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The description of the automorphism groups of finite-dimensional cyclic Leibniz algebras

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In the study of Leibniz algebras, the information about their automorphisms (as well as about endomorphisms, derivations, etc.) is very useful. We describe the automorphism groups of finite-dimensional cyclic Leibniz algebras. In particular, we consider the natural relationships between Leibniz algebras, groups and modules over associative rings.

Keywords: *Leibniz algebra, automorphism group, module over an associative ring.*

Let L be an algebra over a field F with the binary operations + and [,]. Then L is called a *left Leibniz algebra*, if it satisfies the left Leibniz identity

$$[[a,b],c]=[a,[b,c]]-[b,[a,c]]$$

for all $a, b, c \in L$.

Leibniz algebras first appeared in the paper by A. Blokh [1], whereas the term “Leibniz algebra” appeared in the book by J.-L. Loday [2], and his article [3]. In [4], J.-L. Loday and T. Pirashvili began to study the properties of Leibniz algebras. The theory of Leibniz algebras has been eveloped very intensively in many different directions. Some of the results of this theory were presented in book [5]. Note that the Lie algebras are a partial case of the Leibniz algebras. Conversely, if L is a Leibniz algebra, in which $[a, a]=0$ for every $a \in L$, then it is a Lie algebra. Thus, the Lie algebras can be characterized as anticommutative Leibniz algebras. The question about those properties of Leibniz algebras that are absent in Lie algebras and, accordingly, about those types of Leibniz algebras that have essential differences from Lie algebras naturally arises. A lot has already been done in this direction, including some results of the authors of this article. Many new

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results can be found in the survey papers [6—8]. In the study of various types of Leibniz algebras, the information about their endomorphisms and derivations is very useful, as shown, for example, in [9, 10]. The endomorphisms and derivations of infinite-dimensional cyclic Leibniz algebras were investigated in [11].

Let L be a Leibniz algebra over a field F. As usual, a linear transformation f of L is called an *endomorphism* of L, if

$$f([a,b]) = [f(a), f(b)]$$

for all $a, b \in L$. Clearly, a product of two endomorphisms of L is also an endomorphism of L, so that the set of all endomorphisms of L is a semigroup by multiplication. In the same time, the sum of two endomorphisms of L is not necessarily to be an endomorphism of L, so that we cannot speak about the endomorphism ring of L.

Here, we will use the term "*semigroup*" for the set that has an associative binary operation. For a semigroup with an identity element, we will use the term "*monoid*". Clearly, the identical transformation is an endomorphism of L. Therefore, the set $\mathrm{End}_{[\,,\,]}(L)$ of all endomorphisms of L is a monoid by multiplication. As usual, a bijective endomorphism of L is called an *automorphism* of L.

Let f be an automorphism of L. Then the mapping f^{-1} is also an automorphism of L. Indeed, let x, y be arbitrary elements of L. Then there are elements $u, v \in L$ such that $x = f(u)$, $y = f(v)$. We have

$$f^{-1}([x,y]) = f^{-1}([f(u), f(v)]) = f^{-1}(f[u,v]) = [u,v] = [f^{-1}(x), f^{-1}(y)].$$

Thus, the set $\mathrm{Aut}_{[\,,\,]}(L)$ of all automorphisms of L is a group by multiplication.

It should be noted that the endomorphisms of Leibniz algebras have hardly been studied. It is also quite unusual that the structure of cyclic Leibniz algebras was described relatively recently in [12]. We begin the study of the structure of the automorphism groups of finite-dimensional cyclic Leibniz algebras.

Let L be a Leibniz algebra. Define the *lower central series* of L

$$L = \gamma_1(L) \geqslant \gamma_2(L) \geqslant \ldots \gamma_a(L) \geqslant \gamma_{a+1}(L) \geqslant \ldots \gamma_d(L) = \gamma_\infty(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and recursively $\gamma_{a+1}(L) = [L, \gamma_a(L)]$ for all ordinals α and $\gamma_1(L) = \cap_{\mu<\lambda} \gamma_\mu(L)$ for the limit ordinals λ.

