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Logarithmic capacity and Riemann and Hilbert problems for generalized analytic functions

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The study of the Dirichlet problem with arbitrary measurable boundary data for harmonic functions in the unit disk is due to the famous Luzin dissertation. Later on, the known monograph of Vekua was devoted to boundary-value problems for generalized analytic functions, but only with Hölder continuous boundary data. The present paper contains theorems on the existence of nonclassical solutions of Riemann and Hilbert problems for generalized analytic functions with sources whose boundary data are measurable with respect to the logarithmic capacity. Our approach is based on the geometric interpretation of boundary values in comparison with the classical operator approach in PDE. On this basis, one can derive the corresponding existence theorems for the Poincaré problem on directional derivatives to the Poisson equations and, in particular, for the Neumann problem with arbitrary boundary data that are measurable with respect to the logarithmic capacity. These results can be also applied to semilinear equations of mathematical physics in anisotropic inhomogeneous media.

Keywords: *Hilbert and Riemann boundary-value problems, generalized analytic functions, logarithmic capacity.*

1. Introduction. The well-known monograph by Vekua [1] is devoted to the theory of *generalized analytic functions*, i.e., continuous complex valued functions $h(z)$ of the complex-variable $z=x+iy$ with generalized first partial derivatives by Sobolev in domains $D\subseteq\mathbb{C}$ satisfying a.e. equations of the form

$$\partial_{\bar z}h+ah+bh=g,\quad \partial_{\bar z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right), \tag{1}$$

where the complex-valued functions a,b and g belong to a class L^p with $p>2$. If a and $b\equiv 0$ and g is real-valued, then we call h by a *generalized analytic function with the source* g.

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The research of the Dirichlet problem for harmonic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with arbitrary measurable boundary data is due to the Luzin dissertation, see its original text [2] and its reprint [3]. Later on, a series of results on various boundary-value problems have been formulated and proved in terms of the logarithmic capacity, see its definition and properties, e.g., in [4]. The base are the following analogs of the Luzin theorems in [5], see also [6], where *q.e.* means *quasi-everywhere* with respect to the logarithmic capacity.

Theorem A. *Let* $\varphi : [a, b] \to \mathbb{R}$ *be measurable with respect to the logarithmic capacity. Then there is a continuous* $\Phi : [a, b] \to \mathbb{R}$ *with* $\Phi'(x) = \varphi(x)$ *q.e.*

Theorem B. *Let* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be measurable with respect to the logarithmic capacity and finite q.e. Then a space of harmonic functions* u *in* $\mathbb{D}$ *with the angular limits* $u(z) \to \varphi(\zeta)$ *as* $z \to \zeta$ *q.e. on* ∂D *has the infinite dimension.*

On the basis of Theorem B, the following result on the Hilbert problem was obtained:

Theorem C. *Let* $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, *be of bounded variation and* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be measurable with respect to the logarithmic capacity. Then there is a space of analytic functions* $f : \mathbb{D} \to \mathbb{C}$ *of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \quad \text{q.e. on } \partial\mathbb{D}. \tag{2}$$

Then this result was extended to arbitrary smooth (C^1) domains. Moreover, the following result was proved in [7] (see the next section for definitions):

Theorem D. *Let* D *be a Jordan domain with the quasihyperbolic boundary condition,* ∂D *have a tangent q.e.,* $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, *be of countable bounded variation, and let* $\varphi : \partial\mathbb{D} \to \mathbb{R}$ *be measurable with respect to the logarithmic capacity. Then there is a space of analytic functions* $f : \mathbb{D} \to \mathbb{C}$ *of the infinite dimension with the angular limits*

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \quad \text{q.e. on } \partial D. \tag{3}$$

2. Hilbert problem and angular limits. Recall that the classic boundary-value *problem of Hilbert* was formulated as follows: To find an analytic function f in a domain D bounded by a rectifiable Jordan contour C that satisfies the boundary condition

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)} \cdot f(z)\} = \varphi(\zeta) \quad \forall \zeta \in C, \tag{4}$$

where the *coefficient* λ and the *boundary date* φ of the problem are continuously differentiable with respect to the natural parameter s and $\lambda \neq 0$ everywhere on C. The latter allows one to consider that $|\lambda(\zeta)| \equiv 1$ on C. Note that the quantity $\mathrm{Re}\{\overline{\lambda} \cdot f\}$ in (4) means a projection of f into the direction λ interpreted as vectors in $\mathbb{R}^2$, see history comments, e.g., in [5].

