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Dirichlet problem with measurable data for semilinear equations in the plane

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The study of the Dirichlet problem with arbitrary measurable data for harmonic functions in the unit disk $\mathbb{D}$ goes to the known dissertation of Luzin. His result was formulated in terms of angular limits (along nontangent paths) that are a traditional tool for the research of the boundary behavior in the geometric function theory. With a view to further developments of the theory of boundary-value problems for semilinear equations, the present paper is devoted to the Dirichlet problem with arbitrary measurable (over logarithmic capacity) boundary data for quasilinear Poisson equations in such Jordan domains. For this purpose, it is firstly constructed completely continuous operators generating nonclassical solutions of the Dirichlet boundary-value problem with arbitrary measurable data for the Poisson equations $\Delta U = G$ over the sources $G \in L^p$, $p > 1$. The latter makes it possible to apply the Leray—Schauder approach to the proof of theorems on the existence of regular nonclassical solutions of the measurable Dirichlet problem for quasilinear Poisson equations of the form $\Delta U(z) = H(z) \cdot Q(U(z))$ for multipliers $H \in L^p$ with $p > 1$ and continuous functions $Q : \mathbb{R} \to \mathbb{R}$ with $Q(t)/t \to 0$ as $t \to \infty$.

These results can be applied to some specific quasilinear equations of mathematical physics, arising under a modeling of various physical processes such as the diffusion with absorption, plasma states, stationary burning, etc. These results can be also applied to semilinear equations of mathematical physics in anisotropic and inhomogeneous media.

Keywords: *logarithmic capacity, quasilinear Poisson equations, nonlinear sources, Dirichlet problem, measurable boundary data, angular limits, nontangent paths.*

1. Introduction. The research of boundary-value problems with arbitrary measurable data is due to the known dissertation of Luzin, see its reprint [1] with comments of his pupils Bari and Men'shov.

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Following this way, we proved Theorem 7 in [2] on the solvability of the Dirichlet problem for the Poisson equations $\Delta U = G$ with sources in classes $G \in L^p$, $p > 1$ in Jordan domains with arbitrary boundary data that are measurable with respect to the logarithmic capacity.

With a view to further developments of the theory of boundary-value problems for semilinear equations, the present paper is devoted to the Dirichlet problem for quasilinear Poisson equations in Jordan domains with arbitrary boundary data that are measurable over the logarithmic capacity, see, e.g., [2] for its definition and properties.

Later on, we use the *abbreviation q.e. (quasieverywhere)* on a set $E \subset \mathbb{C}$, if the corresponding property holds only for all $\zeta \in E$ except its subset with zero logarithmic capacity.

Recall also that a path in the domain D in $\mathbb{C}$ terminating at $\zeta \in \partial D$ is called *nontangent*, if its part in a neighborhood of ζ lies inside of an angle in D with its vertex at ζ. Hence, the limit along all nontangential paths at $\zeta \in D$ also named *the angular limit* at a point.

2. Definitions and preliminary remarks. Here, we use the designation of the *logarithmic (Newtonian) potential* $\mathcal{N}_G$ *of sources* $G \in L^p(\mathbb{C})$, $p > 1$, with compact supports given by the formula:

$$\mathcal{N}_G(z) := \frac{1}{2\pi} \int_{\mathbb{C}} \ln|z - w| G(w) dm(w), \tag{1}$$

where $dm(w)$ corresponds to the Lebesgue measure in the plane.

Remark 1. As known, N_G with G supported in $\mathbb{D}$ is continuous in $\mathbb{C}$, belongs to the class $W^{2,p}(\mathbb{D})$, and $\Delta N_G = G$ a.e. Moreover, $N_G \in W^{1,q}_{\text{loc}}(\mathbb{C})$ for some $q > 2$, consequently, N_G is locally Hölder-continuous. Furthermore, if $G \in L^p(\mathbb{C})$, $p > 2$, then $N_G \in C^{1,\alpha}_{\text{loc}}(\mathbb{C})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$ (see, e.g., Lemma 3 in [2] or Theorem 2 in [3]).

