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A multiplicity theorem for Fréchet spaces

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This note serves to announce a multiplicity result for Keller C_c^1-functionals on Fréchet spaces which are invariant under the action of a discrete subgroup. For such functionals, we evaluate the minimal number of critical points by applying the Lyusternik—Schnirelmann category.

Keywords: *Fréchet spaces, Lyusternik—Schnirelmann category, Palais—Smale condition, discrete group action.*

We consider a multiplicity problem, namely evaluating the minimal number of the critical orbits of a functional $f : F \to R$ which is invariant under the action of a discrete subgroup G of a Fréchet spaces F. In [1], it was proved that if a functional $f : F \to R$ of the Keller class C_c^1 is bounded from below and satisfies the Palais—Smale condition at the level $c = \inf f$, then c is a critical value for f. Our goal is to significantly improve this result. To this end, we consider functionals which are invariant under a discrete subgroup action. To evaluate the minimal numbers of critical points of such functionals, we employ the Lyusternik–Schnirelmann method.

1. A compactness condition. The initial point of our approach is to introduce a compactness condition of the Palais—Smale type for G-invariant functionals.

We denote by F a Fréchet space whose topology is defined by an increasing sequence of seminorms $(\|\cdot\|_n)$. Moreover, the complete translation-invariant metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}$$

induces the same topology on F. We denote by $B(x, r)$ an open ball with center x and radius $r > 0$ with respect to this metric.

In what follows, we consider only Fréchet spaces over the field R of real numbers. Let E be another Fréchet space, $C(E, F)$ the set of all continuous linear mappings from E to F. A bornology β_E on E is a covering of E satisfying the following:

1. β_E is stable under finite unions;
2. if $A \subseteq \beta_E$ and $B \subseteq A$, then $B \subseteq \beta_E$.

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The compact bornology on E is the family β_{EC} of relatively compact subsets of E having the set of all compact subsets of F as a base, in the sense that every $B\in\beta_{EC}$ is contained in some compact set. We endow the vector space $C(E, F)$ with the β_{EC} -topology which is the topology of uniform convergence on all compact subsets of E. This is a Hausdorff locally convex topology which can be defined by the family of all seminorms obtained as follows:

$$\|L\|_{B,n} = \sup\{\|L(e)\|_n : e\in B\}$$

where $B\in\beta_{EC}$ and $n\in N$. Let U be an open subset of E and $f: E\to F$ be a mapping. If the directional derivatives

$$f(x)h = \lim_{t\to 0}\frac{f(x+th)-f(x)}{t}$$

exist for all $x\in U$ and all $h\in E$, and the induced map $df: U\to C(E,F)$ is continuous for all $x\in U$, then we say that f is a Keller C_c^1-mapping (see [2]).

Let $\boldsymbol{L}$ be a topological group with the identity element $\mathbf{e}$. A continuous action of $\boldsymbol{L}$ on F is a mapping $A:\mathbf{L}\times F\to F$, $A(l,m)$ written as $l\cdot m$, such that $\mathbf{e}\cdot m = m$ and $(l_1 * l_2)\cdot m = l_1\cdot(l_2\cdot m)$ for all $l_1, l_2\in\boldsymbol{L}$ and all $m\in F$ (here, * denotes the operation of $\boldsymbol{L}$). A set $A\in F$ is called $\boldsymbol{L}$-invariant, if $l\cdot m\in A$ for all $m\in A$ and all $l\in\boldsymbol{L}$. A functional $f: F\to R$ is called $\boldsymbol{L}$-invariant, if $f(l\cdot m) = f(m)$ for all $l\in\boldsymbol{L}$ and $m\in F$. A mapping $h: F\to F$ is called $\boldsymbol{L}$-equivalent, if $h(l\cdot m) = l\cdot h(m)$ for all $m\in F$ and all $l\in\boldsymbol{L}$. Let G be a discrete subgroup of a Fréchet space F, and let $q: F\to F/G$ be the canonical surjection. A subset $A\subseteq F$ is called q-saturated, if $A = q^{-1}\circ q(A)$. Suppose the space F_1 generated by G has the dimension n. Let F_2 be a topological complement of F_1, such that F is isomorphic to $F_1\times F_2$. Let $\mathbf{T}^n$ be the n-torus, then $G\simeq Z^n$ and $q(F)\simeq Z^n\times F_2$. Let c be critical point of f. We call the set $q^{-1}(q(c))$ consisting of the critical points of f, a critical orbit of f through c.