A straight line L is said to be *tangent* to a curve Γ in $\mathbb{C}$ at a point $z_0 \in \Gamma$, if

$$\limsup_{z\to z_0,\, z\in\Gamma} \frac{\mathrm{dist}(z, L)}{|z - z_0|} = 0. \tag{5}$$

Let D be a Jordan domain in $\mathbb{C}$ with a tangent at a point $\zeta \in \partial D$. A path in D terminating at ζ is called *nontangential*, if its part in a neighborhood of ζ lies inside of an angle with the vertex at ζ that is less than a straight angle. The limit along all nontangential paths at ζ is called *angular* at the point. Following [7], we say that a Jordan curve Γ in $\mathbb{C}$ is *almost smooth*, if Γ has a tangent *q.e*. In particular, Γ is almost smooth, if Γ has a tangent at all its points except a countable collection.

Recall also that the *quasihyperbolic distance* between points z and z_0 in a domain $D\subset\mathbb{C}$ is the quantity

$$k_D(z,z_0):=\inf_{\gamma}\int_{\gamma}\frac{ds}{d(\zeta,\partial D)}, \tag{6}$$

where $d(\zeta,\partial D)$ denotes the Euclidean distance from the point $\zeta\in D$ to ∂D, and the infimum is taken over rectifiable curves γ joining the points z and z_0 in D.

It is said by [8] that a domain D satisfies the *quasihyperbolic boundary condition*, if there exist constants a and b and a point $z_0\in D$ such that

$$k_D(z,z_0)\leqslant a+b\ln\frac{d(z_0,\partial D)}{d(z,\partial D)}\quad \forall z\in D.$$

Every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition, see e.g., [9] for its discussion.

Given a Jordan domain D in $\mathbb{C}$, we call $\lambda:\partial D\to\mathbb{C}$ a *function of bounded variation*, write $\lambda\in\mathcal{BV}(\partial D)$, if

$$V_\lambda(\partial D):=\sup\sum_{j=1}^{k}\left|\lambda(\zeta_{j+1})-\lambda(\zeta_j)\right|<\infty, \tag{7}$$

where the supremum is taken over all finite collections of points $\zeta_j\in\partial D$, $j=1,\ldots,k$, with the cyclic order meaning that ζ_j lies between ζ_{j+1} and ζ_{j-1} for every $j=1,\ldots,k$. Here, we assume that $\zeta_{k+1}=\zeta_1=\zeta_0$. The quantity $V_\lambda(\partial D)$ is called the *variation of the function* λ.

Now, we call $\lambda:\partial D\to\mathbb{C}$ a function of *countable bounded variation*, write $\lambda\in\mathcal{CBV}(\partial D)$, if there is a countable collection of mutually disjoint arcs γ_n of ∂D, $n=1,2,\ldots$ on each of which the restriction of λ is of bounded variation and the set $\partial D\setminus\cup\gamma_n$ has the zero logarithmic capacity. In particular, the latter holds true, if the set $\partial D\setminus\cup\gamma_n$ is countable. It is clear that such functions can be singular enough.

Theorem 1. *Let D be a Jordan domain with the quasihyperbolic boundary condition, ∂D have a tangent q.e., $\lambda:\partial\mathbb{D}\to\mathbb{C}$, $|\lambda(\zeta)|\equiv 1$, be in $\mathcal{CBV}(\partial D)$, and let $\varphi:\partial\mathbb{D}\to\mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $g:D\to\mathbb{R}$ is in $L^p(D), p>2$. Then there exist generalized analytic functions $h:D\to\mathbb{C}$ with the source g that have the angular limits

$$\lim_{z\to\zeta}\mathrm{Re}\{\overline{\lambda(\zeta)}\cdot h(z)\}=\varphi(\zeta)\quad q.e.\ on\ \partial D. \tag{8}$$

