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O.A. Dovgoshey[1, 2], https://orcid.org/0000-0002-6496-2466
J. Prestin[2], https://orcid.org/0000-0001-5985-7939
I.O. Shevchuk[3], https://orcid.org/0000-0003-1140-373X

[1] Institute of Applied Mathematics and Mechanics of the NAS of Ukraine, Slov'yansk
[2] Institute of Mathematics, University of Lubeck, Germany
[3] Taras Shevchenko National University of Kyiv
E-mail: oleksiy.dovgoshey@gmail.com, prestin@math.uni-luebeck.de, shevchuk@univ.kiev.ua

Bernstein-type characterization of entire functions

Presented by Corresponding Member of the NAS of Ukraine I.O. Shevchuk

Let $\mathcal{E}$ be the set of all entire functions on the complex plane $\mathbb{C}$. Let us consider the class $\mathbf{X_E}$ of all complex Banach spaces X such that $X \supseteq \mathcal{E}$. For $(X, \|\cdot\|) \in \mathbf{X_E}$ and $g \in X$ we write $E_{n,X}(g) = \inf\{\|g - p\| : p \in \Pi_n\}$, where Π_n is the set of all polynomials with degree at most n. We describe all $X \in \mathbf{X_E}$ for which the relation $\lim_{n\to\infty} (E_{n,X}(g))^{1/n} = 0$ holds if and only if $g \in \mathcal{E}$.

Keywords: *Bernstein theorem, entire function, polynomial approximation, Shauder basis, transfinite diameter.*

1. Introduction. ***The initial Bernstein theorem.*** Let f be a real-valued continuous function on $[-1,1]$ and let $E_{n,[-1,1]}(f)$ be the minimum error in the Chebyshev approximation of f on $[-1,1]$ by polynomials of degree at most n.

Theorem 1. (Bernstein theorem). *The equality $\lim_{n\to\infty} E^{1/n}_{n,[-1,1]}(f) = 0$ holds if and only if f is the restriction of an entire function to $[-1,1]$.*

This theorem was published in the classical book [1].

The Introduction briefly describes the early development of Bernstein theorem 1. In Section 2 we formulate two new theorems and two conjectures describing the structure of Banach spaces for which "the Bernstein theorem" remains valid.

The Walsh theorem. In 1926 J. L. Walsh [2] published the following result.

Theorem 2 (Walsh theorem). *Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the one-point compactification of the complex plane, K be a compact subset of $\mathbb{C}$ and let $\overline{\mathbb{C}} \setminus K$ be a simply connected regular for Dirichlet problem domain. Then the following statements are equivalent for every continuous function $f : K \to \mathbb{C}$:*

(i) *f is the restriction to K of an entire function;*

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(ii) *the equality* $\lim_{n\to\infty} E_{n,K}^{1/n}(f) = 0$ *holds, where* $E_{n,K}(f)$ *is the minimum error in uniform approximation of* f *on* K *by polynomials with degree at most* n.

Recall that the domain $\overline{\mathbb{C}} \setminus K$ is regular for Dirichlet problem if and only if it possesses the classical Green function with pole at infinity.

Considering the over-convergence of polynomials of the best uniform approximation J. Walsh and H. Russell obtained (see [3]) a result which implies that the equivalence (i) $\Leftrightarrow$ (ii) in Walsh theorem 2 remains valid if $\overline{\mathbb{C}} \setminus K$ is an arbitrary regular for Dirichlet problem domain.

The extension of Bernstein theorem by R. S. Varga. For more than thirty years, the Bernstein-Walsh-Russell theorems do not actually attract the attention of mathematicians till the paper of R. S. Varga [4] who characterized the order and type of an entire function f by minimum error sequence $(E_{n,[-1,1]}(f))_{n\in\mathbb{N}}$.

It should be noted here that these remarkable characteristics and the results associated with them are not the subject of present paper, and we limit ourselves to studying the equivalence

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0 \text{ iff } f \text{ is entire.}$$

Reformulation of Bernstein-Walsh-Russell theorems. For further it is convenient to give some suitable reformulations of the Bernstein-Walsh-Russell theorems.

