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A.O. Borysyuk, Corresponding Member of the NAS of Ukraine,
https://orcid.org/0000-0002-3878-3915

Institute of Hydromechanics of the NAS of Ukraine, Kyiv
E-mail: aobor@ukr.net

A numerical technique to solve a problem of the fluid motion in a straight plane rigid duct with two axisymmetric rectangular constrictions

A second-order numerical technique is developed to study the steady laminar fluid motion in a straight two-dimensional hard-walled duct with two axisymmetric rectangular constrictions. In this technique, the governing relations are solved via deriving their integral analogs, performing a discretization of these analogs, simplifying the obtained (after making the discretization) coupled nonlinear algebraic equations, and the final solution of the resulting (after making the simplification) uncoupled linear ones. The discretization consists of the spatial and temporal parts. The first of them is performed with the use of the TVD-scheme and a two-point scheme of discretization of the spatial derivatives, whereas the second one is made on the basis of the implicit three-point asymmetric backward differencing scheme. The above-noted uncoupled linear algebraic equations are solved by an appropriate iterative method, which uses the deferred correction implementation technique and the technique of conjugate gradients, as well as the solvers ICCG and Bi-CGSTAB.

Keywords: *fluid motion, flat duct, rectangular constriction, technique.*

Study of flows in ducts is an actual problem in the gas-oil industry, architecture, medicine, municipal economy, etc. Among others, a significant interest is related here to studying the flows in ducts with local constrictions. That is explained by the fact that such irregularities in the duct geometry cause local changes in the flow structure and/or character, etc. Those changes can result in the corresponding consequences not only in a vicinity of, but also far from the irregularities [1].

As analysis of the scientific literature shows, the study of flows in ducts with local constrictions has been paid much attention to. In those studies, straight hard-walled ducts and their constrictions of the simplest geometries were considered. The basic flow (i.e., the flow upstream of a (first) constriction) was laminar, axisymmetric, and steady. As for fluids, they were assumed to be homogeneous, incompressiblem and Newtonian (the other types of ducts, their constric-

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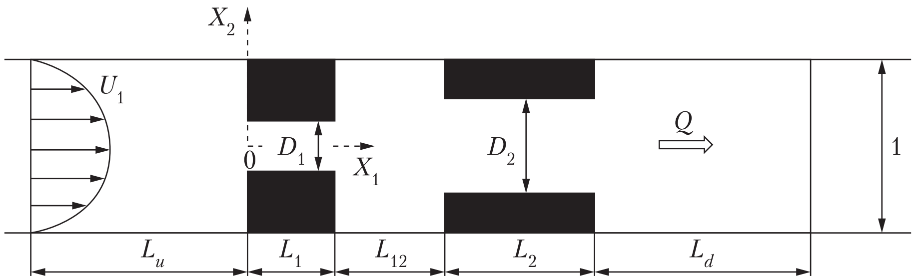

Fig. 1. Geometry of the problem and the computational domain

tions, fluidsm and the basic flow are not considered in this paper, because they were studied much less intensively compared with the noted ones). These allowed one, on the one hand, to study (within the framework of appropriate models chosen and with acceptable accuracy) the influence of the basic parameters of a duct, its constriction(s) and the basic flow on the flow not only nearm but also far downstream of the constriction(s), and, on the other hand, to simplify significantly solutions to the corresponding problems of interest [1-4].

Among the results obtained in those studies, numerical methods, which have been developed to investigate flows around duct constrictions, are of a particular interest. One of the latest of them was presented in [1]. It has been devised to solve a problem of the flow in a straight hard-walled two-dimensional duct with two rigid constrictions of a rectangular axisymmetric shape. That method allows one to study the fluid motion in the noted duct in the stream function–vorticity–pressure variables, has high stability of a solution and the second order of accuracy in the spatial co-ordinates. However, its first order of accuracy in the temporal coordinate should apparently stimulate researchers either to develop more accurate appropriate computational techniques or to improve the method in such a way to make its temporal accuracy higher.

In this study, an alternative technique is presented to solve the same problem. This technique uses the fluid velocity and the pressure as the basic variables, has nearly the same stability of a solution, the same order of accuracy in the spatial coordinates and higher (i.e., the second) order of accuracy in the temporal coordinate. However, due to the large amount of mathematical operations used in this technique, it needs a more computational time to obtain a solution compared to the above one.

Statement of the problem. A straight hard-walled plane duct of dimensionless width 1, having two rigid constrictions of a rectangular axisymmetric shape, is considered (Fig. 1). The constrictions are situated at the distance L_{12} from each other, and have the diameters D_i and the lengths L_i ($i=1,2$). In this duct, a viscous homogeneous Newtonian fluid moves. The fluid has mass density ρ and kinematic viscosity ν. Its flow is characterized by a small Mach number and the rate Q per unit depth of the duct. In addition, the flow upstream of the first constriction (i.e., the basic flow) is steady and laminar. It is necessary to study the flow around the constrictions.