Furthermore, the space of such functions h *has the infinite dimension.*

Later on, we often apply the *logarithmic (Newtonian) potential $\mathcal{N}_G$ of sources $G\in L^p(\mathbb{C})$, $p>2$,* with compact supports given by the formula:

$$\mathcal{N}_G(z) := \frac{1}{2\pi}\int_{\mathbb{C}} \ln|z-w|G(w)\,dm(w). \tag{9}$$

By Lemma 3 in [4], $\mathcal{N}_G \in W^{2,p}_{\text{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{C})$, $\alpha := (p-2)/p$, $\Delta\mathcal{N}_G = G$ a.e.

Proof. Extending the function g by zero outside of D and setting $P = \mathcal{N}_G$ with $G = 2g$, $U = P_x$ and $V = -P_y$, we have that $U_x - V_y = G$ and $U_y + V_x = 0$. Thus, elementary calculations show that $H := U + iV$ is just a generalized analytic function with the source g. Moreover, the function

$$\varphi_*(\zeta) := \lim_{z\to\zeta} \text{Re}\{\overline{\lambda(\zeta)}\cdot H(z)\} = \text{Re}\{\overline{\lambda(\zeta)}\cdot H(\zeta)\}, \quad \forall \zeta \in \partial D, \tag{10}$$

is measurable with respect to the logarithmic capacity, because the function H is continuous in the whole plane $\mathbb{C}$.

By Theorem 2 in [7], see also Theorems 5.1 and 6.1 in [10], there exist analytic functions $\mathcal{A}$ in D with the angular limits

$$\lim_{z\to\zeta} \text{Re}\{\overline{\lambda(\zeta)}\cdot A(z)\} = \Phi(\zeta) \quad \text{q.e. on } \partial D \tag{11}$$

for the function $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta), \zeta \in \partial D$. The space of such analytic functions $\mathcal{A}$ has the infinite dimension, see, e.g., Corollary 8.1 in [10].

Finally, it is clear that the functions $h := \mathcal{A} + H$ are desired generalized analytic functions with the source g satisfying the Hilbert condition (8). Thus, the space of such functions h has really the infinite dimension.

Remark 1. As follows from the proof of Theorems 1, the generalized analytic functions h with a source $g \in L^p$, $p > 2$, satisfying the Hilbert boundary condition (8) q.e. in the sense of the angular limits can be represented in the form of the sums $\mathcal{A} + H$ with analytic functions $\mathcal{A}$ satisfying the corresponding Hilbert boundary condition (11) and a generalized analytic function $H = U + iV$ with the same source g, $U = P_x$ and $V = -P_y$, where P is the logarithmic (Newtonian) potential $\mathcal{N}_G$ with $G = 2g$ in the class $W^{2,p}_{\text{loc}}(\mathbb{C}) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{C})$, $\alpha = (p-2)/p$, that satisfies the equation $\Delta P = G$.

In particular, in the case $\lambda \equiv 1$, we obtain the corresponding consequence of Theorem 1 on the Dirichlet problem for the generalized analytic functions.

3. Hilbert problem and Bagemihl—Seidel systems. Let D be a domain in $\mathbb{C}$, whose boundary consists of a finite collection of mutually disjoint Jordan curves. A family of mutually disjoint Jordan arcs $J_\zeta : [0,1] \to \overline{D}, \zeta \in \partial D$, with $J_\zeta([0,1)) \subset D$ and $J_\zeta(1) = \zeta$ that is continuous in the parameter ζ is called a *Bagemihl—Seidel system* or, in short, of *class* $\mathcal{BS}$, see [11].

Lemma 1. *Let D be a bounded domain in $\mathbb{C}$ whose boundary consists of a finite number of mutually disjoint Jordan curves, $\lambda : \partial\mathbb{D} \to \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, $\varphi : \partial\mathbb{D} \to \mathbb{R}$ and $\psi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ is a family of Jordan arcs of class $\mathcal{BS}$ in D and that a function $g : D \to \mathbb{R}$ is of the class $L^p(D)$ for some $p > 2$. Then there is a generalized analytic function $f : D \to \mathbb{C}$ with the source g such that

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)}\cdot h(z)\} = \varphi(\zeta), \quad (12)$$

$$\lim_{z\to\zeta} \mathrm{Im}\{\overline{\lambda(\zeta)}\cdot h(z)\} = \psi(\zeta) \quad (13)$$

along γ_ζ *q.e. on* ∂D.