Let us denote by $\mathcal{E}$ the set of all entire functions $f:\mathbb{C}\to\mathbb{C}$ and write Π_n for the set of all polynomials of degree at most n. Now we define the class $\mathbf{X_E}$ as follows.

Definition 1. By $\mathbf{X_E}$ we denote the class of all complex Banach linear spaces $(X, \|\cdot\|)$ such that $(X, \|\cdot\|)$ belongs to $\mathbf{X_E}$ if and only if $\mathcal{E} \subseteq X$.

For $(X, \|\cdot\|) \in \mathbf{X_E}$, we define the set $\mathcal{L}_X$ as $\mathcal{L}_X := \{f \in X : \lim_{n\to\infty} (E_{n,X}(f))^{1/n} = 0\}$, where, for every $n \in \mathbb{N}$, $E_{n,X}(f) = \inf\{\|f - p\| : p \in \Pi_n\}$. We will also denote by C_K the set of all continuous complex-valued functions on the compact $K \subseteq \mathbb{C}$ and write $\|f\|_\infty := \sup_{z\in K} |f(z)|$ for $f \in C_K$.

Now the classical results of Bernstein, Walsh, and Walsh-Russell can be formulated as follows.

Theorem 3. (Bernstein theorem). *Let* $(X, \|\cdot\|) = (C_{[-1,1]}, \|\cdot\|_\infty)$. *Then the equality*

$$\mathcal{E}_X = \mathcal{L}_X \tag{1}$$

holds.

Theorem 4. (Walsh theorem). *Let* $(X, \|\cdot\|) = (C_K, \|\cdot\|_\infty)$. *Equality* (1) *holds if* $\overline{\mathbb{C}} \setminus K$ *is a simply connected regular for Dirichlet problem domain.*

Theorem 5. (Walsh-Russell theorem). Let $(X, \|\cdot\|) = (C_K, \|\cdot\|_\infty)$. *Equality* (1) *holds if* $\overline{\mathbb{C}} \setminus K$ *is a regular for Dirichlet problem domain.*

2. The main results. In this section we formulate new Theorems 6, 7, and Conjectures 1, 2.

Theorem 6. *Let* $X \in \mathbf{X_E}$. *Then equality* (1) *holds iff* $0 < \liminf_{n\to\infty} (\tau_{n,X})^{1/n}$ *and* $\limsup_{n\to\infty} (m_{n,X})^{1/n} < \infty$ *hold.*

Theorem 7. *Let a complex Banach space* Y *have a Shauder basis. Then* Y *is linearly isometric to a space* $X \in \mathbf{X_E}$.

By Mazur's theorem, every infinite-dimensional vector normed space contains an infinite-dimensional subspace that has a Shauder basis (see, for example, Theorem 6.3.3 in [5]). Hence, Theorem 7 implies the following.

Corollary 1. *Every infinite-dimensional complex Banach space contains a subspace Y which is linearly isometric to some $X \in \mathbf{X_E}$.*

Let us consider now some corollaries of Theorem 6 for the case of uniform approximation.

It is clear that

$$\boxed{\text{Walsh-Russell theorem}} \Rightarrow \boxed{\text{Walsh theorem}} \Rightarrow \boxed{\text{Bernstein theorem}}$$

In what follows we will use the concept of transfinite diameter.

For $K \subseteq \mathbb{C}$ and $u_1, \ldots, u_n \in K$, we write

$$V(u_1, \ldots, u_n) := \prod_{\substack{k,l \\ k<l}} (u_k - u_l)$$

and

$$V_n = V_n(K) := \sup\{|V(u_1, \ldots, u_n)| : u_j \in K, 1 \leqslant j \leqslant n\}.$$

In accordance with M. Fekete, the *transfinite diameter* of K is the number

$$d(K) = \lim_{n\to\infty} V_n^{\frac{2}{n(n-1)}}.$$

Let $(X, \|\cdot\|) \in \mathbf{X_E}$. If $f_n \in \Pi_n$ is the monomial $f_n(z) = z^n$, we write

$$m_{n,X} = \|f_n\| \quad \text{and} \quad \tau_{n,X} = \inf_{p\in\Pi_{n-1}} \|f - p\|.$$