Furthermore, the collection $\{N_G\}$ is equicontinuous, if the collection $\{G\}$ is bounded by the norm in $L^p(\mathbb{C})$. More precisely, $\| N_G \|_C \leqslant M \cdot \| G \|_p$ on each compact set S in $\mathbb{C}$, where M is a constant depending only on S, and, in particular, the restriction of N_G to $\overline{\mathbb{D}}$ is a completely continuous bounded linear operator (see, e.g., Lemma 2 in [2] or Theorem 1 in [3]).

Let us also recall the following analog of the Luzin theorem on the *antiderivatives* in terms of the logarithmic capacity (see Theorem 3.1 in [4]).

Lemma 1. *Let $\varphi : [a,b] \to \mathbb{R}$ be a measurable function over the logarithmic capacity. Then there is a continuous function $\Phi : [a,b] \to \mathbb{R}$ with $\Phi'(x) = \varphi(x)$ q.e. on (a,b). Furthermore, Φ can be chosen with $\Phi(a) = \Phi(b) = 0$ and $|\Phi(x)| \leqslant \varepsilon$, $x \in [a,b]$ for arbitrary prescribed $\varepsilon > 0$.*

Remark 2. In view of the arbitrariness of $\varepsilon > 0$ in Lemma 1, for each φ, there is an infinite collection of such Φ. Furthermore, it is easy to see by Lemma 3.1 in [4] that the space of such functions Φ has the infinite dimension.

Corollary 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function with respect to the logarithmic capacity. Then the space of continuous functions $\Phi : \partial\mathbb{D} \to [-1,1]$ with $\Phi(1) = 0$, $|\Phi(\zeta)| \leqslant \varepsilon$ for all $\zeta \in \partial\mathbb{D}$ under arbitrary prescribed $\varepsilon > 0$, and $\Phi'(e^{it}) = \varphi(e^{it})$ q.e. on $\mathbb{R}$ has the infinite dimension.*

On this basis, we obtain the following result (see, e.g., Theorem 4.1 in [4]).

Proposition 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function over the logarithmic capacity. Then there is a space of harmonic functions U in a unit disk $\mathbb{D}$ of the infinite dimension with the angular limits $\lim_{z \to \zeta} u(z) = \varphi(\zeta)$ q.e. on $\partial\mathbb{D}$.*

Remark 3. By the proof of Theorem 4.1 in [4], $u(z)=\frac{\partial}{\partial\vartheta}U(z)$, where

$$U(re^{i\vartheta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2}\Phi(e^{it})dt, \tag{2}$$

i.e., for any function Φ from Corollary 1, u can be calculated in the explicit form

$$u(re^{i\vartheta})=-\frac{r}{\pi}\int_0^{2\pi}\frac{(1-r^2)\sin(\vartheta-t)}{(1-2r\cos(\vartheta-t)+r^2)^2}\Phi(e^{it})\,dt. \tag{3}$$

Later on, it was shown by Theorems 1 and 3 in [5] that the functions $u(z)$ can be represented as the *Poisson—Stieltjes integrals*

$$\mathbb{U}_\Phi(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\vartheta-t)\,d\Phi(e^{it}) \quad \forall\, z=re^{i\vartheta}, \quad r\in(0,1), \quad \vartheta\in[-\pi,\pi], \tag{4}$$

where $P_r(\Theta)=(1-r^2)/(1-2r\cos\Theta+r^2)$, $r<1$, $\Theta\in\mathbb{R}$, is the *Poisson kernel*.