Proof. As in the proof of Theorem 1, the function $H = U + iV$ with $U = P_x$ and $V = -P_y$, where $P = N_G$ with $G = 2g$, is a generalized analytic function with the source g. Moreover, the functions

$$\varphi_*(\zeta) := \lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)}\cdot H(z)\} = \mathrm{Re}\{\overline{\lambda(\zeta)}\cdot H(\zeta)\}, \quad \forall\zeta\in\partial D, \quad (14)$$

$$\psi_*(\zeta) := \lim_{z\to\zeta} \mathrm{Im}\{\overline{\lambda(\zeta)}\cdot H(z)\} = \mathrm{Im}\{\overline{\lambda(\zeta)}\cdot H(\zeta)\}, \quad \forall\zeta\in\partial D, \quad (15)$$

are measurable with respect to the logarithmic capacity, because the function H is continuous in the whole plane $\mathbb{C}$.

Next, by Theorem 3 in [12], there is an analytic function $\mathcal{A}$ in D that has the limits along γ_ζ q.e. on ∂D:

$$\lim_{z\to\zeta} \mathrm{Re}\{\overline{\lambda(\zeta)}\cdot A(z)\} = \Phi(\zeta), \quad (16)$$

$$\lim_{z\to\zeta} \mathrm{Im}\{\overline{\lambda(\zeta)}\cdot A(z)\} = \Psi(\zeta) \quad (17)$$

for the functions $\Phi(\zeta) := \varphi(\zeta) - \varphi_*(\zeta), \zeta\in\partial D$, and $\Psi(\zeta) := \psi(\zeta) - \psi_*(\zeta), \zeta\in\partial D$. Thus, the function $h := \mathcal{A} + H$ is a desired generalized analytic function with the source g.

Remark 2. As follows from the proof of Lemma 1, the generalized analytic functions h with a source $g\in L^p$, $p>2$, satisfying the Hilbert boundary condition (12) q.e. in the sense of the limits along γ_ζ can be represented in the form of the sums $\mathcal{A} + H$ with analytic functions $\mathcal{A}$ satisfying the corresponding Hilbert boundary condition (16) and a generalized analytic function $H = U + iV$ with the same source g, $U = P_x$ and $V = -P_y$, where P is the logarithmic (Newtonian) potential N_G with $G = 2g$ in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{C})\cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{C})$, $\alpha = (p-2)/p$, that satisfies the equation $\Delta P = G$.

The space of all solutions h of the Hilbert problem (12) in the given sense has the infinite dimension for any such prescribed φ, λ and $\{\gamma_\zeta\}_{\zeta\in\partial D}$, because the space of all functions $\psi:\partial D\to\mathbb{R}$ which are measurable with respect to the logarithmic capacity has the infinite dimension.

The latter is valid even for its subspace of continuous functions $\psi:\partial D\to\mathbb{R}$. Indeed, by the Fourier theory, the space of all continuous functions $\tilde{\psi}:\partial D\to\mathbb{R}$, equivalently, the space of all continuous 2π-periodic functions $\psi_*:\mathbb{R}\to\mathbb{R}$, has the infinite dimension.

Theorem 2. *Let* D *be a bounded domain in* $\mathbb{C}$ *whose boundary consists of a finite number of mutually disjoint Jordan curves, and* $\lambda:\partial\mathbb{D}\to\mathbb{C}$, $|\lambda(\zeta)|\equiv 1$, *and* $\varphi:\partial\mathbb{D}\to\mathbb{R}$ *be measurable functions with respect to the logarithmic capacity.*

Suppose that $\{\gamma_\zeta\}_{\zeta\in\partial D}$ *is a family of Jordan arcs of class* $\mathcal{BS}$ *in D and that a function* $g:D\to\mathbb{R}$ *is of the class* $L^p(D)$, $p>2$.