As usual, we say that a Leibniz algebra L is called *nilpotent*, if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be a *nilpotent of the nilpotentency class c*, if $\gamma_{c+1}(L) = \langle 0 \rangle$, but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively, *right*) *center* $\zeta^{\mathrm{left}}(L)$ (respectively, $\zeta^{\mathrm{right}}(L)$) of a Leibniz algebra L is defined by the following rule:

$$\zeta^{\mathrm{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for every } y \in L\}$$

(respectively,

$$\zeta^{\mathrm{right}}(L) = \{x \in L \mid [y, x] = 0 \text{ for every } y \in L\}).$$

The left center of L is an ideal of L, but it is not true for the right center. Moreover, $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$, so that $L / \zeta^{\mathrm{left}}(L)$ is a Lie algebra. The right center of L is a subalgebra of L, and, in general, the left and right centers are different. They can even have different dimensions (see [13]).

The *center* $\zeta(L)$ of L is defined by the following rule:

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [x, y] \text{ for every } y \in L\}.$$

The center is an ideal of L. Define the *upper central series*

$$\langle 0\rangle = \zeta_0(L) \leqslant \zeta_1(L) \leqslant \ldots \zeta_\alpha(L) \leqslant \zeta_{\alpha+1}(L) \leqslant \ldots \zeta_\gamma(L) = \zeta_\infty(L)$$

of a Leibniz algebra L by the following rule: $\zeta_1(L)=\zeta(L)$ is the center of L, and, recursively, $\zeta_{\alpha+1}(L)/\zeta_\alpha(L)=\zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L)=\cup_{\mu<\lambda}\zeta_\mu(L)$ for the limit ordinals λ.

Let L be a cyclic Leibniz algebra over a field $F, L=\langle a\rangle$, and suppose that L has finite dimension over F. Then there exists a positive integer n such that L has a basis $\{a_1, \ldots, a_n\}$, where $a_1=a$, $a_2=[a_1,a_1], \ldots, a_n=[a_1,a_{n-1}], [a_1,a_n]=\alpha_2a_2+\ldots+\alpha_na_n$ [12]. Moreover, $[L,L]=\mathrm{Leib}(L)=Fa_2+\ldots+Fa_n$. We fix these designations.

The study of the automorphism groups of finite-dimensional cyclic Leibniz algebras consists of the following three stages.

1. The automorphism group of a cyclic Leibniz algebra of type (I). The first case: $[a_1,a_n]=0$. In this case, L is nilpotent, and we will say that L is a *cyclic algebra of type* (I). The structure of the automorphism group of a cyclic Leibniz algebra of type (I) is described in Theorem A.

We start from the elementary properties of automorphisms and endomorphisms of Leibniz algebras.

Lemma 1. *Let L be a Leibniz algebra over a field F, and let f be an automorphism of L. Then* $f(\zeta^{\mathrm{left}}(L))=\zeta^{\mathrm{left}}(L)$, $f(\zeta^{\mathrm{right}}(L))=\zeta^{\mathrm{right}}(L)$, $f(\zeta(L))=\zeta(L)$, $f([L,L])=[L,L]$.

Corollary 1. *Let L be a Leibniz algebra over a field F, and let f be an automorphism of L. Then* $f(\zeta_\alpha(L))=\zeta_\alpha(L)$, $f(\gamma_\alpha(L))=\gamma_\alpha(L)$ *for all ordinals* α. *In particular,* $f(\zeta_\infty(L))=\zeta_\infty(L)$, $f(\gamma_\infty(L))=\gamma_\infty(L)$.

Lemma 2. *Let L be a Leibniz algebra over a field F, and let f be an endomorphism of L. Then* $f(\gamma_a(L))\leqslant\gamma_a(L)$ *for all ordinals a. In particular,* $f(\gamma_\infty(L))\leqslant\gamma_\infty(L)$.