The corresponding analytic functions $\mathcal{A}(z)$ in $\mathbb{D}$ with the real parts $u(z)$ can be represented as the *Schwartz—Stieltjes integrals*

$$\mathbb{S}_\Phi(z)=\frac{1}{2\pi}\int_{\partial\mathbb{D}}\frac{\zeta+z}{\zeta-z}\,d\Phi(\zeta), \quad z\in\mathbb{D}, \tag{5}$$

because the Poisson kernel is the real part of the (analytic in the variable z) *Schwartz kernel* $(\zeta+z)/(\zeta-z)$. Integrating (5) by parts (see Lemma 1 and Remark 1 in [5]), we obtain the more convenient form of the representation

$$\mathbb{S}_\Phi(z)=\frac{z}{\pi}\int_{\partial\mathbb{D}}\frac{\Phi(\zeta)}{(\zeta-z)^2}\,d\zeta, \quad z\in\mathbb{D}. \tag{6}$$

3. On completely continuous Dirichlet operators. By Proposition 1, there is a space of harmonic functions u in a unit disk $\mathbb{D}$ of the infinite dimension with the angular limits q.e. on $\partial\mathbb{D}$

$$\lim_{z\to\zeta}u(z)=\psi_G(\zeta):=\varphi(\zeta)-\varphi_G(\zeta), \qquad \varphi_G(\zeta):=N_G(\zeta). \tag{7}$$

By Remark 1, $U:=u+N_G|_{\mathbb{D}}$ with such u are continuous solutions of the Poisson equation $\Delta U=G$ a.e. in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})\cap W^{1,q}_{\mathrm{loc}}(\mathbb{D})$, $q>2$, with the angular limits

$$\lim_{z\to\zeta}U(z)=\varphi(\zeta) \text{ q.e. on } \partial\mathbb{D}. \tag{8}$$

By Remark 3, such a harmonic function $u:\mathbb{D}\to\mathbb{R}$ can be obtained in the form of the real part of the analytic function

$$\mathbb{S}_{\Psi}(z) := \frac{z}{\pi}\int_{\partial\mathbb{D}} \frac{\Psi(\zeta)}{(\zeta - z)^2}\, d\zeta, \quad z \in \mathbb{D}, \tag{9}$$

where Ψ is an antiderivative of the function ψ_G from Corollary 1.

Consequently, such a harmonic function u can be represented in the form

$$u(z) = u_0(z) - u_G(z), \quad u_0(z) := \mathrm{Re}\, \mathbb{S}_{\Phi}(z), \quad u_G(z) := \mathrm{Re}\, \mathbb{S}_{\Phi_G}(z), \tag{10}$$

where Φ and Φ_G are antiderivatives of φ and φ_G in Corollary 1, correspondingly. Note that the harmonic function u_0 does not depend on the sources G at all.

Let us choose the function Φ_G in a suitable way to guarantee that the correspondence $G \mapsto u + N_G|_{\mathbb{D}}$ is a Dirichlet operator $\mathcal{D}_G$ that is completely continuous on compact sets in $\mathbb{D}$ generating solutions of the Poisson equation $\Delta U = G$ a.e. in the class $C \cap W^{2,p}_{\mathrm{loc}}(\mathbb{D})$ with the Dirichlet boundary condition (8).

Namely, the following function Φ_G is an antiderivative for the function φ_G:

$$\Phi_G(\zeta) := \int_0^{\vartheta} N_G(e^{i\theta})\, d\theta - S(\vartheta), \quad \zeta = e^{i\vartheta}, \quad \theta,\ \vartheta \in [0, 2\pi], \tag{11}$$

where $S : [0, 2\pi] \to \mathbb{C}$ is either zero or a singular function of the form

$$S(\vartheta) := C(\vartheta) \int_0^{2\pi} N_G(e^{i\theta})\, d\theta, \quad \zeta = e^{i\vartheta}, \quad \theta,\ \vartheta \in [0, 2\pi], \tag{12}$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, i.e., C is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ q.e. Recall that the existence of such functions C follows from Lemma 3.1 in [4].