Then there exist generalized analytic functions $h: D \to \mathbb{C}$ *with the source g that have the limits (12) along* γ_ζ *q.e. on* ∂D. *Furthermore, the space of such functions* h *has the infinite dimension.*

In particular, in the case $\lambda \equiv 1$, we obtain the corresponding consequence on the Dirichlet problem for the generalized analytic functions with the source g along any prescribed Bagemihl—Seidel system.

4. Riemann problem and Bagemihl—Seidel systems. The classical setting of the **Riemann problem** in a smooth Jordan domain D of the complex plane $\mathbb{C}$ was to find analytic functions $f^+: D \to \mathbb{C}$ and $f^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ that admit continuous extensions to ∂D and satisfy the condition

$$f^+(\zeta) = A(\zeta) \cdot f^-(\zeta) + B(\zeta) \quad \forall \zeta \in \partial D \tag{18}$$

with prescribed Hölder continuous functions $A: \partial D \to \mathbb{C}$ and $B: \partial D \to \mathbb{C}$.

Recall also that the *Riemann problem with shift* in ∂D is to find analytic functions $f^+: D \to \mathbb{C}$ and $f^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ satisfying the condition

$$f^+(\alpha(\zeta)) = A(\zeta) \cdot f^-(\zeta) + B(\zeta) \quad \forall \zeta \in \partial D \tag{19}$$

where $\alpha: \partial D \to \partial D$ was a one-to-one sense preserving correspondence having the non-vanishing Hölder continuous derivative with respect to the natural parameter on ∂D. The function α is called a *shift function*. The special case $A \equiv 1$ gives the so-called *jump problem*, and $B \equiv 0$ gives the *problem on gluing* of analytic functions.

Arguing similarly to the proof of Theorem 1, we obtain by Theorem 8 in [12] on the Riemann problem for analytic functions the following statement.

Theorem 3. *Let* D *be a domain in* $\mathbb{C}$ *whose boundary consists of a finite number of mutually disjoint Jordan curves,* $A: \partial D \to \mathbb{C}$ *and* $B: \partial D \to \mathbb{C}$ *be functions that are measurable with respect to the logarithmic capacity, and let* $\{\gamma_\zeta^+\}_{\zeta \in \partial D}$ *and* $\{\gamma_\zeta^-\}_{\zeta \in \partial D}$ *be families of Jordan arcs of class* $\mathcal{BS}$ *in* D *and* $\mathbb{C} \setminus \overline{D}$, *correspondingly.*

Suppose that $g: \mathbb{C} \to \mathbb{R}$ *is a function with compact support in the class* $L^p(\mathbb{C})$ *with some* $p > 2$. *Then there exist generalized analytic functions* $f^+: D \to \mathbb{C}$ *and* $f^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ *with the source* g *that satisfy (18) q.e. on* $\zeta \in \partial D$, *where* $f^+(\zeta)$ *and* $f^-(\zeta)$ *are limits of* $f^+(z)$ *and* $f^-(z)$ *as* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-, *correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for every couple* (A, B) *and any collections* γ_ζ^+ *and* γ_ζ^-, $\zeta \in \partial D$.

Theorem 3 is a special case of the following lemma based on Lemma 3 in [12] on the Riemann problem with shift that may be of independent interest.

Lemma 2. *Under the hypotheses of Theorem 3, let, in addition,* $\alpha: \partial D \to \partial D$ *be a homeomorphism keeping components of* ∂D *such that* α *and* α^{-1} *have the N-property by Luzin with respect to the logarithmic capacity.*

Then there exist generalized analytic functions $f^+: D \to \mathbb{C}$ *and* $f^-: \mathbb{C} \setminus \overline{D} \to \mathbb{C}$ *with the source* g *that satisfy (19) for a.e.* $\zeta \in \partial D$ *with respect to the logarithmic capacity, where* $f^+(\zeta)$ *and* $f^-(\zeta)$ *are the limits of* $f^+(z)$ *and* $f^-(z)$ *az* $z \to \zeta$ *along* γ_ζ^+ *and* γ_ζ^-, *correspondingly.*

Furthermore, the space of all such couples (f^+, f^-) *has the infinite dimension for every couple* (A, B) *and any collections* γ_ζ^+ *and* γ_ζ^-, $\zeta \in \partial D$.