The fluid motion in the duct is governed by the dimensionless momentum and continuity equations, viz.

$$\frac{\partial U_i}{\partial T}+U_j\frac{\partial U_i}{\partial X_j}=-\frac{\partial P}{\partial X_i}+\frac{1}{\mathrm{Re}}\frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right), \quad i=1,2, \tag{1}$$

$$\partial U_i / \partial X_i = 0 . \tag{2}$$

The boundary conditions consist in the absence of a fluid motion at the channel wall, S_{ch}, and on both constrictions, S_j, $(j = 1, 2)$. Also, the flow rate Q must be invariable along the duct axis, viz.

$$U_i \big|_{S_{ch}, S_j} = 0 , \quad \partial Q / \partial X_1 = 0 , \quad Q = 1 , \quad i, j = 1, 2 . \tag{3}$$

Apart from these, the parabolic velocity profile is specified outside the disturbed flow region due to the constrictions[1], viz.

$$U_1 \big|_{X_1 = -L_u, L_1 + L_{12} + L_2 + L_d} = 1.5(1 - 4X_2^2) , \qquad U_2 \big|_{X_1 = -L_u, L_1 + L_{12} + L_2 + L_d} = 0 . \tag{4}$$

As for the pressure P, it is assumed to be constant both sufficiently far upstream of the first constriction ($P\big|_{X_1 = -L_u} = P_u$), and far downstream of the second one ($P\big|_{X_1 = L_1 + L_{12} + L_2 + L_d} = P_d$). In addition, the corresponding pressure drop, $\Delta P = P_u - P_d > 0$, should ensure the existence of the given laminar regime of the basic flow. Also, without loss of generality, the pressure P_d is taken to be zero[2], and the magnitude P_u, like the pressure in the whole duct, needs to be found. Apart from these, the normal pressure derivative is zero on the rigid walls of the channel and both its constrictions, viz.

$$(\partial P / \partial n)_{S_{ch}, S_j} = 0 , \quad j = 1, 2 . \tag{5}$$

Regarding the initial conditions, they are in the absence of a fluid motion in the channel at the time instant $T = 0$ [1], viz.

$$P \big|_{T=0} = 0 , \quad U_i \big|_{T=0} = 0 . \tag{6}$$

In (1)-(6), X_1, X_2, X_3 are the rectangular Cartesian coordinates shown in Fig. 1 (here, the axis X_3 is normal to the plane X_1X_2 and directed to us); T the time; U_i the local fluid velocities in the directions X_i; $\mathrm{Re} = U_a D / \nu$ the Reynolds number of the cross-sectionally averaged basic flow; U_a its velocity; and the values of the distances L_u and L_d are given in the next section. In addition, hereinafter, the vector $\vec{n}$ denotes the outward unit normal to the appropriate surface, and the summation over repeated indices is assumed throughout the paper. As for the scaling factors used in (1)-(6), these are the duct width D as the length scale, the velocity U_a as the velocity scale, the product $U_a D$ as the flow rate scale, the ratio D / U_a as the time scale, and the product ρU_a^2 as the pressure scale.

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[1] This is the region before the constrictions, where the flow is still undisturbed by them, and far behind them, where the flow is already undisturbed (i.e., where the flow disturbances disappear, and it becomes like the basic one).

[2] A choice of the value of P_d always can be compensated for by the choice of the corresponding value of P_u in such a way that the corresponding pressure drop ΔP (which governs fluid motion in the duct) remains unchangeable.

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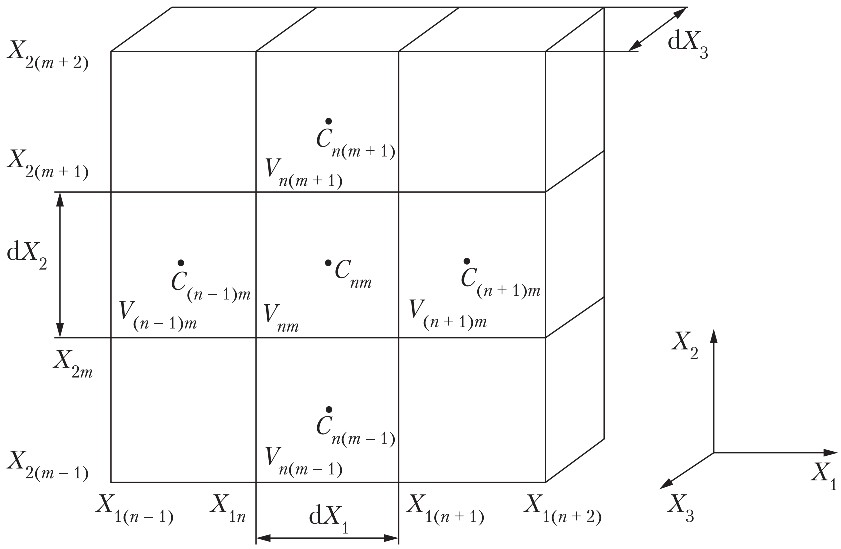