Setting $u_G = \mathrm{Re}\, \mathbb{S}_{\Phi_G}$, it is easy to see in view of the second part of Remark 1 that

$$|\Phi_G(\zeta)| \leqslant 4\pi M \cdot \|G\|_p \qquad \forall \zeta \in \partial\mathbb{D} \qquad 13)$$

and, by (6), that, for the constants C_r and C_r^* depending only on $r \in (0, 1)$,

$$|u_G(z)| \leqslant |\mathbb{S}_{\Phi_G}(z)| \leqslant C_r \cdot \|G\|_p, \quad \forall\, z \in \mathbb{D}_r, \tag{14}$$

$$|u_G(z_1) - u_G(z_2)| \leqslant |\mathbb{S}_{\Phi_G}(z_1) - \mathbb{S}_{\Phi_G}(z_2)| \leqslant C_r^* \|G\|_p |z_1 - z_2|, z_1, z_2 \in \mathbb{D}_r. \tag{15}$$

Consequently, the operator $u_G := \mathrm{Re}\, \mathbb{S}_{\Phi_G}$ is completely continuous on compact sets in $\mathbb{D}$ by the Arzela—Ascoli theorem (see, e.g., Theorem IV.6.7 in [6]). Thus, we obtain the next conclusion.

Lemma 2. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable over the logarithmic capacity. Then there is a Dirichlet operator $\mathcal{D}_G$ over $G : \mathbb{D} \to \mathbb{C}$ in $L^p(\mathbb{D})$, $p > 1$, generating continuous solutions $U : \mathbb{D} \to \mathbb{R}$ of the Poisson equation $\Delta U = G$ in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$ with the Dirichlet boundary condition* (8) *in the sense of angular limits q.e. on $\partial\mathbb{D}$, which is completely continuous over $\mathbb{D}_r$ for each $r \in (0, 1)$.*

Remark 4. Note that the nonlinear operator $\mathcal{D}_G$ constructed above is not bounded except the trivial case $\Phi \equiv 0$ because then $\mathcal{D}_0 = \mathbb{S}_\Phi \neq 0$. However, the restriction of the operator $\mathcal{D}_G$ to $\mathbb{D}_r$ under each $r \in (0,1)$ is bounded at infinity in the sense that $\max\limits_{z\in\mathbb{D}_r} |\mathcal{D}_G(z)| \leqslant M \cdot \|G\|_p$ for some $M > 0$ and all G with large enough $\|G\|_p$. Note also that, by Corollary 1, we are able always to choose Φ for any φ, including $\varphi \equiv 0$, which is not identically 0 in the unit disk $\mathbb{D}$.

Moreover, by the above construction, $U := \mathcal{D}_G$ belongs to the class $W^{1,q}_{\mathrm{loc}}(\mathbb{D})$ for some $q > 2$. Consequently, U is locally Hölder continuous. Furthermore, if $G \in L^p(\mathbb{D})$, $p > 2$, then $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

4. The Dirichlet problem in a unit disk. In this section, we study the solvability of the Dirichlet problem for semilinear Poisson equations of the form $\Delta U(z) = H(z) \cdot Q(U(z))$ in the unit disk $\mathbb{D}$.

Theorem 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable with respect to the logarithmic capacity. Suppose that $H : \mathbb{D} \to \mathbb{R}$ is a function in the class $L^p(\mathbb{D})$ for $p > 1$ with compact support in $\mathbb{D}$, and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with*

$$\lim_{t\to\infty} \frac{Q(t)}{t} = 0. \tag{16}$$

Then there is a function $U : \mathbb{D} \to \mathbb{R}$ in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$ such that

$$\Delta U(z) = H(z) \cdot Q(U(z)) \quad \text{a.e. in } \mathbb{D} \tag{17}$$

with the angular limits

$$\lim_{z\to\zeta} U(z) = \varphi(\zeta) \quad \text{q.e. on } \partial\mathbb{D}. \tag{18}$$

Moreover, U belongs to the class $W^{1,q}_{\mathrm{loc}}(\mathbb{D})$ for some $q > 2$; consequently, U is locally Hölder continuous. Furthemore, if $G \in L^p(\mathbb{D})$, $p > 2$, then $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ for $\alpha := (p-2)/p$, and for all $\alpha \in (0,1)$ under $p = \infty$.

