given. By $W_p^n := W_p^n([a, b]; \mathbb{C})$, we denote a complex Sobolev space and set $W_p^0 := L_p$. By $(W_p^s)^m := W_p^s([a, b]; \mathbb{C}^m)$ and $(W_p^n)^{m \times m} := W_p^n([a, b]; \mathbb{C}^{m \times m})$, we denote the Sobolev spaces of

vector functions and matrix functions, respectively, with elements from the function space W_p^n. By $\|\cdot\|_{n,\,p}$, we denote the norms in these spaces. They are defined as the sums of the corresponding norms of all elements of a vector-valued or matrix-valued function in W_p^n. The space of functions (scalar functions, vector functions, or matrix functions) in which the norm is introduced is always clear from the context. For $m=1$, all these spaces coincide. It is known that W_p^n are separable Banach spaces.

We denote, by $W_p^s := W_p^s([a,b]; \mathbb{C})$, where $1 \leqslant p < \infty$ and $s > 1$ is not integer, the Sobolev—Slobodetskiy space of all complex-valued functions belonging to Sobolev space $W_p^{[s]}$ and satisfying the condition

$$\|f\|_{n,\,p} := \|f\|_{[s],\,p} + \left(\int\limits_a^b \int\limits_a^b \frac{\left| f^{[s]}(x) - f^{[s]}(y) \right|^p}{|x-y|^{1+\{s\}p}} dx dy \right)^{1/p} < +\infty,$$

where $[s]$ is the integer part, and $\{s\}$ is the fractional part of the number s. Here, we recall that $\|\cdot\|_{[s],\,p}$ is the norm in the Sobolev space $W_p^{[s]}$. This equality defines the norm $\|f\|_{s,\,p}$ in the space W_p^s.

Consider a linear boundary-value problem on a finite interval (a,b) for the system of m first-order scalar differential equations

$$(Ly)(t) := y'(t) + A(t)y(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c, \tag{2}$$

where the matrix function $A(\cdot)$ belongs to the space $(W_p^s)^{m \times m}$, the vector function $f(\cdot)$ belongs to the space $(W_p^s)^m$, the vector c belongs to the space $\mathbb{C}^l$, and B is a linear continuous operator

$$B: (W_p^{s+1})^m \to \mathbb{C}^l. \tag{3}$$

The boundary condition (2) consists of l scalar boundary conditions for the system of m differential equations of the first order. We represent vectors and vector functions in the form of columns. A solution to the boundary-value problem (1), (2) is understood as a vector function $y \in (W_p^{s+1})^m$ satisfying Eq. (1) for $s > 1 + 1/p$ everywhere and, for $s \leqslant 1 + 1/p$, almost everywhere on (a,b) and equality (2) specifying l scalar boundary conditions. The solutions to Eq. (1) fill the space $(W_p^{s+1})^m$, if its right-hand side $f(\cdot)$ runs through the space $(W_p^s)^m$. Hence, the boundary condition (2) is the most general condition for this equation and includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multipoint problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives of integer or fractional order k of required vector — functions, where $0 < k < s+1$.

The main purpose of this work is to establish whether the boundary-value problem (1), (2) has the Fredholm property; to find its index and the dimension of the cokernel and the kernel of the operator of an inhomogeneous boundary-value problem in terms of the properties of a special rectangular numerical matrix and to investigate its stability. In the case of Sobolev spaces of integer order, similar results were obtained in [6].

Main results. We rewrite the inhomogeneous boundary-value problem (1), (2) in the form of a linear operator equation

$$(L, B)y = (f, c),$$

where (L, B) is a linear operator in the pair of Banach spaces

$$(L, B): (W_p^{s+1})^m \to (W_p^s)^m \times \mathbb{C}^l. \tag{5}$$

Let X and Y be Banach spaces. Recall that a linear continuous operator $T: X \to Y$ is called a Fredholm operator, if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If the operator is a Fredholm one, then its range $T(X)$ is closed in Y, and the index

$$\operatorname{ind} T := \dim \ker T - \dim(Y/T(X))$$

is finite (see, e.g., [7], Lemma 19.1.1).

Theorem 1. *The linear operator* (5) *is a bounded Fredholm operator with index* $m - l$.

Denote, by $Y(\cdot) \in (W_p^s)^{m \times m}$, the unique solution to a linear homogeneous matrix equation

$$Y'(t) + A(t)Y(t) = O_m, \quad t \in (a, b), \tag{6}$$

with the initial condition

$$Y(a) = I_m. \tag{7}$$

Here, O_m are zero matrices, and I_m are identity $(m \times m)$ matrices. The unique solution to the Cauchy problem (6), (7) belongs to the space $(W_p^{s+1})^{m \times m}$.

By $[BY]$, we denote a numerical matrix of dimension $(m \times l)$ whose i-th column is a result of the action of the operator B from (3) on i-th column of the matrix function $Y(\cdot)$, $i \in \{1, \ldots, m\}$.