Fig. 2. A scheme of fragmentation of the computational domain into small volumes

Computational domain. The domain, in which a solution to the problem should be found, is shown in Fig. 1. It is restricted by the duct sections $X_1 = -L_u$, $X_1 = L_1 + L_{12} + L_2 + L_d$ and $X_3 = X_{3a}$, $X_3 = X_{3a} + \mathrm{d}X_3$ (where $\mathrm{d}X_3 << 1$, and X_{3a} is an arbitrary value of the coordinate X_3). Herewith, the boundary $X_1 = -L_u$ is taken upstream of the first constriction, where the flow is still undisturbed by it, and the boundary $X_1 = L_1 + L_{12} + L_2 + L_d$ behind the second constriction, where the flow disturbances already disappear, and the flow redevelops into the basic state at $X_1 = -L_u$. As for the distances L_u and L_d, for the basic flow velocities considered in this study, their values should vary in the ranges $L_u \leqslant 0.5$ and $L_d \leqslant 12$ [1, 2].

The chosen computational domain is divided into the small volumes V_{nm} by the duct sections $X_1 = X_{1n}$ and $X_2 = X_{2m}$ (where $X_{1n} = X_{1(n-1)} + \mathrm{d}X_1$, $\mathrm{d}X_1 << 1$, and $X_{2m} = X_{2(m-1)} + \mathrm{d}X_2$, $\mathrm{d}X_2 << 1$), as shown in Fig. 2. Herewith, in order to have a smooth velocity profile in an arbitrary duct cross-section, the steps $\mathrm{d}X_1$ and $\mathrm{d}X_2$ are reduced in an appropriate manner as one approaches either the duct or constrictions' walls.

Integral equations and their discrete analogs. Integral analogs of Eqs. (1) and (2) are obtained by their integrating over the control volumes V_{nm} (in making this operation, the appropriate conservation laws take place in each volume V_{nm}). It gives

$$\frac{\partial}{\partial T}\iiint\limits_{\mathrm{V}_{nm}} U_i \mathrm{d}V + \iiint\limits_{\mathrm{V}_{nm}} U_j \frac{\partial U_i}{\partial X_j}\mathrm{d}V = -\iiint\limits_{\mathrm{V}_{nm}} \frac{\partial P}{\partial X_i}\mathrm{d}V + \frac{1}{\mathrm{Re}}\iiint\limits_{\mathrm{V}_{nm}} \frac{\partial}{\partial X_j}\left(\frac{\partial U_i}{\partial X_j}\right)\mathrm{d}V\,, \tag{7}$$

$$\iiint\limits_{\mathrm{V}_{nm}} (\partial U_i / \partial X_i)\mathrm{d}V = 0\,. \tag{8}$$

The application (wherever possible) of the Gauss theorem to the terms of Eqs. (7) and (8), and/or the expansion (wherever needed) of their integrands (which are denoted by $\vec{f}(\vec{r})$) in the

Taylor series around the mass center C_{nm} of the volume[3] V_{nm} (Fig. 2), further use of the first two terms of these series (i.e., $\vec{f}(\vec{r}) = \vec{f}(\vec{r}_{c_{nm}}) + \vec{\nabla}(\vec{f})\big|_{\vec{r}=\vec{r}_{c_{nm}}} \cdot (\vec{r} - \vec{r}_{c_{nm}})$), making the discretization of the temporal and spatial derivatives on the basis of the implicit three-point non-symmetric backward differencing and two-point schemes [5], respectively, viz.

$$\frac{\partial \vec{f}(\vec{r}_{c_{nm}}, T)}{\partial T} = \frac{1.5\vec{f}^{\,k}_{c_{nm}} - 2\vec{f}^{\,k-1}_{c_{nm}} + 0.5\vec{f}^{\,k-2}_{c_{nm}}}{\Delta T}, \qquad \vec{\nabla} f\big|_{\vec{r}=\vec{r}_{c^{(j)}_{nm}}} = \vec{e}_i (\partial f / \partial X_i)_{\vec{r}=\vec{r}_{c^{(j)}_{nm}}},$$

$$\left.\frac{\partial f}{\partial X_1}\right|_{\vec{r}=\vec{r}_{c^{(1)}_{nm}}} = \frac{f(\vec{r}_{c_{(n+1)m}}) - f(\vec{r}_{c_{nm}})}{\mathrm{d}X_1}, \qquad \left.\frac{\partial f}{\partial X_1}\right|_{\vec{r}=\vec{r}_{c^{(2)}_{nm}}} = \frac{f(\vec{r}_{c_{nm}}) - f(\vec{r}_{c_{(n-1)m}})}{\mathrm{d}X_1},$$