Remark 3. Some investigations were devoted also to the nonlinear Riemann problems with boundary conditions of the form

$$\Phi(\zeta, f^{+}(\zeta), f^{-}(\zeta)) = 0 \quad \forall \zeta \in \partial D . \tag{20}$$

It is natural, as above, to weaken such conditions to the following one:

$$\Phi(\zeta, f^{+}(\zeta), f^{-}(\zeta)) = 0 \quad \text{q.e.} \quad \zeta \in \partial D . \tag{21}$$

It is easy to see that the proposed approach makes it possible to reduce such problems to the algebraic measurable solvability of the relations

$$\Phi(\zeta, v, w) = 0 . \tag{22}$$

with respect to complex-valued functions $v(\zeta)$ and $w(\zeta)$.

Further, we say "*C*-measurable" in short instead of the expression "measurable with respect to the logarithmic capacity".

Example 1. For instance, correspondingly to the scheme given above, special nonlinear problems of the form

$$f^{+}(\zeta) = \varphi(\zeta, f^{-}(\zeta)) \quad \text{q.e. } \mathit{on} \ \zeta \in \partial D \tag{23}$$

are always solved, if the function $\varphi : \partial D \times \mathbb{C} \to \mathbb{C}$ satisfies the **Carathéodory conditions** with respect to the logarithmic capacity, that is, if $\varphi(\zeta, w)$ is continuous in the variable $w \in \mathbb{C}$ for a.e. $\zeta \in \partial D$ with respect to the logarithmic capacity, and it is *C*-measurable in the variable $\zeta \in \partial D$ for all $w \in \mathbb{C}$.

The spaces of solutions of such problems always have the infinite dimension. Indeed, by the Egorov theorem, see, e.g., Theorem 2.3.7 in [13], see also Section 17.1 in [14], the function $\varphi(\zeta, \psi(\zeta))$ is *C*-measurable in $\zeta \in \partial D$ for every *C*-measurable function $\psi : \partial D \to \mathbb{C}$, if the function φ satisfies the Carathéodory conditions, and the space of all *C*-measurable functions $\psi : \partial D \to \mathbb{C}$ has the infinite dimension, see, e.g., arguments in Remark 2 above.

Furthermore, applying Lemma 2 with $A \equiv 0$ in (19), we able to solve nonlinear boundary-value problems with shifts of the type (even with arbitrary measurable $f^{-}(\zeta)$ with respect to the logarithmic capacity)

$$f^{+}(\alpha(\zeta)) = \varphi(\zeta, f^{-}(\zeta)) \quad \text{q.e. on } \zeta \in \partial D . \tag{24}$$

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ЛОГАРИФМІЧНА ЄМНІСТЬ І ЗАДАЧІ РІМАНА ТА ГІЛЬБЕРТА
ДЛЯ УЗАГАЛЬНЕНИХ АНАЛІТИЧНИХ ФУНКЦІЙ

Вивчення задачі Діріхле з довільними вимірюваними граничними даними для гармонічних функцій в одиничному крузі має витоки з відомої дисертації Лузіна. Пізніше Векуа дослідив узагальнені аналітичні функції, але тільки для граничних даних, неперервних за Гельдером. Ця робота містить теореми існування некласичних розв'язків задач Рімана і Гільберта для узагальнених аналітичних функцій з джерелом, граничні дані яких є вимірюваними відносно логарифмічної ємності. Наш підхід заснований на геометричній інтерпретації граничних значень на відміну від класичного операторного підходу в теорії рівнянь з частинними похідними. На цій основі можна отримати відповідні теореми існування задачі Пуанкаре для похідної за напрямком для рівняння Пуассона і, зокрема, для задачі Неймана з довільними граничними даними, вимірюваними відносно логарифмічної ємності. Ці результати можуть бути застосовані до напівлінійних рівнянь математичної фізики в анізотропних і неоднорідних середовищах.

Ключові слова: *крайові задачі Рімана і Гільберта, узагальнені аналітичні функції, логарифмічна ємність.*