Definition 1. Let $f: F\to R$ be a G-invariant functional of the Keller class C_{c}^1. We say that f satisfies the Palais—Smale condition, $\mathrm{PS}_{\mathbf{G}}$-condition for short, if, for every sequence $(x_n)\subset F$ for which $f(x_n)$ is bounded and $f'(x)\to 0$, the sequence $q(x_n)$ contains a convergent subsequence.

2. A multiplicity theorem. To locate critical points, we will apply the strong version of the Ekeland variational principle (see [3]). It states the existence of a certain minimizing sequence on a complete metric space along which we reach the infimum value of the minimization problem.

The Lyusternik—Schnirelmann category $\mathrm{Cat}_T A$ of a subset A of a topological space T is the minimal number of closed sets that cover A and each of which is contractible to a point in T. If $\mathrm{Cat}_T A$ is not finite, we write $\mathrm{Cat}_T A = \infty$. Let $\mathrm{Co}(F)$ be the set of compact subsets of F. Define the sets

$$A_i = \{A\subset F : A\in \mathrm{Co}(T), \mathrm{Cat}_{q(F)} q(A)\geqslant i\}.$$

From [4, Proposition 2.2], it follows that each A_i is a deformation invariant class of subsets of F. The i-th Lyusternik—Schnirelmann minimax value of f is defined by

$$\mu_i = \inf_{A\in A_i}\sup_{x\in A} f(x).$$

The proofs of the following two lemmas are based on the standard arguments, see, for example [4, Lemma 3.2, Lemma 3.3]. Let $CB(F)$ be the family of all nonempty closed and bounded subsets of F. We define the Hausdorff metric on $CB(F)$ by

$$d_H(A,B)=\max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}.$$

Lemma 1. *The space* (A_i, d_H) *is a complete metric space.*

Proof. The space $(CB(F), d_H)$ is complete since F is complete (cf. [5]). Thus, we only need to prove that A_i is closed in $CB(F)$. Let $(A_k)\subset A_i$. Suppose $A\in CB(F)$ and $d_H(A_k, A)\to 0$. By [6, Corollary 1.2.13] and [6, Proposition 1.2.14], the space $q(F)\simeq \mathbf{T}^n\times F_2\simeq (\mathbf{S}^1)^n\times F_2$ is an ANR. So, by [7, Theorem 6.3], there exists a closed neighborhood U of A such that $\mathrm{Cat}_{q(F)}(q(A))=\mathrm{Cat}_{q(F)}U$. As $q^{-1}(U)$ is a closed neighborhood of the compact set A, there exists k such that $A_k\subset U$. Thereby, $\mathrm{Cat}_{q(F)}(q(A))=\mathrm{Cat}_{q(F)}U\geqslant \mathrm{Cat}_{q(F)}(q(A_k))\geqslant i$. Therefore $A\in A_i$.

Lemma 2. *Let* $f: F\to R$ *be a G-invariant functional of the Keller class* C_c^1. *Then, the function* $\Psi: A_i\to R$ *defined by* $\Psi(A)=\max_{x\in A} f(x)$ *is lower semicontinuous.*

Proof. Let $(B_k)\subset A_i$. Suppose $B\in A_i$ and $d_H(B_k, B)\to 0$. For each $x_0\in B$, there exists a sequence $(x_j)\subset B_k$ such that $x_j\to x_0$. Thus,

$$f(x_0)=\lim_{j\to\infty} f(x_j)\leqslant \lim_{k\to\infty}\Psi(B_k)$$

and, as $x_0\in B$ is arbitrary, we have $\psi(B)\leqslant \varliminf_{k\to\infty}\Psi(B_k)$.