Proof. If $\|H\|_p = 0$ or $\|Q\|_C = 0$, then any harmonic function from Theorem 7.2 in [7] gives the desired solution of (17). Thus, we may assume that $\|H\|_p \neq 0$ and $\|Q\|_C \neq 0$. Set $Q_*(t) = \max\limits_{|\tau|\leqslant t} |Q(\tau)|$, $t \in \mathbb{R}^+ := [0,\infty)$. Then the function $Q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing. Moreover, by (16),

$$\lim_{t\to\infty} \frac{Q_*(t)}{t} = 0. \tag{19}$$

By Lemma 2 and Remark 4, we obtain the family of operators $F(G;\tau) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions $G \in L^p(\mathbb{D})$ with supports in the support of H,

$$F(G;\tau) := \tau H \cdot Q(\mathcal{D}_G) \qquad \forall \tau \in [0,1] \tag{20}$$

which satisfies hypothesis H1-H3 of Theorem 1 in [8] (see also [2]). Indeed:

H1). First of all, by Lemma 2, the function $F(G;\tau)\in L_H^p(\mathbb{D})$ for all $\tau\in[0,1]$ and $G\in L_H^p(\mathbb{C})$ because the function $Q(\mathcal{D}_G)$ is continuous. Furthermore, the operators $F(\cdot;\tau)$ are completely continuous for each $\tau\in[0,1]$ and even uniformly continuous in the parameter $\tau\in[0,1]$.

H2). The index of the operator $F(\cdot;0)$ is obviously equal to 1.

H3). Let us assume that solutions of the equations $G=F(G;\tau)$ are not bounded in $L_H^p(\mathbb{D})$, i.e., there is a sequence of functions $G_n\in L_H^p(\mathbb{D})$ with $\|G_n\|_p\to\infty$ as $n\to\infty$ such that $G_n=F(G_n;\tau_n)$ for some $\tau_n\in[0,1]$, $n=1,2,\ldots$. However, by Remark 4, we have that, for some constant $M>0$,

$$\|G_n\|_p\leqslant\|H\|_p\,Q_*(M\|G_n\|_p)$$

and, consequently,

$$\frac{Q_*(M\|G_n\|_p)}{M\|G_n\|_p}\geqslant\frac{1}{M\|H\|_p}>0 \qquad (21)$$

for all large enough n. The latter is impossible in view of condition (19). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [8], there is a function $G\in L_H^p(D)$ with $F(G;1)=G$, and, by Lemma 2, the function $U:=\mathcal{D}_G$ gives the desired solution of (16). The rest properties of the given solution U follow from Remark 4.

Remark 5. By the construction in the above proof, $U=\mathcal{D}_G$, where $\mathcal{D}_G$ is the completely continuous Dirichlet operator described in the last section, and the support of G is in the support of H. The upper bound of $\|G\|_p$ depends only on $\|H\|_p$ and on the function Q. Moreover, $G:\mathbb{D}\to\mathbb{R}$ is a fixed point of the nonlinear operator $\Omega_G:=H\cdot Q(\mathcal{D}_G):L_H^p(\mathbb{D})\to L_H^p(\mathbb{D})$, where $L_H^p(\mathbb{D})$ consists of functions G in $L^p(\mathbb{D})$ with supports in the support of H.

5. Extension of the main result to Jordan domains. Here, we extend the result to domains D with the *quasihyperbolic boundary condition by Gehring—Martio* (see [9]). Recall that, by the discussion in [10] and [7], every smooth (or Lipschitz) domain satisfies the quasihyperbolic boundary condition. But such boundaries can be even nowhere locally rectifiable, and without the so-called $(A)-$*condition by Ladyzhenskaya—Ural'tseva* in [11], which was standard in the theory of boundary-value problems for PDEs.

