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Derivations and automorphisms of locally matrix algebras and groups

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We describe derivations and automorphisms of infinite tensor products of matrix algebras. Using this description, we show that, for a countable–dimensional locally matrix algebra A over a field F, the dimension of the Lie algebra of outer derivations of A and the order of the group of outer automorphisms of A are both equal to $|F|^{\aleph_0}$, *where |F| is the cardinality of the field F.*

Let A be the group of invertible elements of a unital locally matrix algebra A. We describe isomorphisms of groups [A*, A*]. In particular, we show that inductive limits of groups SLn(F) are determined by their Steinitz numbers.*

Keywords: *locally matrix algebra, derivation, automorphism.*

Let F be a ground field. Following [1], we call an associative F-algebra A a *locally matrix algebra,* if, for each finite subset of A, there exists a subalgebra $B \subset A$ containing this subset such that B is isomorphic to some matrix algebra $M_n(F)$ for $n \geqslant 1$. We call a locally matrix algebra A *unital,* if it contains a unit 1.

Let N be the set of all positive integers, and let P be the set of all primes. An infinite formal product of the form $s = \prod_{p \in P} p^{r_p}$, where $r_p \in N \cup \{0, \infty\}$ for all $p \in P$, is called *Steinitz number* (see [2]).

J.G. Glimm [3] proved that every countable-dimensional unital locally matrix algebra is uniquely determined by its Steinitz number. In [4, 5], we showed that this is no longer true for unital locally matrix algebras of uncountable dimensions.

S.A. Ayupov and K.K. Kudaybergenov [6] constructed an outer derivation of the countable-dimensional unital locally matrix algebra of Steinitz number 2^∞ and used it as an example of an outer derivation in a von Neumann regular simple algebra. In [7], H. Strade studied derivations of locally finite-dimensional locally simple Lie algebras over a field of characteristic 0.

Recall that a linear map $d : A \to A$ is called a *derivation*, if $d(xy) = d(x)y + x\,d(y)$ for arbitrary elements x, y from A.

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For an element $a \in A$, the adjoint operator $\mathrm{ad}_A(a): A \to A, x \to [a, x]$, is an *inner derivation* of the algebra A.

Let $\mathrm{Der}(A)$ be the Lie algebra of all derivations of the algebra A, and let $\mathrm{Inder}(A)$ be the ideal of all inner derivations. The factor algebra $\mathrm{Outder}(A) = \mathrm{Der}(A)/\mathrm{Inder}(A)$ is called the algebra of *outer derivations* of A.

Let $\mathrm{Aut}(A)$ and $\mathrm{Inn}(A)$ be the group of automorphisms and the group of inner automorphisms of the algebra A, respectively. The factor group $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$ is called the group of *outer automorphisms* of A.

Along with automorphisms of the algebra A, we consider the semigroup $P(A)$ of injective endomorphisms (embeddings) of A, $\mathrm{Aut}(A) \subseteq P(A)$.

The set $\mathrm{Map}(A, A)$ of all mappings $A \to A$ is equipped with the Tykhonoff topology (see [8]).

Theorem 1. *Let A be a locally matrix algebra.*

1) *The ideal* $\mathrm{Inder}(A)$ *is dense in* $\mathrm{Der}(A)$ *in the Tykhonoff topology*.

2) *Let the algebra A contain* 1. *Then the completion of* $\mathrm{Inn}(A)$ in $\mathrm{Map}(A, A)$ *in the Tykhonoff topology is the semigroup* $P(A)$. *In particular*, $\mathrm{Inn}(A)$ *is dense in* $\mathrm{Aut}(A)$.

G. Köthe [9] proved that every countable-dimensional unital locally matrix algebra is isomorphic to a tensor product of matrix algebras.

We describe derivations of infinite tensor products of matrix algebras.

Let I be an infinite set, and let $\mathbf{P}$ be a system of nonempty finite subsets of I. We say that the system $\mathbf{P}$ is *sparse*, if:

1) for any $S \in \mathbf{P}$, all nonempty subsets of S also lie in $\mathbf{P}$,

2) an arbitrary element $i \in I$ lies in no more than finitely many subsets from $\mathbf{P}$.