Theorem 1. *Let G be a discrete subgroup of a Fréchet spaces F. Assume that the dimension of the space generated by G is a finite number n. Let* $f: F\to R$ *be a G-invariant functional of the Keller class* C_c^1. *If f is bounded from below and satisfies the Palais—Smale condition, then f has* $n+1$ *critical orbits.*

Proof. Consider the increasing sequence of the Lyusternik—Schnirelmann minimax values μ_i, $1\leqslant i\leqslant n+1$. Define the sets

$$S_{\mu_i}=\{x\in F: f'(x)=0, f(x)=\mu_i\}.$$

We claim that if $\mu_i=\mu_k=\mu$ for some $k, i\leqslant k\leqslant n+1$, then S_{μ_i} contains $k-i+1$ critical orbits. This concludes the proof of the theorem.

We prove the claim by contradiction. Suppose that S_μ contains m distinct critical orbits $q(x_1),\cdots,q(x_m)$ and $m\leqslant k-1$. Pick the positive number r_0 so that, on the balls $B(x_j, 2r_0)$, $1\leqslant j\leqslant m$ the canonical surjection q is injective. Define the set

$$B_{r_0}=\bigcup_{j=1}^{m}\bigcup_{g\in G} B(x_j+g, r).$$

We show that there exists ε, $0<\varepsilon^2<r_0^2$ such that

$$\|f'(x)\|_B>\varepsilon,\quad \forall B\in\beta_{FC} \tag{1}$$

if $x \in f^{-1}([\mu-\varepsilon^2, \mu+\varepsilon^2]) \setminus B_{r_0}$. Because, if (1) is not valid, then there exists a sequence $(x_j) \subset F \setminus B_{r_0}$ such that

$$\left| f(x_j) \right| \leqslant \mu + 1/j \quad \text{and} \quad \|f'(x)\|_{B,n} \leqslant 1/j, \ \forall n \in N, B \in \beta_{FC}.$$

Since f satisfies the $\mathrm{PS}_{\mathbf{G}}$-condition, we may assume that $q(x_j) \to q(\bar{x})$ for some $\bar{x} \in F$. Since f and f' are G-invariant, we may suppose that $x_j \in [0,1]^n \times F_2$.

Whence, $x_j \to \bar{x}$ yields $\bar{x} \in F \setminus B_{r_0}$ and $f(\bar{x}) = \mu$ and $f'(\bar{x}) = 0$ which is impossible, because B_{r_0} is a neighborhood of S_μ. There exists $A \in A_i$ such that

$$\Psi(A) = \max_A f \leqslant \mu + \varepsilon^2.$$

This is achievable by the definition of μ_k. Let $A_0 = A \setminus B_{2r_0}$. By [4, Proposition 2.2], we obtain

$$k \leqslant \mathrm{cat}_{q(F)} q(A) \leqslant \mathrm{cat}_{q(F)} (q(A_0) \cup q(B_{2r_0})) \leqslant \mathrm{cat}_{q(F)} q(A_0) + \mathrm{cat}_{q(F)} q(B_{2r_0}) \leqslant$$

$$\leqslant \mathrm{cat}_{q(F)} q(A_0) + m \leqslant \mathrm{cat}_{q(F)} q(A_0) + k - i.$$

Thus, $A_0 \in A_i$. By Lemma 1, the space (A_i, d_H) is complete. Also, by Lemma 2, the function $\Psi : A_i \to R$ is lower semicontinuous. So, we can employ the Ekeland variational theorem [3, Theorem 4.7]. By the latter theorem, there exists $C \in A_i$ such that

(P1) $\psi(C) \leqslant \psi(A_0) \leqslant \psi(A) \leqslant \mu + \varepsilon^2$,

(P2) $d_H(C, A_0) \leqslant \varepsilon$,

(P3) $\psi(S) > \psi(C) - \varepsilon d_H(C, S), \ \forall S \in A_i, \ S \neq C$.