$$\left.\frac{\partial f}{\partial X_2}\right|_{\vec{r}=\vec{r}_{c^{(3)}_{nm}}} = \frac{f(\vec{r}_{c_{n(m+1)}}) - f(\vec{r}_{c_{nm}})}{\mathrm{d}X_2}, \qquad \left.\frac{\partial f}{\partial X_2}\right|_{\vec{r}=\vec{r}_{c^{(4)}_{nm}}} = \frac{f(\vec{r}_{c_{nm}}) - f(\vec{r}_{c_{n(m-1)}})}{\mathrm{d}X_2},$$

as well as the application (wherever necessary) of the following TVD-scheme [5]

$$\vec{f}(\vec{r}_{c^{(j)}_{nm}}) = \vec{f}^{(j)}_1 + \Phi(\vec{f}^{(j)}_2 - \vec{f}^{(j)}_1),$$

$$\vec{f}^{(j)}_1 = \begin{cases} \vec{f}(\vec{r}_{c_{nm}}), & F^{(j)}_{nm} \geqslant 0, \\ \vec{f}(\vec{r}_{c_j}), & F^{(j)}_{nm} < 0, \end{cases} \quad C_1 = C_{(n+1)m},\ C_2 = C_{(n-1)m},\ C_3 = C_{n(m+1)},\ C_4 = C_{n(m-1)},$$

$$\vec{f}^{(j)}_2 = \alpha_j \vec{f}(\vec{r}_{c_{nm}}) + (1-\alpha_j)\vec{f}(\vec{r}_{c_j}),\ \alpha_j = \left|\vec{r}_{c^{(j)}_{nm}} - \vec{r}_{c_j}\right| / \left|\vec{r}_{c_{nm}} - \vec{r}_{c_j}\right|,$$

$$\Phi(\eta_j) = \max(0, \min(4\eta_j, 1)),\ \eta_j = \left|\vec{U}(\vec{r}_{c^{(j)}_{nm}}) - \vec{U}(\vec{r}_{c_{nm}})\right| / \left|\vec{U}(\vec{r}_{c_j}) - \vec{U}(\vec{r}_{c^{(j)}_{nm}})\right|,$$

allows one to proceed to considering the discrete analogs of Eqs. (7) and (8), viz.

$$\frac{1.5U^k_{ic_{nm}} - 2U^{k-1}_{ic_{nm}} + 0.5U^{k-2}_{ic_{nm}}}{\Delta T}\left|V_{nm}\right| + \sum_{j=1}^{4} F^{(j)k}_{nm} U^k_{ic^{(j)}_{nm}} - \frac{1}{\mathrm{Re}}\sum_{j=1}^{4} \vec{\nabla} U^k_{ic^{(j)}_{nm}} \cdot \vec{n}_j \left|S^{(j)}_{nm}\right| =$$

$$= -\begin{cases} \displaystyle\sum_{j=1}^{4} P^k_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| \\ (\partial P / \partial X_i)^k_{C_{nm}} \left|V_{nm}\right| \end{cases}, \tag{9}$$

[3] Since the fluid in the duct is homogeneous (see the problem formulation), the mass center of the volume V_{nm} coincides with its geometrical center. The analogous situation is with the mass center of each side face of the volume V_{nm}.

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$$\sum_{j=1}^{4}\vec{U}^{\,k}_{c^{(j)}_{nm}}\cdot\vec{n}_j\left|S^{(j)}_{nm}\right|=\sum_{j=1}^{4}U^{\,k}_{ic^{(j)}_{nm}}n_{ji}\left|S^{(j)}_{nm}\right|=\sum_{j=1}^{4}F^{(j)k}_{nm}=0, \tag{10}$$

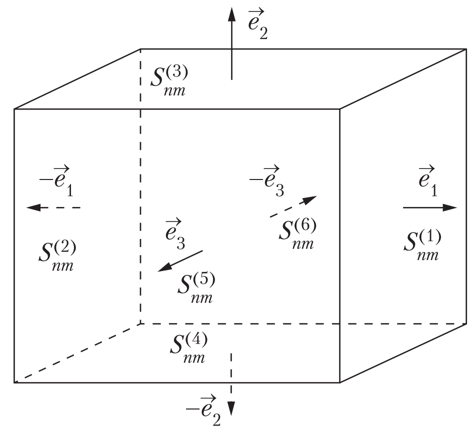

Fig. 3. The small volume V_{nm}, its side faces $S^{(i)}_{nm}$ and their outward unit normals $\vec{n}_i$ $(i=1,...,6)$