Definition 1. *A rectangular numerical matrix*

$$M(L, B) = [BY] \in \mathbb{C}^{m \times l}, \tag{8}$$

is called the characteristic matrix for the inhomogeneous boundary-value problem (1), (2).

Here, m is the number of scalar differential equations of system (1), and l is the number of scalar boundary conditions.

Theorem 2. *The dimensions of the kernel and cokernel of operator* (5) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix* (8), *respectively:*

$$\dim \ker(L, B) = \dim \ker(M(L, B)),$$

$$\dim \operatorname{coker}(L, B) = \dim \operatorname{coker}(M(L, B)).$$

A criterion for the invertibility of the operator (L, B) follows from Theorem 2, i.e., the condition under which problem (1), (2) possesses a unique solution, and this solution continuously depends on the right-hand sides of the differential equation and the boundary condition.

Corollary 1. *Operator* (5) *is invertible, if and only if* $l = m$, *and the square matrix* $M(L, B)$ *is nondegenerate.*

Application. In addition to problem (1), (2), we consider the sequence of inhomogeneous boundary-value problems

$$L(k)\,y(t,k) := y'(t,k) + A(t,k)\,y(t,k) = f(t,k), \quad t \in (a,b),\tag{9}$$

$$B(k)\,y(\cdot,k) = c(k), \quad k \in \mathbb{N},\tag{10}$$

where the matrix functions $A(\cdot,k)$, the vector functions $f(\cdot,k)$, the vectors $c(k)$ and linear continuous operators $B(k)$ satisfy the above conditions for problem (1), (2).

With the boundary-value problem (9), (10), we associate a sequence of linear continuous operators $(L(k), B(k)) : (W_p^{s+1})^m \to (W_p^s)^m \times \mathbb{C}^l$
and a sequence of characteristic matrices

$$M(L(k), B(k)) := [B(k)Y(\cdot,k)] \subset \mathbb{C}^{m \times l}$$

depending on the parameter $k \in \mathbb{N}$.

We now formulate a sufficient condition for the convergence of the characteristic matrices $M(L(k), B(k))$ to the matrix $M(L, B)$.

Theorem 3. *If the sequence of operators $(L(k), B(k))$ converges strongly to the operator (L, B) for $k \to \infty$, then the sequence of characteristic matrices $M(L(k), B(k))$ converges to the matrix $M(L, B)$.*

Corollary 2. *Under the assumptions from Theorem 3, the following inequalities hold*

$$\dim\ \ker(L(k), B(k)) \leqslant \dim\ \ker(L, B),$$

$$\dim\ \operatorname{coker}(L(k), B(k)) \leqslant \dim\ \operatorname{coker}(L, B).$$

for sufficiently large k.

In particular:

1) If $l = m$ and the operator (L, B) is invertible, then the operators $(L(k), B(k))$ are also invertible for large *k;*

2) If the boundary-value problem (1), (2) has a solution for any values of the right-hand sides, then the boundary-value problems (9), (10) also have a solution for large *k;*

3) If the boundary-value problem (1), (2) has a unique solution, then problems (9), (10) also have a unique solution for each sufficiently large *k.*

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ПРО РОЗВ'ЯЗНІСТЬ НЕОДНОРІДНИХ КРАЙОВИХ ЗАДАЧ У ПРОСТОРАХ СОБОЛЄВА–СЛОБОДЕЦЬКОГО

Досліджено найбільш широкий клас нетерових одновимірних крайових задач у просторах Соболєва–Слободецького. Крайові умови в них можуть містити похідні розв'язку цілого або дробового порядку. Встановлено, що кожній із таких крайових задач відповідає деяка прямокутна числова характеристична матриця, вимірність ядра і коядра якої збігаються відповідно з вимірністю ядра і коядра крайової задачі. Знайдені достатні умови збіжності послідовності характеристичних матриць розглянутих крайових задач.

Ключові слова: *неоднорідна крайова задача, простір Соболєва–Слободецького, нетерів оператор, індекс оператора.*

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О РАЗРЕШИМОСТИ НЕОДНОРОДНЫХ КРАЕВЫХ ЗАДАЧ В ПРОСТРАНСТВАХ СОБОЛЕВА–СЛОБОДЕЦКОГО

Исследуется наиболее широкий класс нетеровых одномерных краевых задач в пространствах Соболева–Слободецкого. Краевые условия в них могут содержать производные решения целого или дробного порядка. Показано, что каждой из таких краевых задач соответствует некоторая прямоугольная числовая характеристическая матрица, размерность ядра и коядра которой совпадают соответственно с размерностью ядра и коядра краевой задачи. Найдены достаточные условия сходимости последовательности характеристических матриц рассмотренных краевых задач.

Ключевые слова: *неоднородная краевая задача, пространство Соболева–Слободецкого, нетеров оператор, индекс оператора.*