Fekete [6] proved that $\lim_{n\to\infty} (\tau_{n,X})^{1/n}$ (*the Chebyshev constant*) exists for $(X, \|\cdot\|) = (C_K, \|\cdot\|_\infty)$. In this case he also showed in [7] that

$$\lim_{n\to\infty} (\tau_{n,X})^{1/n} = d(K). \tag{2}$$

The existence of Green function for the domain $\overline{\mathbb{C}} \setminus K$ implies that the Robin constant $\gamma(\overline{\mathbb{C}} \setminus K)$ is strictly positive,

$$\gamma(\overline{\mathbb{C}} \setminus K) > 0. \tag{3}$$

Now, from the equality

$$d(K) = \gamma(\overline{\mathbb{C}} \setminus K), \tag{4}$$

we have

$$\boxed{\text{Theorem 6 \& (3) \& (4)}} \Rightarrow \boxed{\text{Walsh-Russell theorem}}$$

Remark 1. Inequality (3) and equality (4) follow, respectively, from Theorem 1 and Theorem 2 of Goluzin's book [8, p. 311].

Using Faber's polynomials A. V. Batyrev [9] proved the following.

Theorem 8 (Batyrev theorem)**.** *A function* f, *holomorphic on a compact set* $K \subseteq \mathbb{C}$ *with the positive transitive diameter* $d(K)$, *and with the simply connected* $\overline{\mathbb{C}} \setminus K$, *can be extended to an entire function if and only if* $\lim\limits_{n\to\infty} E_{n,K}^{1/n}(f) = 0$, *where* $E_{n,K}(f) = \inf\{\|f-p\| : p \in \Pi_n\}$.

Batyrev theorem was extended by T. Winiarski [10] for the case when $\overline{\mathbb{C}} \setminus K$ is not necessarily simply connected. Using our notation we can formulate this result as follows.

Theorem 9 (Winiarski theorem)**.** If K is a compact subset of $\mathbb{C}$ with $d(K) > 0$, then (1) holds for $(X, \|\cdot\|) = (C_K, \|\cdot\|_\infty)$.

Thus, we obtain

Theorem 6 & (2)	$\Rightarrow$	Winiarski theorem	$\Rightarrow$	Batyrev theorem
		$\Downarrow$		
		Walsh-Russell theorem		

Theorem 6 and (2) also imply the following result which shows that the converse to Winiarski theorem is valid.

Corollary 2 [11]. *Let* K *be a compact set in* $\mathbb{C}$ *with* $|K| = \infty$. *Then, for* $(X, \|\cdot\|) = (C_K, \|\cdot\|_\infty)$, *equality* (1) *holds if and only if* $d(K) > 0$.

Corollary 2 can be strengthen as follows.

Theorem 10 [12]. *Let* K *be a compact subset of* $\mathbb{C}$. *Then the following statements are equivalent for the space* $(X, \|\cdot\|) = (C_K, \|\cdot\|_\infty)$:

(i) *the equality* $\mathcal{L}_X = \{f|_K : f \text{ is holomorphic on } K\}$ *holds;*

(ii) *the transfinite diameter of* K *equals zero.*

The original formulation of Theorem 10 contains the condition: "The logarithmic capacity of K is zero" instead of statement (ii); but it was shown by P.J. Myrberg [13] that the logarithmic capacity coincides with the transfinite diameter for every compact $K \subseteq \mathbb{C}$.

We conclude this brief survey of "uniform" generalizations of the Bernstein theorem by following.

Theorem 11 [10]**.** *Let* $K \subseteq \mathbb{C}$ *be a compact set with* $|K| = \infty$ *and let* $f \in (C_K, \|\cdot\|_\infty)$. *The function* f *can be extended to an entire function if and only if*

$$\lim_{n\to\infty}\left[E_{n,K}(f)\frac{V_{n+1}(K)}{V_{n+2}(K)}\right]^{1/n} = 0.$$

The last theorem is valid even if $d(K) = 0$. This result and the equality

$$d(K) = \lim_{n\to\infty}\left(\frac{V_{n+1}(K)}{V_n(K)}\right)^{1/n}$$

imply Winiarski theorem.