which have the second order of accuracy. Here $\vec{\nabla}$ is the gradient; $\vec{r}=X_i\vec{e}_i$ and $\vec{r}_{c_{nm}}=X_{ic_{nm}}\vec{e}_i$ the position vectors of an arbitrary point in the region V_{nm} and its mass center C_{nm}, respectively; the point in the Taylor series indicates the scalar product of the corresponding magnitudes; $\vec{e}_i$ the unit directivity vector of the axis X_i; ΔT the small time step; $\vec{f}^{\,k}_{c_{nm}}$ and $\vec{f}^{\,k}_{c^{(j)}_{nm}}$ the values of the function $\vec{f}$ at the points C_{nm} and $C^{(j)}_{nm}$ at the time instant $T=k\Delta T$; $C^{(j)}_{nm}$ the mass center of the side face $S^{(j)}_{nm}$ of the volume V_{nm}; $\vec{f}^{\,k-1}_{c_{nm}}$ and $\vec{f}^{\,k-2}_{c_{nm}}$ the known values of the function $\vec{f}$ at the point C_{nm} at the moments $T=(k-1)\Delta T$ and $T=(k-2)\Delta T$, respectively; $|V_{nm}|$ the volume of the region V_{nm}; $\left|S^{(j)}_{nm}\right|$ and $\vec{n}_j=n_{ji}\vec{e}_i$ the area and the outward unit normal to the face $S^{(j)}_{nm}$ ($\vec{n}_1=\vec{e}_1, \vec{n}_2=-\vec{e}_1, \vec{n}_3=\vec{e}_2, \vec{n}_4=-\vec{e}_2, \vec{n}_5=\vec{e}_3, \vec{n}_6=-\vec{e}_3$; Fig. 3); and $F^{(j)k}_{nm}=\vec{U}^{\,k}_{c^{(j)}_{nm}}\cdot\vec{n}_j\left|S^{(j)}_{nm}\right|$ the fluid flow across the face $S^{(j)}_{nm}$ at the moment $T=k\Delta T$.

Discrete analogs of conditions (3)-(6) and their application to Eqs. (9) and (10). The discrete analogs of conditions (3)-(5) on the boundary of the computational domain are as follows:

$$U^k_1\Big|_{X_1=-L_u,\,L_1+L_{12}+L_2+L_d}=1.5(1-4X_2^2), \qquad U^k_2\Big|_{X_1=-L_u,\,L_1+L_{12}+L_2+L_d}=0,$$

$$U^k_i\Big|_{S_{ch},S_j}=0, \quad \partial Q^k/\partial X_1=0, \quad Q^k=1, \quad L_u\geqslant 0.5, \quad L_d\geqslant 12, \quad i,j=1,2,$$

$$P^k_{c^{(1)}_{nm}}\Big|_{X_1=L_1+L_{12}+L_2+L_d}=0, \quad (\partial P^k/\partial X_2)_{S_{ch}}=0, \quad (\partial P^k/\partial n)_{S_j}=0.$$

They allow one to find $F^{(j)k}_{nm}$ and $\vec{\nabla}U^{\,k}_{ic^{(j)}_{nm}}$ on the noted boundary in Eqs. (9) and (10), viz.

$$F^{(j)k}_{nm}\Big|_{S_{ch},S_1,S_2}=0, \quad F^{(3)k}_{nm}\Big|_{X_1=-L_u,\,L_1+L_{12}+L_2+L_d}=F^{(4)k}_{nm}\Big|_{X_1=-L_u,\,L_1+L_{12}+L_2+L_d}=0,$$

$$F^{(2)k}_{nm}\Big|_{X_1=-L_u}=-1.5(1-4X_2^2)\mathrm{d}X_2\mathrm{d}X_3, \quad F^{(1)k}_{nm}\Big|_{X_1=L_1+L_{12}+L_2+L_d}=1.5(1-4X_2^2)\mathrm{d}X_2\mathrm{d}X_3,$$

$$\vec{\nabla}U^{\,k}_{ic^{(3)}_{nm}}\Big|_{X_2=1/2}=-\vec{e}_2U^{\,k}_{ic_{nm}}/\mathrm{d}X_2, \quad \vec{\nabla}U^{\,k}_{ic^{(4)}_{nm}}\Big|_{X_2=-1/2}=\vec{e}_2U^{\,k}_{ic_{nm}}/\mathrm{d}X_2,$$

$$-L_u\leqslant X_1\leqslant 0, \quad L_1\leqslant X_1\leqslant L_1+L_{12}, \quad L_1+L_{12}+L_2\leqslant X_1\leqslant L_1+L_{12}+L_2+L_d,$$

$$\vec{\nabla} U^{k}_{1c^{(2)}_{nm}}\Big|_{X_1=-L_u} = -12 X_2 \vec{e}_2\,, \quad \vec{\nabla} U^{k}_{1c^{(1)}_{nm}}\Big|_{X_1=L_1+L_{12}+L_2+L_d} = -12 X_2 \vec{e}_2\,,$$

