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Algebraic theory of measure algebras

Presented by Academician of the NAS of Ukraine M.O. Perestyuk

A. Horn and A. Tarski initiated the abstract theory of measure algebras. Independently V. Sushchansky, B. Oliynyk and P. Cameron studied the direct limits of Hamming spaces. In the current paper, we introduce new examples of locally standard measure algebras and complete the classification of countable locally standard measure algebras. Countable unital locally standard measure algebras are in one-to-one correspondence with Steinitz numbers. Given a Steinitz number s such measure algebra is isomorphic to the Boolean algebra of s-periodic sequences of 0 and 1. Nonunital locally standard measure algebras are parametrized by pairs (s, r), where s is a Steinitz number and r is a real number greater or equal to 1. We also show that an arbitrary (not necessarily locally standard) measure algebra is embeddable in a metric ultraproduct of standard Hamming spaces. In other words, an arbitrary measure algebra is sofic.

Keywords: *measure algebra, locally matrix algebra, Boolean algebra, Hamming spaces, Steinitz number.*

Let $\mathbf{F}_2$ be the field of order 2. By a *Boolean algebra* we mean an associative commutative algebra over the field $\mathbf{F}_2$ satisfying the identity $x^2 = x$.

Let $[0, \infty)$ denote the set of nonnegative real numbers. Let H be a Boolean algebra. We call a function $\mu : H \to [0, \infty)$ a *measure* if

(1) $\mu(a) = 0$ if and only if $a = 0,\ a \in H$;

(2) if $a, b \in H$ and $a \cdot b = 0$, then $\mu(a+b) = \mu(a) + \mu(b)$.

Following A. Horn and A. Tarski [1], we call a Boolean algebra H with a measure $\mu : H \to [0, \infty)$ a *measure algebra*. For more information on measure algebras (see [1—3]). If (H, μ) is a measure algebra, then the distance $d_H(a, b) = \mu(a - b)$ makes it a metric space.

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Example 1. The Boolean algebra $\mathbf{St}_n = \mathbf{F}_2^n$, $\mathbf{F}_2 = \{0, 1\}$, with the function

$$\mu_n(x_1, ..., x_n) = \frac{1}{n}(x_1 + \cdots + x_n) \text{ for all } x_1, ..., x_n \in \{0, 1\}$$

is a measure algebra. We call the measure algebra $(\mathbf{St}_n, \mu_n)$ *standard*. For all elements $a, b \in \mathbf{St}_n$ the distance $d_{\mathbf{St}_n}(a, b)$ equals the number of coordinates, where a and b differ, divided by n.

Example 2. Let N be the set of positive integers. For a sequence $\mathbf{a} = (a_1, a_2, ...) \in \{0, 1\}^N$ define the pseudomeasure function

$$\tilde{\mu}_n(\mathbf{a}) = \lim_{n\to\infty} \sup \frac{1}{n}(a_1, \ldots, a_n).$$

Then $I = \{\mathbf{a} \in \{0, 1\}^N \mid \tilde{\mu}(\mathbf{a}) = 0\}$ is an ideal of the Boolean algebra $\mathbf{F}_2^N$.

Consider the Boolean algebra $B = \mathbf{F}_2^N / I$ and the measure

$$\mu(\mathbf{a} + I) = \tilde{\mu}(\mathbf{a}), \quad \mathbf{a} \in \mathbf{F}_2^N.$$

The measure algebra (B, μ) is called *Besicovich measure algebra* (see [4]).

Example 3. Let X be an infinite set and let H be the Boolean algebra of finite subsets of X, including the empty one. The measure $\mu(a) = \# a$, $a \in H$, makes (H, μ) a measure algebra. If the set X is countable, then we denote the measure algebra (H, μ) as $H(\infty)$.

In order to introduce the next series of examples we need to start with the concept of a Steinitz number.

Definition 1. A *Steinitz number* [5] is an infinite formal product of the form

$$\prod_{p\in \mathrm{P}} p^{r_p},$$

where P is the set of all primes, $r_p \in N \cup \{0, \infty\}$ for all $p \in \mathrm{P}$. We can define the product of two Steinitz numbers by the rule:

$$\prod_{p\in \mathrm{P}} p^{r_p} \cdot \prod_{p\in \mathrm{P}} p^{k_p} = \prod_{p\in \mathrm{P}} p^{r_p + k_p}, \quad r_p,\, k_p \in N \cup \{0, \infty\},$$

where we assume, that

$$r_p + k_p = \begin{cases} r_p + k_p & \text{if } r_p < \infty \text{ and } k_p < \infty, \\ \infty & \text{in other cases.} \end{cases}$$

By symbol $\mathbf{S}N$ we denote the set of all Steinitz numbers. The set of all positive integers N is the subset of $\mathbf{S}N$. The elements of the set $\mathbf{S}N \setminus N$ are called *infinite Steinitz numbers*.