Lemma 3. *Let L be a cyclic finite-dimensional Leibniz algebra over a field F. Let S be a subset of all endomorphisms f of L such that* $f(x)\in[L,L]$ *for every* $x\in L$. *Then* $S=\{f \mid f\in\mathrm{End}_{[\,,\,]}(L),\ f^2=0\}$, *and S is an ideal of* $\mathrm{End}_{[\,,\,]}(L)$ *with zero multiplication* $f\circ g=0$ *for every* $f,g\in S$.

Lemma 4. *Let L be a cyclic Leibniz algebra of type* (I) *over a field F. Then the linear mapping f is an endomorphism of L, if and only if*

$$\begin{aligned}
&f(a_1)=\gamma_1a_1+\gamma_2a_2+\ldots+\gamma_na_n,\\
&f(a_2)=\gamma_1^2a_2+\gamma_1\gamma_2a_3+\ldots+\gamma_1\gamma_{n-1}a_n,\\
&f(a_3)=\gamma_1^3a_3+\gamma_1^2\gamma_2a_4+\ldots+\gamma_1^2\gamma_{n-2}a_n,\\
&f(a_4)=\gamma_1^4a_4+\gamma_1^3\gamma_2a_5+\ldots+\gamma_1^3\gamma_{n-3}a_n,\\
&\ldots\\
&f(a_{n-1})=\gamma_1^{n-1}a_{n-1}+\gamma_1^{n-2}\gamma_2a_n,\\
&f(a_n)=\gamma_1^na_n.
\end{aligned}$$

Corollary 2. *Let L be a cyclic Leibniz algebra of type* (I) *over a field F. Then* $\mathrm{End}_{[,]}(L)$ *is isomorphic to a submonoid of* $\mathbf{M}_n(F)$ *consisting of all matrices having the following form*:

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \gamma_2 & \gamma_1^2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \gamma_3 & \gamma_1\gamma_2 & \gamma_1^3 & 0 & \ldots & 0 & 0 & 0 \\ \gamma_4 & \gamma_1\gamma_3 & \gamma_1^2\gamma_2 & \gamma_1^4 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_{n-2} & \gamma_1\gamma_{n-3} & \gamma_1^2\gamma_{n-4} & \gamma_1^3\gamma_{n-5} & \ldots & \gamma_1^{n-2} & 0 & 0 \\ \gamma_{n-1} & \gamma_1\gamma_{n-2} & \gamma_1^2\gamma_{n-3} & \gamma_1^3\gamma_{n-4} & \ldots & \gamma_1^{n-3}\gamma_2 & \gamma_1^{n-1} & 0 \\ \gamma_n & \gamma_1\gamma_{n-1} & \gamma_1^2\gamma_{n-2} & \gamma_1^3\gamma_{n-3} & \ldots & \gamma_1^{n-3}\gamma_3 & \gamma_1^{n-2}\gamma_2 & \gamma_1^n \end{pmatrix}.$$

Corollary 3. *Let L be a cyclic Leibniz algebra of type* (I) *over a field F. Then the automorphism group* $\mathrm{Aut}_{[\,,\,]}(L)$ *is isomorphic to a subgroup* $\mathbf{AC}(n)$ *of* $\mathbf{GL}_n(F)$ *consisting of all matrices having the following form*:

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \gamma_2 & \gamma_1^2 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \gamma_3 & \gamma_1\gamma_2 & \gamma_1^3 & 0 & \ldots & 0 & 0 & 0 \\ \gamma_4 & \gamma_1\gamma_3 & \gamma_1^2\gamma_2 & \gamma_1^4 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \gamma_{n-2} & \gamma_1\gamma_{n-3} & \gamma_1^2\gamma_{n-4} & \gamma_1^3\gamma_{n-5} & \ldots & \gamma_1^{n-2} & 0 & 0 \\ \gamma_{n-1} & \gamma_1\gamma_{n-2} & \gamma_1^2\gamma_{n-3} & \gamma_1^3\gamma_{n-4} & \ldots & \gamma_1^{n-3}\gamma_2 & \gamma_1^{n-1} & 0 \\ \gamma_n & \gamma_1\gamma_{n-1} & \gamma_1^2\gamma_{n-2} & \gamma_1^3\gamma_{n-3} & \ldots & \gamma_1^{n-3}\gamma_3 & \gamma_1^{n-2}\gamma_2 & \gamma_1^n \end{pmatrix}$$

where $\gamma_1 \neq 0$.