Let $\mathbf{A} = \bigotimes_{i \in I} A_i$ and let all algebras A_i be isomorphic to finite-dimensional matrix algebras over F. For a subset $S = \{i_1, \ldots, i_r\} \subset I$, the subalgebra $A_S := A_{i_1} \otimes \cdots \otimes A_{i_r}$ is a tensor factor of the algebra $\mathbf{A}$.

Let $\mathbf{P}$ be a system of nonempty finite subsets of I. Let f_S, $S \in \mathbf{P}$, be a system of linear operators $A \to A$. The sum $\sum_{S \in \mathbf{P}} f_S$ converges in the Tykhonoff topology if for an arbitrary element $a \in \mathbf{A}$ the set $\{S \in \mathbf{P} \mid f_S(a) \neq 0\}$ is finite. In this case, the operator $a \to \sum_{S \in \mathbf{P}} f_S(a)$ is a linear operator. Moreover, if every summand f_S is a derivation of the algebra $\mathbf{A}$, then this sum is also a derivation of the algebra $\mathbf{A}$.

Let $\mathbf{P}$ be a sparse system. For each subset $S \in \mathbf{P}$, we choose an element $a_S \in A_S$. The sum $\sum_{S \in \mathbf{P}} \mathrm{ad}_{\mathbf{A}}(a_S)$ converges in the Tykhonoff topology to a derivation of $\mathbf{A}$. Indeed, choose an arbitrary element $a \in \mathbf{A}$. Let $a \in A_{i_1} \otimes \cdots \otimes A_{i_r}$. Because of the sparsity of the system $\mathbf{P}$, for all but finitely many subsets $S \in \mathbf{P}$, we have $\{i_1, \ldots, i_r\} \cap S = \varnothing$, and therefore $\mathrm{ad}_{\mathbf{A}}(a_S)(a) = 0$. Let $D_{\mathbf{P}}$ be the vector space of all such sums, $D_{\mathbf{P}} \subseteq \mathrm{Der}(\mathbf{A})$.

For each algebra A_i, $i \in I$, choose a subspace A_i^0 such that $A_i = F \cdot 1_{A_i} \dot{+} A_i^0$ is a direct sum and 1_{A_i} is a unit element of A_i. Let E_i be a basis of A_i^0. For a subset $S = \{i_1, \ldots, i_r\}$ of the set I let $E_S := E_{i_1} \otimes \cdots \otimes E_{i_r} = \{a_1 \otimes \cdots \otimes a_r \mid a_k \in E_{i_k}, 1 \leqslant k \leqslant r\}$ and $\mathrm{ad}_{\mathbf{A}}(E_S) := \{\mathrm{ad}_{\mathbf{A}}(e) \mid e \in E_S\}$.

A description of derivations of the algebra $\mathbf{A}$ is given by the following theorem.

Theorem 2. 1) *Suppose that the set I is countable. Then* $\mathrm{Der}(\mathbf{A}) = \bigcup_{\mathbf{P}} D_{\mathbf{P}}$, *where the union is taken over all sparse systems of subsets of I.*

2) *Let I be an infinite (not necessarily countable) set. Let* **P** *be a sparse system of subsets of I. Then the union of finite sets of operators* $\bigcup_{S\in\mathbf{P}} \mathrm{ad}_{\mathbf{A}}(E_S)$ *is a topological basis of* $D_{\mathbf{P}}$.

Using this description, we prove the analog of the result of H. Strade [7] for locally matrix algebras.

Theorem 3. *Let A be a countable-dimensional locally matrix algebra. Then the Lie algebra* Outder(A) *is not locally finite-dimensional.*

We describe automorphisms and unital injective endomorphisms of a countable-dimensional unital locally matrix algebra A. We note that by the result of A.G. Kurosh ([1, Theorem 10]), the semigroup $P(A)$ of unital injective homomorphisms is strictly bigger than Aut(A).

The starting point here is again Köthe's theorem [9] stating that every countable-dimensional unital locally matrix algebra A is isomorphic to a countable tensor product of matrix algebras. Therefore $A \cong \bigotimes_{i=1}^{\infty} A_i,\ A_i \cong M_{n_i}(F),\ n_i \geqslant 1$.