Example 4. An infinite sequence $\mathbf{a} = (a_1, a_2, ...) \in \{0, 1\}^N$ is said to be *periodic* if there exists a positive integer $k \in N$ such that the equality $a_i = a_{i+k}$ holds for all $i \in N$. In this case the number k is called a *period* of the sequence $\mathbf{a}$.

Let s be a Steinitz number. A periodic sequence $\mathbf{a}$ is called *s-periodic* if its minimum period is a divisor of s.

Let $\mathcal{H}(s)$ be the set of all s-periodic sequences. Clearly, $\mathcal{H}(s)$ is a Boolean subalgebra of $\{0,1\}^N$. The function

$$\mu_{\mathcal{H}(s)}(a_1, a_2, \ldots) = \frac{1}{k}(a_1 + \cdots + a_k),$$

where k is a period of the sequence $(a_1, a_2, \ldots)$, makes $(\mathcal{H}(s), \mu_{\mathcal{H}(s)})$ a measure algebra.

A measure algebra (H, μ) is called *unital* if the Boolean algebra H contains 1 and $\mu(1) = 1$. In this case, it is easy to see that $\mu(H) \subseteq [0,1]$ and 1 is the only element of measure 1.

The measure algebras of examples 1, 2, and 4 are unital. The measure algebras of example 3 are not unital.

If (H, μ) is a measure algebra and $h \in H$ is a nonzero element, then hH is a unital Boolean algebra. The function

$$\mu_h : hH \to [0,1], \qquad \mu_h(a) = \frac{\mu(a)}{\mu(h)}, \qquad a \in hH,$$

makes $H_h = (hH, \mu_h)$ a unital measure algebra.

Definition 2. We say that two measure algebras (H_1, μ_1) and (H_2, μ_2) are *scalar equivalent* if there exists a positive number $\alpha > 0$ and an isomorphism $\varphi : H_1 \to H_2$ of Boolean algebras such that $\mu_2(\varphi(a)) = \alpha\mu_1(a)$ for an arbitrary element $a \in H_1$.

If measure algebras are scalar equivalent and unital, then they are isomorphic.

Definition 3. We call a measure algebra (H, μ) *locally standard* if every finite subset of H is contained in a measure subalgebra of (H, μ) that is scalar equivalent to $\mathbf{St}_n$ for some $n \geqslant 1$.

If the measure algebra (H, μ) is unital and locally standard, then every infinite subset of H is contained in a measure subalgebra that is isomorphic to $\mathbf{St}_n$ for some $n \geq 1$.

The measure algebras of Examples 1, 3, and 4 are locally standard. The Besicovich measure algebra is not locally standard because it contains elements of irrational measure.

In Sec. 1, we review the classification of unital countable locally standard measure algebras and their connections to locally matrix algebras. In Sec. 2, we introduce new examples of non-unital locally standard measure algebras and proceed with the classification of countable, not necessarily unital, locally standard measure algebras. In Sec. 3, we discuss the property of unital measure algebras, not necessarily locally standard, to be sofic.

1. Unital locally standard algebras.

Definition 4. Let H be a unital locally standard measure algebras, and let $D(H) = \{n \geqslant 1 \mid 1 \in H' \subset H, H' \cong \mathbf{St}_n\}$. The least common multiple of the set $D(H)$ is called the *Steinitz number of the measure algebra H* and is denoted as $\mathbf{st}(H)$.

Theorem 1. *If H is a countable unital locally measure algebra and* $\mathbf{st}(H) = s$, *then* $H \cong \mathcal{H}(s)$ (see Example 4 above).

In particular, every countable unital locally standard measure algebra is uniquely determined by its Steinitz number.

The theory of locally standard measure algebras is parallel to the theory of locally matrix algebras. From this point of view, Theorem 1 is an analogue of the theorem of J.G. Glimm [6].

Definition 5. Let F be a field. An associative F-algebra A is called a *locally matrix algebra* if an arbitrary finite collection of elements $a_1, \ldots, a_n \in A$ is contained in a subalgebra $A' \subset A$ that is isomorphic to a matrix algebra $M_n(F)$ for some $n \geqslant 1$. If $1 \in A$, then we say that A is a *unital locally matrix algebra*.

Definition 6. For a unital locally matrix algebra A, let $D(A)$ be the set of all positive integers $n \in N$ such that there exists a subalgebra $A',\ 1 \in A' \subset A,\ A' \cong M_n(F)$. The least common multiple of the set $D(A)$ is called the *Steinitz number* $\mathbf{st}(A)$ of the algebra A (see [7, 8]).