$$\vec{\nabla} U^{k}_{2c^{(2)}_{nm}}\Big|_{X_1=-L_u} = 0\,, \quad \vec{\nabla} U^{k}_{2c^{(1)}_{nm}}\Big|_{X_1=L_1+L_{12}+L_2+L_d} = 0\,,$$

$$\vec{\nabla} U^{k}_{ic^{(1)}_{nm}}\Big|_{\substack{X_1=0,\, D_1/2\leqslant X_2\leqslant 1/2,\, -1/2\leqslant X_2\leqslant -1/2+D_1/2\\ X_1=L_1+L_{12},\, D_2/2\leqslant X_2\leqslant 1/2,\, -1/2\leqslant X_2\leqslant -1/2+D_2/2}} = -\vec{e}_1 U^{k}_{ic_{nm}} \,/\, \mathrm{d}X_1,$$

$$\vec{\nabla} U^{k}_{ic^{(2)}_{nm}}\Big|_{\substack{X_1=L_1,\, D_1/2\leqslant X_2\leqslant 1/2,\, -1/2\leqslant X_2\leqslant -1/2+D_1/2\\ X_1=L_1+L_{12}+L_2,\, D_2/2\leqslant X_2\leqslant 1/2,\, -1/2\leqslant X_2\leqslant -1/2+D_2/2}} = \vec{e}_1 U^{k}_{ic_{nm}} \,/\, \mathrm{d}X_1,$$

$$\vec{\nabla} U^{k}_{ic^{(3)}_{nm}}\Big|_{0\leqslant X_1\leqslant L_1,\, X_2=D_1/2;\, L_1+L_{12}\leqslant X_1\leqslant L_1+L_{12}+L_2,\, X_2=D_2/2} = -\vec{e}_2 U^{k}_{ic_{nm}} \,/\, \mathrm{d}X_2\,,$$

$$\vec{\nabla} U^{k}_{ic^{(4)}_{nm}}\Big|_{0\leqslant X_1\leqslant L_1,\, X_2=-1/2+D_1/2;\, L_1+L_{12}\leqslant X_1\leqslant L_1+L_{12}+L_2,\, X_2=-1/2+D_2/2} = \vec{e}_2 U^{k}_{ic_{nm}} \,/\, \mathrm{d}X_2\,.$$

As for the discrete analogs of conditions (6) (i.e., $U_i^{k=0}=0\,, P^{k=0}=0$), they give one the possibility to compute the appropriate terms in (9) and (10) at the initial time instant in the computational domain, viz. $F_{nm}^{(j)k=0}=0\,,\ \vec{\nabla} U^{k=0}_{ic^{(j)}_{nm}}=0\,,\ (\partial P/\partial X_i)^{k=0}_{C_{nm}}=0\,.$

A method of solution to Eqs. (9) and (10). The system of equations (9), (10) is solved numerically. In making this, one comes across the two significant problems. The first of them is connected with a nonlinearity of Eq. (9), whereas the second one is due to the absence of an equation for the pressure which is available in the right part of (9).

In order to solve the first problem, the flow $F_{nm}^{(j)k}$ is modified in the appropriate way. More specifically, the velocity components in it are initially replaced by their values found at the previous time step. After that, the components are replaced by their known previous approximations. These replacements allow one to proceed from solving the coupled systems of nonlinear algebraic equations to the corresponding uncoupled linear ones.

The second problem is solved via introducing the pressure in Eq. (10) and the subsequent agreeing of the velocity and the pressure with each other, when making the noted modification of the flow $F_{nm}^{(j)k}$. The velocity and pressure values, which are obtained in making this, are corrected at each step by performing the appropriate operations. Let us demonstrate the abovesaid in more details.

If one formally solves (9) with respect to $U^{k}_{ic_{nm}}$, one obtains the equation

$$U^{k}_{ic_{nm}} = A^{0}_{ic_{nm}} + A^{k}_{ic_{nm}} - A^{p}_{ic_{nm}} \begin{cases} \left(1/\left|\mathrm{V}_{nm}\right|\right)\sum\limits_{j=1}^{4} P^{k}_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right|, \\ (\partial P/\partial X_i)^{k}_{C_{nm}}. \end{cases} \tag{11}$$

Here $A^{0}_{ic_{nm}}$ is a rational function whose numerator contains the known values $U^{k-1}_{ic_{nm}}$ and $U^{k-2}_{ic_{nm}}$. Its denominator involves $F^{(j)k}_{nm}$. The term $A^{k}_{ic_{nm}}$ also is a rational function whose denominator only differs from that of the function $A^{0}_{ic_{nm}}$ in the multiplier $|\mathrm{V}_{nm}|/\Delta T$. Its numerator has the unknown quantities $U^{k}_{ic_j}$ and $F^{(j)k}_{nm}U^{k}_{ic_j}$. As for the fractional multiplier $A^{p}_{ic_{nm}}$, its numerator only consists of the time step ΔT, whereas the denominator coincides with that of the function $A^{0}_{ic_{nm}}$.