Theorem 2. *Let D be a Jordan domain in $\mathbb{C}$ with the quasihyperbolic boundary condition, ∂D have a tangent q.e., and $\varphi:\partial D\to\mathbb{R}$ be measurable over the logarithmic capacity. Suppose that $H:D\to\mathbb{R}$ is a function in the class $L^p(D)$ for $p>1$ with compact support in D, and $Q:\mathbb{R}\to\mathbb{R}$ is a continuous function with*

$$\lim_{t\to\infty}\frac{Q(t)}{t}=0. \qquad (22)$$

Then there is a continuous function $U:D\to\mathbb{R}$ in class $W^{2,p}_{\text{loc}}(D)$ such that

$$\Delta U(\xi)=H(\xi)\cdot Q(U(\xi)) \qquad \text{a.e. in } D \qquad (23)$$

with the angular limits

$$\lim_{\xi\to\omega} U(\xi) = \varphi(\omega) \qquad q.e. \ \ on \ \partial D. \tag{24}$$

Moreover, U belongs to the class $W^{1,q}_{\mathrm{loc}}(D)$ for some $q>2$. Consequently, U is locally Hölder continuous in D. Furthermore, if $G\in L^p(D)$, $p>2$, then $U\in C^{1,\alpha}_{\mathrm{loc}}(D)$ for $\alpha:=(p-2)/p$, and for all $\alpha\in(0,1)$ under $p=\infty$.

Proof. Let c be a conformal mapping of D onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see, e.g., Theorem II.2.1 in [12]. Now, by the Caratheodory theorem (see, e.g., Theorem II.3.4 in [12]), c is extended to a homeomorphism $\tilde{c}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. Furthermore, by Corollary of Theorem 1 in [13], $c_* := \tilde{c}\,|_{\partial D} : \partial D \to \partial\mathbb{D}$ and its inverse function are Hölder continuous. Then $\tilde{\varphi} := \varphi\circ c_*^{-1}$ is measurable over the logarithmic capacity (see, e.g., Remarks 2.1 and 2.2 in [7]).

Now, set $\tilde{H} = |C'|^2\cdot H\circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D}\to D$. Then it is clear by the hypotheses of Theorem 2 that $\tilde{H}$ has compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 1, there is a continuous function $\tilde{U} : \mathbb{D}\to\mathbb{R}$ in the class $W^{2,p}_{\mathrm{loc}}(\mathbb{D})$ such that

$$\Delta\tilde{U}(z) = \tilde{H}(z)\cdot Q(\tilde{U}(z)) \qquad \text{a.e. in } \mathbb{D} \tag{25}$$

with the angular limits

$$\lim_{z\to\zeta} \tilde{U}(z) = \tilde{\varphi}(\zeta) \qquad \text{q.e. on } \partial\mathbb{D}. \tag{26}$$

Moreover, $\tilde{U} = \mathcal{D}_{\tilde{G}}$, where $\mathcal{D}_{\tilde{G}}$ is the completely continuous Dirichlet operator described in Section 3, and the support of $\tilde{G}$ is in the support of $\tilde{H}$. The upper bound of $\|\tilde{G}\|_p$ depends only on $\|\tilde{H}\|_p$ and on the function Q.

Next, setting $U = \tilde{U}\circ c$, we obtain by simple calculations that $\Delta U = |c'|^2\cdot\Delta\tilde{U}\circ c$ and, consequently, the continuous function $U : D\to\mathbb{C}$ is in the class $W^{1,p}_{\mathrm{loc}}(D)$ that satisfies Eq. (23) a.e. Moreover, $U(\xi) = \mathcal{D}_{\tilde{G}}(c(\xi))$, where $\mathcal{D}_{\tilde{G}}$ is the completely continuous Dirichlet operator from Section 3. Hence, by Remark 4, U belongs to the class $W^{1,q}_{\mathrm{loc}}(D)$ for some $q>2$. Consequently, U is locally Hölder continuous in D and, if $G\in L^p(D)$, $p>2$, then $U\in C^{1,\alpha}_{\mathrm{loc}}(D)$ for $\alpha:=(p-2)/p$, and for all $\alpha\in(0,1)$ under $p=\infty$.