Theorem 11 can be derived also from the results A. G. Naftalevich, whose paper [14], apparently, is the first attempt to consider the polynomial approximation of entire functions on compact sets of zero transfinite diameter.

Let us turn to the weighted polynomial approximation.

Let K be a bounded subset of $\mathbb{C}$ and let $w: K \to [0, \infty)$ be a weight on K. We denote by $X = X_w$ the set of all functions $f: K \to \mathbb{C}$ such that

$$\|f\|_{\infty, w} = \sup_{z \in K} |f(z) w(z)| < \infty.$$

Then $\|\cdot\|_{\infty, w}: X \to [0, \infty)$ is a seminorm on X. Furthermore, the space $(\quad , \|\cdot\|_{\infty} \quad)$ belongs to $\mathbf{X_E}$ if and only if the set $K \setminus w^{-1}(0)$ has an infinite cardinality.

Conjecture 1. Let K be a bounded subset of $\mathbb{C}$ and let $w: K \to [0, \infty)$ be a weight on K such that $|K \setminus w^{-1}(0)| = \infty$. Then the following statements are equivalent:

(i) equality (1) holds for $(X, \|\cdot\|_{\infty, w})$;

(ii) there is a constant $c \in (0, \infty)$ such that $\liminf_{n \to \infty} V_n^{2/(n(n-1))}(K_n) > 0,$ where

$$K_n = \{z \in K : w(z) \geqslant c^n\}.$$

We conclude the paper by the following conjecture that can be considered as a "weighted generalization" of the Walsh-Russell theorem.

Conjecture 2. The following statements are equivalent for every compact $K \subseteq \mathbb{C}$ with $|K| = \infty$ and connected $\overline{\mathbb{C}} \setminus K$.

(i) Equality (1) holds for every $(X, \|\cdot\|_{\infty, w})$ with continuous $w(z) \not\equiv 0$.

(ii) The domain $\overline{\mathbb{C}} \setminus K$ is regular for Dirichlet problem.

We conclude the paper by the following.

Problem 1. Does every $X \in \mathbf{X_E}$ have a Shauder basis?

Remark 2. The first example of separable Banach space which does not have any Shauder basis was constructed by P. Enflo [15]. So, if the above formulated problem has a positive solution, then using Theorem 6 we can characterize the complex Banach spaces with a basis as spaces linearly isometric to $\mathbf{X_E}$-spaces.

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О.А. Довгоший[1, 2], https://orcid.org/0000-0002-6496-2466
Ю. Престін[2], https://orcid.org/0000-0001-5985-7939
І.О. Шевчук[3], https://orcid.org/0000-0003-1140-373X

[1] Інститут прикладної математики і механіки НАН України, Слов'янськ

[2] Інститут математики університету Любека, Німеччина

[3] Київський національний університет ім. Тараса Шевченка

E-mail: oleksiy.dovgoshey@gmail.com, prestin@math.uni-luebeck.de, shevchuk@univ.kiev.ua

ХАРАКТЕРИЗАЦІЯ ЦІЛИХ ФУНКЦІЙ НЕРІВНОСТЯМИ ТИПУ БЕРНШТЕЙНА

Нехай $\mathcal{E}$ — це множина усіх цілих функцій, що задані на комплексній площині $\mathbb{C}$. Розглянемо клас $\mathbf{X_E}$ усіх Банахових комплексних просторів X таких, що $X \supseteq \mathcal{E}$. Для $X \in \mathbf{X_E}$ і $g \in X$ позначено $E_{n,X}(g) = \inf\{\|g-p\| : p \in \Pi_n\}$, де Π_n — це множина всіх многочленів степеня не вище *n*. Описано усі $X \in \mathbf{X_E}$, для яких співвідношення $\lim\limits_{n\to\infty}(E_{n,X}(g))^{1/n} = 0$ виконується тоді і тільки тоді, коли $g \in \mathcal{E}$.

Ключові слова: *теорема Бернштейна, ціла функція, наближення многочленами, базис Шаудера, трансфінітний діаметр.*