Let $H_n,\ n_i \geqslant 1,$ be the subgroup of the group Inn(A) generated by conjugations by invertible elements from $\bigotimes_{i\geqslant n} A_i$. Clearly, $H_n \cong \mathrm{Inn}(\bigotimes_{i\geqslant n} A_i)$ and $\mathrm{Inn}(A) = H_1 > H_2 > \cdots$. For each $n \geqslant 1$, choose a system of representatives of left cosets $hH_{n+1},\ h \in H_n$, and denote it as X_n. We assume that each X_n contains the identical automorphism.

For an arbitrary sequence of automorphisms $\varphi_n \in X_n,\ n \geqslant 1$, the infinite product $\varphi = \varphi_1\varphi_2\cdots$ converges in the Tykhonoff topology. Clearly, $\varphi \in P(A)$.

Theorem 4. *An arbitrary unital injective endomorphism* $\varphi \in P(A)$ *can be uniquely represented as* $\varphi = \varphi_1\varphi_2\cdots$, *where* $\varphi_n \in X_n$ *for each* $n \geqslant 1$.

We call a sequence of automorphisms $\varphi_n \in H_n,\ n \geqslant 1,$ *integrable*, if, for an arbitrary element $a \in A$, the subspace spanned by all elements $\varphi_n\varphi_{n-1}\cdots\varphi_1(a),\ n \geqslant 1,$ is finite-dimensional.

Theorem 5. *An injective endomorphism* $\varphi = \varphi_1\varphi_2\cdots$, *where* $\varphi_n \in H_n,\ n \geqslant 1$, *is an automorphism, if and only if the sequence* $\{\varphi_n^{-1}\}_{n\geqslant 1}$ *is integrable.*

Using Theorems 3, 4, we determine dimensions of Lie algebras Der(A) and Outder(A) and orders of groups Aut(A) and Out(A), where A is a countable-dimensional locally matrix algebra.

We denote the cardinality of a set X as $|X|$. For two sets X and Y, let Map(Y, X) denote the set of all mappings from Y to X. Given two cardinals α, β and sets X, Y such that $|X| = \alpha$, $|Y| = \beta$ we define $\alpha^{\beta} = |\mathrm{Map}(Y,X)|$. As always $\aleph_0$ stands for the countable cardinality.

Theorem 6. *Let* $\mathbf{A} = \bigotimes_{i\in I} A_i$, *where I is an infinite set, and each algebra* A_i *is isomorphic to a matrix algebra over a field F of the dimension* >1. *Then* $\dim_F \mathrm{Der}(A) = \dim_F \mathrm{Outder}(A) = |F|^{|I|}$.

Theorem 7. *Let A be a countable-dimensional locally matrix algebra over a field F. Then* $\dim_F \mathrm{Der}(A) = \dim_F \mathrm{Outder}(A) = |F|^{\aleph_0}$.

Theorem 8. *Let A be a countable—dimensional locally matrix algebra over a field F. Then* $|\mathrm{Aut}(A)| = |\mathrm{Out}(A)| = |F|^{\aleph_0}$.

Consider the algebra $M_N(F)$ of $N\times N$ matrices over the ground field F having finitely many nonzero elements in each column.

Following [10], we call an $N\times N$ matrix *periodic* (more precisely: n-periodic), if it is block-diagonal diag($a, a, \ldots$), where a is an $n\times n$ matrix.

Let $M_n^p(F)$ be the subalgebra of $M_N(F)$ that consists of all n-periodic matrices. Clearly, $M_n^p(F) \cong M_n(F)$.

Let s be a Steinitz number. Then $M_s^p(F) = \bigcup_{n \in N,\, n|s} M_n^p(F)$ is a subalgebra of $M_N(F)$ (see [10]). By the Theorem of J. Glimm [3], $M_s^p(F)$ is the only (up to isomorphism) unital locally matrix algebra of Steinitz number s.

Let $GL_n^p(F)$ be the group of invertible elements of $M_n^p(F)$, $SL_n^p(F) = [GL_n^p(F), GL_n^p(F)]$. Clearly, $GL_n^p(F) \cong GL_n(F)$, $SL_n^p(F) \cong SL_n(F)$.