It remains to show that (26) implies (23). Indeed, by the Lindelöf theorem, see, e.g., Theorem II.C.2 in [14], if ∂D has a tangent at a point ω, then $\arg[c_*(\omega)-c(\xi)]-\arg[\omega-\xi]\to\mathrm{const}$ as $\xi\to\omega$. In other words, the images under the conformal mapping c of sectors in D with a vertex at $\omega\in\partial D$ is asymptotically the same as sectors in $\mathbb{D}$ with a vertex at $\zeta = c_*(\omega)\in\partial\mathbb{D}$. Consequently, the nontangential paths in D are transformed under c into nontangential paths in $\mathbb{D}$ and inversely q.e. on ∂D and $\partial\mathbb{D}$, respectively, because ∂D has a tangent q.e. and c_* and c_*^{-1} keep sets of the logarithmic capacity zero.

Theorem 2 can be applied as in [10] to problems of mathematical physics appearing under modeling various types of physical and chemical absorptions with diffusion, plasma states, stationary burning, etc. Finally, due to the factorization theorem in [15], we are able, by the quasiconformal replacements of variables, to extend the above results to semilinear equations of the

Poisson type describing the corresponding physical phenomena in anisotropic and inhomogeneous media that shall be published elsewhere.

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ЗАДАЧА ДІРІХЛЕ З ВИМІРНИМИ ДАНИМИ ДЛЯ НАПІВЛІНІЙНИХ РІВНЯНЬ НА ПЛОЩИНІ

Вивчення задачі Діріхле з довільними вимірними даними для гармонічних функцій в одиничному колі $\mathbb{D}$ сходить до відомої дисертації Лузіна. Його результат був сформульований у термінах кутових границь (уздовж недотичних шляхів), які є традиційним інструментом для дослідження граничної поведінки відображень у геометричній теорії функцій. Слідуючи цим шляхом, раніше ми довели теорему про розв'язність задачі Діріхле для рівнянь Пуассона $\Delta U = G$ з джерелами в класах $G \in L^p$, $p > 1$, в жорданових областях з довільними граничними даними, вимірними відносно логарифмічної ємності. При цьому передбачалося, що області задовольняють квазігіперболічну граничну умову Герінга—Мартіо, взагалі кажучи, без відомої (А)-умови Ладиженської—Уральцевої і, зокрема, без умови зовнішнього конуса, які були стандартними для крайових задач в теорії диференціальних рівнянь у частинних похідних. Відзначимо, що такі жорданові області можуть бути навіть локально неспрямлюваними.

З метою подальшого розвитку теорії крайових задач для напівлінійних рівнянь у роботі досліджується задача Діріхле з довільними вимірними (відносно логарифмічної ємності) граничними даними для квазілінійних рівнянь Пуассона в таких областях. Для цього спочатку будуються повністю неперервні оператори, які породжують некласичні розв'язки крайової задачі Діріхле з довільними вимірними даними для рівнянь Пуассона $\Delta U = G$ з джерелами $G \in L^p$, $p > 1$. Останнє дає змогу застосувати підхід Лере—Шаудера до доведення теорем про існування регулярних некласичних розв'язків вимірної задачі Діріхле для квазілінійних рівнянь Пуассона виду $\Delta U(z) = H(z) \cdot Q(U(z))$ для множників $H \in L^p$ з $p > 1$ і неперервних функцій $Q : \mathbb{R} \to \mathbb{R}$ з $Q(t)/t \to 0$ для $t \to \infty$. Ці результати можуть бути застосовані до деяких конкретних квазілінійних рівнянь математичної фізики, що виникають під час моделювання різних фізичних процесів, таких як дифузія з абсорбцією, стани плазми, стаціонарне горіння і т .д., а також до напівлінійних рівнянь математичної фізики в анізотропних і неоднорідних середовищах.

Ключові слова: *логарифмічна ємність, квазілінійне рівняння Пуассона, нелінійні джерела, задача Діріхле, вимірні граничні дані, кутові границі, недотичні шляхи.*