From relation (11), one can obtain an equation for $U^{k}_{ic^{(j)}_{nm}}$. That equation, after substituting into the expression for $F^{(j)k}_{nm}$ and then the obtained relation into (10), yields an equation for the pressure, viz.

$$\sum_{j=1}^{4} A^{p}_{ic^{(j)}_{nm}} (\partial P/\partial X_i)^{k}_{C^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| = \sum_{j=1}^{4} (A^{0}_{ic^{(j)}_{nm}} + A^{k}_{ic^{(j)}_{nm}}) n_{ji} \left|S^{(j)}_{nm}\right|. \qquad (12)$$

Then we begin to solve (11), (12) with finding the first approximations of the velocities $U^{k*}_{ic_{nm}}$. For this purpose, the unknown quantities $P^{k}_{c^{(j)}_{nm}}$ in (11) are replaced with the known ones $P^{k-1}_{c^{(j)}_{nm}}$, and the functions $A^{\cdots}_{ic_{nm}}$ are modified by replacing the unknown velocities in the flow $F^{(j)k}_{nm}$ with their known values $U^{k-1}_{ic_{nm}}$. This results in the following system of linear algebraic equations for $U^{k*}_{ic_{nm}}$:

$$U^{k*}_{ic_{nm}} = A^{0l}_{ic_{nm}} + A^{kl}_{ic_{nm}} - (A^{pl}_{ic_{nm}} / |\mathrm{V}_{nm}|) \sum_{j=1}^{4} P^{k-1}_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| \qquad (13)$$

in which $A^{\cdots l}_{ic_{nm}}$ are the modified functions $A^{\cdots}_{ic_{nm}}$.

Once the quantities $U^{k*}_{ic_{nm}}$ are found from (13) (the method of solution of this system is described below), they are further used to obtain the corresponding values of the operators $A^{\cdots}_{ic^{(j)}_{nm}}$, which are then substituted into (12). This yields the system of linear algebraic equations for the first approximation of the pressure, viz.

$$\sum_{j=1}^{4} A^{p*}_{ic^{(j)}_{nm}} (\partial P/\partial X_i)^{k*}_{C^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| = \sum_{j=1}^{4} (A^{0*}_{ic^{(j)}_{nm}} + A^{k*}_{ic^{(j)}_{nm}}) n_{ji} \left|S^{(j)}_{nm}\right| \qquad (14)$$

in which $A^{\cdots}_{ic^{(j)}_{nm}}$ denote the values of the operators $A^{\cdots}_{ic^{(j)}_{nm}}$ found with the use of $U^{k*}_{ic_{nm}}$.

Further, one applies a procedure which is similar to the just described one. More specifically, the first approximations of the pressure, $P^{k*}_{c^{(j)}_{nm}}$, found from (14) are substituted into (11) instead of $P^{k}_{c^{(j)}_{nm}}$. Also, in the functions $A^{\cdots}_{ic_{nm}}$ in (11), the flow $F^{(j)k}_{nm}$ is modified by replacing the unknown velocities in it with their first approximations obtained from (13). This results in the systems of linear algebraic equations for the second approximations (or the first corrections) of the velocities $U^{k**}_{ic_{nm}}$, viz.

$$U^{k**}_{ic_{nm}} = A^{0l*}_{ic_{nm}} + A^{kl*}_{ic_{nm}} - (A^{pl*}_{ic_{nm}} / |\mathrm{V}_{nm}|) \sum_{j=1}^{4} P^{k*}_{c^{(j)}_{nm}} n_{ji} \left|S^{(j)}_{nm}\right| \qquad (15)$$

(here $A_{ic_{nm}}^{\cdots l*}$ are the functions $A_{ic_{nm}}^{\cdots}$ in which the just noted flow modification has been performed).

After that, the second approximations of the velocities, obtained from (15), are used to obtain the values $A_{ic_{nm}^{(j)}}^{\cdots **}$ of the operators $A_{ic_{nm}^{(j)}}^{\cdots}$. The subsequent substitution of these values into (12) instead of $A_{ic_{nm}^{(j)}}^{\cdots}$ allows one to write a system of equations for the second approximation (or the first correction) of the pressure, $P_{c_{nm}^{(j)}}^{k**}$, which is similar to (14) viz.

$$\sum_{j=1}^{4} A_{ic_{nm}^{(j)}}^{p**} (\partial P / \partial X_i)_{C_{nm}^{(j)}}^{k**} n_{ji} \left| S_{nm}^{(j)} \right| = \sum_{j=1}^{4} (A_{ic_{nm}^{(j)}}^{0**} + A_{ic_{nm}^{(j)}}^{k**}) n_{ji} \left| S_{nm}^{(j)} \right|. \qquad (16)$$

If the accuracy of the second approximations of the velocities and the pressure is not satisfactory, then the just-described procedure must be carried out until the accuracy becomes as desired.