Corollary 4. *Let L be a cyclic Leibniz algebra of type* (I) *over a field F. Then a monoid of all endomorphisms of L is a union of an ideal*

$$S = \{f \mid f \in \mathrm{End}_{[\,,\,]}(L),\ f^2 = 0\}$$

and an automorphism group $\mathrm{Aut}_{[,]}(L)$. *Moreover, S is an ideal with zero multiplication* $f \circ g = 0$ *for every* $f, g \in S$.

Lemma 5. *Let L be a cyclic finite-dimensional Leibniz algebra over a field F. Let* $G = \mathrm{Aut}_{[,]}(L)$ *and U be a subset of all automorphisms f of L such that* $f(a_1) = a_1 + u$ *for some element* $u \in [L, L]$. *Then U is a normal subgroup of G and G/U is isomorphic to a subgroup of the multiplicative group of a field F.*

Corollary 5. *Let L be a cyclic Leibniz algebra of type* (I) *over a field* $F, G = \mathrm{Aut}_{[\,,\,]}(L)$. *Then G is a semidirect product of a normal subgroup U consisting of all automorphisms f of L such that* $f(a_1) = a_1 + u$ *for some* $u \in [L, L]$, *and a subgroup*

$$D = \{f \mid f \in \mathrm{Aut}_{[\,,\,]}(L), f(a_1) = \gamma a_1, 0 \neq \gamma \in F\}.$$

Moreover, D is isomorphic to the multiplicative group of a field F, and U is isomorphic to a subgroup $\mathbf{UC}(n)$ *of* $\mathbf{AC}(n)$ *consisting of matrices having the form*

$$E+\gamma_2\Sigma_{1\leqslant k\leqslant n-1}E_{k+1,k}+\gamma_3\Sigma_{1\leqslant k\leqslant n-2}E_{k+2,k}+\ldots+\gamma_n E_{n,1}.$$

Consider now a polynomial ring $F[X]$. Denote, by $R(n)$, the ideal of $F[X]$ generated by the polynomial X^n. Put $z=X+R(n)$. Then every element of a factor-ring $F[X]/R(n)$ has the form

$$\alpha_0+\alpha_1 z+\alpha_2 z^2+\ldots+\alpha_{n-1}z^{n-1},$$

$\alpha_0,\alpha_1,\alpha_2,\ \ldots,\alpha_{n-1}\in F$, and this representation is unique. It is possible to show that

$$\mathbf{U}(F[X]/R(n))=\{\alpha_0+\alpha_1 z+\alpha_2 z^2+\ldots+\alpha_{n-1}z^{n-1}\mid\alpha_0\neq 0\}.$$

Put

$$\mathbf{I}(F[X]/R(n))=\{1+\alpha_1 z+\alpha_2 z^2+\ldots+\alpha_{n-1}z^{n-1}\mid\alpha_0,\alpha_1,\alpha_2,\ \ldots,\alpha_{n-1}\in F\}.$$

Then it is not difficult to show that $\mathbf{I}(F[X]/R(n))$ is a subgroup of $\mathbf{U}(F[X]/R(n))$.

The following theorem describes the structure of the automorphism group of a cyclic Leibniz algebra of type (I).

Theorem A. *Let L be a cyclic Leibniz algebra of type* (I) *over a field F. Then* $\mathrm{Aut}_{[\,,\,]}(L)$ *is a semidirect product of a normal subgroup* $U\cong\mathbf{I}(F[X]/R(n))$ *and a subgroup* $D\cong\mathbf{U}(F)$.