As $A_0 \cap B_{2r_0} = \varnothing$ and $d_H(C, A_0) \leqslant \varepsilon \leqslant r_0$, then $A \cap B_{2r_0} = \varnothing$. Also, the set $D = \{s \in C : \mu - \varepsilon^2 \leqslant f(s)\}$ is a subset of $f^{-1}([\mu-\varepsilon^2, \mu+\varepsilon^2]) \setminus B_{r_0}$. The set D is closed and, as f is continuous, then it is compact. By (3) for each $y \in D$, there exists $h_{B,y} \in B$ such that

$$\langle f'(y), h_{B,y} \rangle < -\varepsilon. \tag{2}$$

Since f' is continuous, it follows from (2) that there exists $r_y > 0$ such that, for all $g \in G$ and all $h \in F$ with $\|h\|_n < r_y$, we have

$$\langle f'(y+g+h), h_{B,y} \rangle < -\varepsilon.$$

Since D is compact, we can find a subcovering $D_1, \cdots, D_p$ defined by

$$D_i = B(y_i, r_{y_i}).$$

Define the functions $\Phi_i : F \to [0,1]$ by

$$\phi_i(x) = \begin{cases} \dfrac{\sum\limits_{g\in G} d(x+g, \complement D_i)}{\sum\limits_{k=1}^{p}\sum\limits_{g\in G} d(x+g, \complement D_k)}, & x \in \bigcup\limits_{j=1}^{p} D_j, \\ 0, & \textit{otherwise}. \end{cases}$$

Fix a G-invariant continuous function $\Phi : F \to [0,1]$ such that

$$\phi(x) = \begin{cases} 1, & \mu \leqslant f(x), \\ 0, & f(x) \leqslant \mu - \varepsilon^2. \end{cases}$$

Let $r_{\min} = \min\limits_{1\leqslant i\leqslant p} r_{y_i}$. Define the continuous curve $\lambda : [0,1]\times F \to F$

$$\lambda(t,x) = x + tr_{\min}\phi(x)\sum_{i=1}^{p}\psi_i(x)(h_{B,y_i}).$$

For all $x \in F$, all $g \in G$ and all $t \in [0,1]$, we have $\lambda(t, x+g) = \lambda(t,x) + g$.

It follows from [4, Proposition 2.2] that $\mathrm{cat}_{q(F)}(q(\lambda(1,C))) \geqslant \mathrm{cat}_{q(F)}(q(C)) \geqslant i$, whence, as $\lambda(1,C)$ is compact, $\lambda(1,C) \in A_i$. By the mean-value theorem (see [2]) and (P3) for each $y \in D$, there is $T \in (0,1)$ such that

$$f(\lambda(1,C)) - f(x) = \left\langle f'(\lambda(T,C)), r_{\min}\Phi(x)\sum_{i=1}^{p}\Psi_i(x)(h_{B,y_i})\right\rangle =$$

$$= r_{\min}\Phi(x)\sum_{i=1}^{p}\Psi_i(x)\left\langle f'\left(x + Tr_{\min}\Phi(x)\sum_{i=1}^{p}\Psi_i(x)(h_{B,y_i})\right), h_{B,y_i}\right\rangle \leqslant$$

$$\leqslant -\varepsilon r_{\min}\Phi(x).$$

If $x \in D$, then $\Phi(x) = 0$ and $f(\lambda(1,x)) = f(x)$.

Let $y_0 \in C$ so that $f(\lambda(1,y_0)) = \psi(D)$. Then, $\mu \leqslant f(\lambda(1,y_0)) - f(y_0) \leqslant -\varepsilon r_{\min}$. So, $y_0 \in D$ and $\Phi(y_0) = 1$ which imply that $f(\lambda(1,y_0)) - f(y_0) \leqslant -\varepsilon r_{\min}$. Therefore,

$$\psi(S) + \varepsilon d_H(C,S) \leqslant \psi(C).$$

However, $d_H(C,S) \leqslant r_{\min}$ by the definition of S. Hence, $\psi(S) + \varepsilon d_H(C,S) \leqslant \psi(C)$ which contradicts (P3) and concludes the proof.

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ТЕОРЕМА КРАТНОСТІ ДЛЯ ПРОСТОРІВ ФРЕШЕ

У статті сформульовано теорему кратності для функціоналів з класу Келлера C_c^1 на просторах Фреше. Для таких функціоналів ми даємо мінімальну кількість критичних точок, застосовуючи категорію Люстерника–Шнірельмана.

Ключові слова*: простори Фреше, категорія Люстерника–Шнірельмана, умова Палаіса–Смейла, дія дискретної групи.*