Solution of Eqs. (13)-(16). The systems of linear algebraic equations (SLAEs) (13)-(16) can be rewritten in the generalized form, with the unknown quantities $\xi_{c_{nm}}^{k}$ and $\xi_{c_i}^{k}$

$$a_{c_{nm}}^{k} \xi_{c_{nm}}^{k} + \sum_{i=1}^{4} a_{c_i}^{k} \xi_{c_i}^{k} = b_{c_{nm}}^{k}. \qquad (17)$$

Such systems are solved either by direct or iterative methods. Usually, the direct methods are applied to small systems of equations and give good results. However, when one deals with big SLAEs (especially with systems whose matrices are rarified), the direct methods need a huge amount of time to obtain their solutions. Therefore, their application is unreasonable here. The iterative methods, when applied to big SLAEs, need much less computational memory and time, save the rarefaction degree of their matrices (when the matrices are rarified) and give satisfactory results.

Proceed from the just-said, as well as from the dimension and the rarefaction degree of the matrix of system (17), an iterative method is chosen in this paper to solve the system. Within its framework, an initial approximation of the solution is chosen initially, which is then improved by making iterations until its accuracy reaches the desired value. Herewith, the attention is paid to the following two features. The first of them concerns with the necessity of providing domination of the diagonal terms in the matrix of system (17). In this study, it is realized by applying the deferred correction implementation method [5] to the convective term. In accordance with this method, the part of the convective term, which corresponds to the scheme for $\vec{f}_1^{(j)}$ (it is given before (9)), is inserted into the matrix, whereas its remainder is placed into the right part of SLAE (17).

The second feature is related to a desire to have the as minimal as possible number of iterations. Here, it is made by the use of the method of conjugate gradients [5]. This method allows one to solve a SLAE via the iterations' whose number does not exceed the number of its unknown values. Herewith, if a successful choice of the initial approximation is made, the number of iterations sharply decreases. Also, the preconditioning results in a significant reduction of the number of iterations. For this purpose, the solvers ICCG and Bi-CGSTAB [5] are used.

Conclusions

1. A second-order numerical technique has been developed to study the steady laminar fluid motion in a straight flat hard-walled duct with two axisymmetric rectangular constrictions.

2. In this technique, the governing relations are solved via deriving their integral analogs, performing a discretization of these analogs, simplifying the obtained (after making the discretization) coupled nonlinear algebraic equations, and the final solution of the resulting (after making the simplification) uncoupled linear ones.

3. The discretization consists of the spatial and temporal parts. The first of them is performed with the use of the TVD-scheme and the two-point scheme of discretization of the spatial derivatives, whereas the second one is made on the basis of the implicit three-point asymmetric backward differencing scheme.

4. The above-noted uncoupled linear algebraic equations are solved by an appropriate iterative method, which uses the deferred correction implementation technique and the technique of conjugate gradients, as well as the solvers ICCG and Bi-CGSTAB.

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А. О. Борисюк, https://orcid.org/0000-0002-3878-3915

Інститут гідромеханіки НАН України, Київ
E-mail: aobor@ukr.net

ЧИСЕЛЬНИЙ МЕТОД РОЗВ'ЯЗУВАННЯ ЗАДАЧІ ПРО РУХ РІДИНИ У ПРЯМОМУ ПЛОСКОМУ ЖОРСТКОМУ КАНАЛІ З ДВОМА ОСЕСИМЕТРИЧНИМИ ПРЯМОКУТНИМИ ЗВУЖЕННЯМИ

Розроблено чисельний метод розв'язування задачі про стаціонарний ламінарний рух рідини у прямому плоскому жорсткому каналі з двома осесиметричними прямокутними звуженнями. Цей метод має другий порядок точності. У ньому співвідношення, що описують зазначений рух, розв'язуються шляхом одержання їхніх інтегральних аналогів, дискретизації цих аналогів, зведення зв'язаних нелінійних алгебраїчних рівнянь (одержаних внаслідок дискретизації) до відповідних незалежних лінійних і подальшого розв'язування останніх. Зазначена дискретизація складається із просторової та часової частин. Перша з них виконується на основі використання TVD-схеми, а також двоточкової схеми дискретизації просторових похідних. При проведенні ж другої частини дискретизації застосовується неявна триточкова несиметрична схема з різницями назад. Що стосується методу розв'язування вказаних незалежних лінійних рівнянь, то це — відповідний ітераційний метод, який використовує методи відкладеної корекції та спряжених градієнтів, а також солвери ICCG та Bi-CGSTAB.

Ключові слова: *рух рідини, плоский канал, прямокутне звуження, метод.*