2. The automorphism group of a cyclic Leibniz algebra of type (II). Consider now the second type of cyclic Leibniz algebras. In this case, $[a_1,a_n]=\alpha_2 a_2+\ldots+\alpha_n a_n$ and $\alpha_2\neq 0$. Put $c=\alpha_2^{-1}(\alpha_2 a_1+\ldots+\alpha_n a_{n-1}-a_n)$. Then $[c,c]=0$, $Fc=\zeta^{\mathrm{right}}(L)$, $L=[L,L]\oplus Fc$, $[c,b]=[a_1,b]$ for every $b\in[L,L]$ [12]. In particular, $a_3=[c,a_2],\ldots,a_n=[c,a_{n-1}]$, $[c,a_n]=\alpha_2 a_2+\ldots+\alpha_n a_n$. In this case, we say that L is a *cyclic algebra of type* (II). For the description of the automorphism group of such algebra we consider the relationships between Leibniz algebras and the modules over an associative ring.

Let L be a Leibniz algebra over a field F, and let A be an Abelian ideal of L. Denote, by $\mathrm{End}_F(A)$, the set of all linear transformations of A. Then $\mathrm{End}_F(A)$ is an associative algebra by the operations + and $[\circ,\circ]$. As usual, $\mathrm{End}_F(A)$ is a Lie algebra by the operations + and $[\circ,\circ]$, where $[f,g]=f\circ g-g\circ f$ for all $f,g\in\mathrm{End}_F(A)$.

Let u be an arbitrary element of L. Consider the mapping $\mathbf{l}_u: A\to A$, defined by the rule $\mathbf{l}_u(x)=[u,x]$, $x\in A$. For every $u,v\in L$ and $\lambda\in F$. We have

$$\mathbf{l}_u(x+y)=[u,x+y]=[u,x]+[u,y]=\mathbf{l}_u(x)+\mathbf{l}_u(y),$$
$$\mathbf{l}_u(\alpha x)=[u,\alpha x]=\alpha[u,x]=\alpha\mathbf{l}_u(x).$$

Hence, $\mathbf{l}_u$ is a linear transformation of A. Furthermore,

$$\beta\mathbf{l}_u(x)=\beta[u,x]=[\beta u,x]=\mathbf{l}_{\beta u}(x)$$

for every $x \in A$, which implies that $\beta\mathbf{l}_u = \mathbf{l}_{\beta u}$. Moreover,

$$(\mathbf{l}_u + \mathbf{l}_v)(x) = \mathbf{l}_u(x) + \mathbf{l}_v(x) = [u, x] + [v, x] = [u+v, x] = \mathbf{l}_{u+v}(x),$$

which yields $\mathbf{l}_u + \mathbf{l}_v = \mathbf{l}_{u+v}$. Consider the mapping $\vartheta : L \to \mathrm{End}_F(A)$ defined by the rule $\vartheta(u) = \mathbf{l}_u, u \in L$. By the above equalities, this mapping is linear. A subspace $\mathrm{Im}(\vartheta)$ is a Lie subalgebra of the Lie algebra associated with $\mathrm{End}_F(A)$. Denote, by $\mathbf{S}_L(A)$, the associative subalgebra of $\mathrm{End}_F(A)$ generated by $\mathrm{Im}(\vartheta)$. Then the action of L on A can be extended in a natural way to the action of $\mathbf{S}_L(A)$. Then A becomes to module over the associative ring $\mathbf{S}_L(A)$. This relationship can be used in a following way.

Put $A = [L, L]$. A linear transformation $\mathbf{l}_c : A \to A$ in the basis $\{a_2, \ldots, a_n\}$ has the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \alpha_2 \\ 1 & 0 & 0 & \ldots & 0 & \alpha_3 \\ 0 & 1 & 0 & \ldots & 0 & \alpha_4 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & \alpha_{n-1} \\ 0 & 0 & 0 & \ldots & 1 & \alpha_n \end{pmatrix}.$$

This matrix is non-degenerate. Hence, $\mathbf{l}_c$ is an F-automorphism of a linear space A. We will consider A as an $F\langle g\rangle$-module, where $\langle g\rangle$ is an infinite cyclic group, and the action of g on A is defined by the following rule: $ga = \mathbf{l}_c(a) = [c, a]$ for every $a \in A$.