Let $n_1, n_2, \ldots$ be a sequence of positive integers such that $n_i \mid n_{i+1}$, $i \geqslant 1$, and let s be the least common multiple of the numbers $(n_i, i \geq 1)$. Then

$$GL_{n_1}^p(F) \subset GL_{n_2}^p(F) \subset \cdots, \ \bigcup_{i \geqslant 1} GL_{n_i}^p(F) = GL_s^p(F),$$

$$SL_{n_1}^p(F) \subset SL_{n_2}^p(F) \subset \cdots, \ \bigcup_{i \geqslant 1} SL_{n_i}^p(F) = SL_s^p(F).$$

Our aim is to describe isomorphisms between groups $SL_s^p(F)$. We will do it in a more general context of unital locally matrix algebras.

Recall that, for an arbitrary associative unital F-algebra R and an arbitrary positive integer $n \geqslant 2$, the elementary linear group $E_n(R)$ is the group generated by all transvections $t_{ij}(a) = I_n + e_{ij}(a)$, $1 \leqslant i \neq j \leqslant n$, where I_n is the identity $n \times n$ matrix, $a \in R$, $e_{ij}(a)$ is the $n \times n$ matrix having the element a at the (i,j)-position and zero elsewhere. Denote, by R^*, the group of invertible elements of algebra R.

Let A be an infinite-dimensional unital locally matrix algebra. Let a subalgebra $1 \in B \subset A$ be isomorphic to some matrix algebra $M_n(F)$ for $n \geqslant 4$ and let C be a centralizer of the subalgebra B in A. By the theorem of H.M. Wedderbun (see [11]), $A \cong M_n(C)$. We show that, in this case, $[A^*, A^*] \cong E_n(C)$. After that, it is sufficient to apply the description of isomorphisms of elementary linear groups over rings due to I.Z. Golubchik and A.V. Mikhalev [12, 13] and E.I. Zelmanov [14] in order to prove the following theorems.

Theorem 9. *Let A, B be unital locally matrix algebras. If the groups $[A^*, A^*]$ and $[B^*, B^*]$ are isomorphic, then the rings A and B are isomorphic or anti-isomorphic. Moreover, for any isomorphism $\varphi : [A^*, A^*] \to [B^*, B^*]$, either there exists a ring isomorphism $\theta_1 : A \to B$ such that φ is the restriction of θ_1 to $[A^*, A^*]$ or there exists a ring anti-isomorphism $\theta_2 : A \to B$ such that, for an arbitrary element $g \in [A^*, A^*]$, we have $\varphi(g) = \theta_2(g^{-1})$.*

If the algebras A, B are countable-dimensional, then Theorem 9 can be strengthened. In this case, without loss of generality, we assume that $A = M_s^p(F)$, where s is the Steinitz number of the algebra A. The algebra $M_s^p(F)$ is closed with respect to the transposition $t : M_s^p(F) \to M_s^p(F)$, $g \to g^t$, which is an anti-isomorphism.

Theorem 10. *Let A, B be countable-dimensional unital locally matrix algebras. If the groups $[A^*, A^*]$ and $[B^*, B^*]$ are isomorphic, then the F-algebras A and B are isomorphic. Moreover, an arbitrary isomorphism $\varphi : [A^*, A^*] \to [B^*, B^*]$ either extends to a ring isomorphism $\theta_1 : A \to B$ or there exists a ring isomorphism $\theta_2 : A \to B$ such that $\varphi(g) = \theta_2((g^{-1})^t)$ for all elements $g \in [A^*, A^*]$.*

Corollary. *Let s_1, s_2 be Steinitz numbers. Then $SL_{s_1}^p(F) \cong SL_{s_2}^p(F)$, if and only if $s_1 = s_2$.*

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ДИФЕРЕНЦІЮВАННЯ ТА АВТОМОРФІЗМИ
ЛОКАЛЬНО МАТРИЧНИХ АЛГЕБР І ГРУП

Описано диференціювання та автоморфізми нескінченних тензорних добутків матричних алгебр. З використанням цього опису показано, що для зліченновимірної локально матричної алгебри A над полем F розмірності алгебри Лі зовнішніх диференціювань A і порядок групи зовнішніх автоморфізмів A збігаються і дорівнюють $|F|^{\aleph_0}$, де $|F|$ означає потужність поля F.

Нехай A^* — група оборотних елементів унітальної локально матричної алгебри A. Описано ізоморфізми групи $[A^*, A^*]$. Зокрема, показано, що індуктивні границі груп $SL_n(F)$ визначаються їх числами Стейніца.

Ключові слова: *локально матрична алгебра, диференціювання, автоморфізм.*