Consider now the dual situation. Let A be a vector space over a field F, and let R be a subalgebra of the associative algebra $\mathrm{End}_F(A)$ of all F-endomorphisms of A. Then we can consider R as a Lie algebra by the operation $[f, g] = f \circ g - g \circ f,\ f, g \in R$. In th Lie algebra R, let us choose some Lie-subalgebra S. Put $L = A \oplus S$ and define an operation $[\circ, \circ]$ on L by the following rule:

$[f, g] = f \circ g - g \circ f$ for all $f, g \in S$,

$[a, b] = 0$ for all $a, b \in A$,

$[a, f] = 0$ for all $a \in A,\ f \in S$,

$[f, a] = f(a)$ for all $a \in A,\ f \in S$.

By such definition, the left center of L includes A. The direct check shows that L is a Leibniz algebra. If the subalgebra R is commutative, then S as a Lie algebra is Abelian. In this case, the right center of L includes S.

Let now A be a finite-dimensional vector space over a field F, and let c be an F-automorphism of A. Let $R = F\langle c\rangle$ be an associative subalgebra of $\mathrm{End}_F(A)$ generated by the automorphism c. This subalgebra is commutative. Therefore, R as a Lie algebra is Abelian. Then a subspace Fc is a Lie subalgebra of R. Using the above construction, we can construct a Leibniz algebra $L = A \oplus Fc$. By this way, we come to a cyclic Leibniz algebra of type (II).

Using these constructions, we obtain the description of the automorphism group of a cyclic Leibniz algebra of type (II).

Lemma 6. *Let L be a cyclic Leibniz algebra of type* (II) *over a field F, and let D be a centralizer of a subspace Fc in a monoid* $\mathrm{End}_{[\,,\,]}(L)$. *Then D is a submonoid of* $\mathrm{End}_{[\,,\,]}(L)$. *Moreover,* $C = D \cap \mathrm{Aut}_{[\,,\,]}(L)$ *is a normal subgroup of* $\mathrm{Aut}_{[\,,\,]}(L)$.

Lemma 7. *Let L be a cyclic Leibniz algebra of type* (II) *over a field F, and let D be a centralizer of a subspace Fc in a monoid* $\mathrm{End}_{[\,,\,]}(L)$. *Then D is isomorphic to a multiplicative monoid of a factor-ring* $F[X]/\mathbf{a}(X)F[X]$ *where*

$$\mathbf{a}(X) = \alpha_2 + \alpha_3 X + \ldots + \alpha_n X^{n-2} - X^{n-1}.$$

Theorem B. *Let L be a cyclic Leibniz algebra of type* (II) *over a field F. Then* $\mathrm{Aut}_{[\,,\,]}(L) = G$ *includes a normal subgroup C, which is isomorphic to* $\mathbf{U}(F[X]/\mathbf{a}(X)F[X])$, *where* $\mathbf{a}(X) = \alpha_2 + \alpha_3 X + \ldots + \alpha_n X^{n-2} - X^{n-1}$ *such that G/C is isomorphic to a subgroup of a multiplicative group of a field F.*

3. The automorphism group of a cyclic Leibniz algebra of type (III). Consider now the third type of cyclic Leibniz algebras. In this case, $[a_1, a_n] = \alpha_2 a_2 + \ldots + \alpha_n a_n$ and $\alpha_2 = 0$. Let t be the first index such that $a_t \neq 0$. In other words,

$$[a_1, a_n] = a_t a_t + \ldots + a_n a_n.$$

By our condition, $t > 2$. Then

$$[a_1, a_n] = \alpha_t [a_1, a_{t-1}] + \ldots + \alpha_n [a_1, a_{n-1}] = [a_1, \alpha_t a_{t-1} + \ldots + \alpha_n a_{n-1}],$$

which implies that $\alpha_t a_{t-1} + \ldots + \alpha_n a_{n-1} - a_n \in \mathrm{Ann}_{\mathrm{L}}^{\mathrm{right}}(a_1)$. Since $\alpha_t \neq 0, \alpha_t^{-1} \neq 0$ and

$$d_{t-1} = \alpha_t^{-1}(\alpha_t a_{t-1} + \ldots + \alpha_n a_{n-1} - a_n) = a_{t-1} + \beta_t a_t + \ldots + \beta_n a_n \in \mathrm{Ann}_{\mathrm{L}}^{\mathrm{right}}(a_1).$$

Put

$$d_{t-2} = a_{t-2} + \beta_t a_{t-1} + \ldots + \beta_n a_{n-1},$$
$$d_{t-3} = a_{t-3} + \beta_t a_{t-2} + \ldots + \beta_n a_{n-2}, \ \ldots,$$
$$d_1 = a_1 + \beta_t a_2 + \ldots + \beta_n a_{n-t+1}.$$

Then

$$[d_1, d_1] = [a_1, d_1] = d_2,$$
$$[d_1, d_2] = [a_1, d_2] = d_3, \ldots,$$
$$[d_1, d_{t-2}] = [a_1, d_{t-2}] = d_{t-1},$$
$$[d_1, d_{t-1}] = [a_1, d_{t-1}] = 0.$$

It follows that the subspace $U = Fd_1 \oplus Fd_2 \oplus \ldots \oplus Fd_{t-1}$ is a nilpotent subalgebra. Moreover, the subspace $[U, U] = Fd_2 \oplus \ldots \oplus Fd_{t-1}$ is an ideal of L. Put further $d_t = a_t, d_{t+1} = a_{t+1}, \ldots, d_n = a_n$. The following matrix corresponds to this transaction:

$$\begin{pmatrix} 1 & \beta_t & \beta_{t+1} & \ldots & \beta_n & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & \beta_t & \ldots & \beta_{n-1} & \beta_n & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & \beta_{n-2} & \beta_{n-1} & \beta_n & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & \beta_t & \beta_{t+1} & \ldots & \beta_{n-1} & \beta_n & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & \beta_t & \ldots & \beta_{n-2} & \beta_{n-1} & \beta_n \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.$$

This matrix is non-singular, which shows that the elements $\{d_1, \ldots, d_n\}$ form a new basis. We note that the subspace $V = Fd_t \oplus \ldots \oplus Fd_n$ is a subalgebra. Moreover, V is an ideal of L, because $[a_1, d_t] = d_{t+1}, \ldots, [a_1, d_{n-1}] = d_n, [a_1, d_n] = \alpha_t d_t + \ldots + \alpha_n d_n$. Moreover, $[a_1, d_j] = [d_1, d_j]$ for all $j \geqslant t$ [12]. In this case, we say that L is a *cyclic algebra of type* (III).

Thus, $L = A \oplus Fd_1$, $A = V \oplus [U, U]$ is the direct sum of two ideals, $U = [U, U] \oplus Fd_1$ is a nilpotent cyclic subalgebra, i.e. is an algebra of type (I), and $V \oplus Fd_1$ is a cyclic subalgebra of type (II). The structure of the automorphism group of a cyclic Leibniz algebra of type (III) is described in the following theorem.

Theorem C. *Let L be a cyclic Leibniz algebra of type* (III) *over a field F. Then* $\mathrm{Aut}_{[\,,\,]}(L)$ *is a subdirect product of groups* G_1 *and* G_2, *where* G_1 *and* G_2 *are the groups described in Theorems A, and B, respectively.*

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ОПИС ГРУП АВТОМОРФІЗМІВ
СКІНЧЕННОВИМІРНИХ ЦИКЛІЧНИХ АЛГЕБР ЛЕЙБНІЦА

Для вивчення алгебр Лейбніца інформація про їх автоморфізми (а також про ендоморфізми, диференціювання та ін.) є дуже корисною. Описано групи автоморфізмів скінченновимірних циклічних алгебр Лейбніца. Зокрема, розглянуто природні зв'язки між алгебрами Лейбніца, групами та модулями над асоціативними кільцями.

Ключові слова: *алгебра Лейбніца, група автоморфізмів, модуль над асоціативним кільцем.*