J.G. Glimm [6] showed that if $\dim_F A \leqslant \aleph_0$ and

$$\mathbf{st}(A) = \prod_{p_i \in \mathrm{P}} p_i, \quad \text{then} \quad A \cong \underset{p_i \in \mathrm{P}}{\otimes} M_{p_i}(F).$$

In particular, every countable-dimensional unital locally matrix algebra is uniquely determined by its Steinitz number.

For an element a of a unital locally matrix algebra A choose a subalgebra $A' \subset A$ such that $1 \in A',\ A' \cong M_n(F)$. Let $r_{A'}(a)$ be the range of the matrix a in $M_n(F)$. As shown by V.M. Kurochkin [9], the ratio

$$r(a) = \frac{1}{n} r_{A'}(a)$$

does not depend on the choice of the subalgebra A'. We call $r(a)$ the *relative range* of the element a. If $a, b \in A$ are orthogonal idempotents, then $r(a + b) = r(a) + r(b)$.

Let C be a commutative subalgebra of a locally matrix algebra A and $1 \in C$. Let $E(C)$ be the set of all idempotents from C. For $e, f \in E(C)$ let ef and $e + f - 2ef$ be their Boolean product and Boolean sum, respectively. The Boolean algebra $E(C)$ with the relative range function $r: E(C) \to [0, 1]$ is a measure algebra.

A subalgebra H of the matrix algebra $M_n(F)$ is called a *Cartan subalgebra* if $H \cong F \oplus \cdots \oplus F$ (n summands), in other words, H is spanned by n pairwise orthogonal idempotents. It is well known that every Cartan subalgebra is a conjugate of the diagonal subalgebra of $M_n(F)$.

A *Cartan subalgebra* of A is a subalgebra $H \subset A$ with decompositions

$$A = \bigotimes_{i=1}^{\infty} A_i \quad \text{and} \quad H = \bigotimes_{i=1}^{\infty} H_i$$

in which all A_i are finite-dimensional matrix algebras and H_i are Cartan subalgebras of A_i. Any two Cartan subalgebras of A are conjugate via an automorphism.

In [10], it is shown that an arbitrary countable unital locally standard measure algebra M is isomorphic to $E(C)$, where C is a Cartan subalgebra of a countable-dimensional unital locally matrix algebra A and $\mathbf{st}(M) = \mathbf{st}(A)$.

2. Classification of non-unital locally standard measure algebras. In [10], we showed that given two measure algebras (H_1, μ_1) and (H_2, μ_2) there exists a unique measure μ on the Boolean algebra $H_1 \otimes_{\mathbf{F}_2} H_2$ such that $\mu(a \otimes b) = \mu_1(a)\mu_2(b)$ for arbitrary elements $a \in H_1,\ b \in H_2$.

Remark 1. In [10], we assumed the unitality of the Boolean algebra H_1 and H_2. However, this unitality has never been used in the definition of tensor product.

Example 5. Let s be an infinite Steinitz number and let $1 \leqslant r < \infty$ be a real number. Choose a sequence $b_1, b_2, \ldots$ of positive integers such that b_i divides b_{i+1}, $i \geqslant 1$, and all these numbers divide s.

Let $m_i = [r\, b_i]$, $i \geqslant 1$, and let

$$m_i^+ = \begin{cases} [rb_i] & \text{if} \quad rb_i \notin N, \\ rb_i - 1 & \text{if} \quad rb_i \in N. \end{cases}$$

For each $i \geqslant 1$ consider the unital countable measure algebras $H(s/b_i)$. Let

$$M_i = \mathbf{St}_{m_i} \otimes H(s/b_i), \qquad M_i^+ = \mathbf{St}_{m_i^+} \otimes H(s/b_i), \qquad i \geqslant 1.$$

The locally standard unital measure algebras $H(s/b_i)$ and $\mathbf{St}_{b_{i+1}/b_i} \otimes H(s/b_{i+1})$ have equal Steinitz numbers. Hence,

$$H(s/b_i) \cong \mathbf{St}_{b_{i+1}/b_i} \otimes H(s/b_{i+1}).$$

This implies

$$M_i = \mathbf{St}_{m_i} \otimes H(s/b_i) \cong \mathbf{St}_{m_i} \otimes \mathbf{St}_{b_{i+1}/b_i} \otimes H(s/b_{i+1}) \cong \mathbf{St}_{m_i \frac{b_{i+1}}{b_i}} \otimes H(s/b_{i+1}),$$

and, similarly,

$$M_i^+ \cong \mathbf{St}_{m_i^+ \frac{b_{i+1}}{b_i}} \otimes H(s/b_{i+1}).$$

We have

$$m_i \cdot \frac{b_{i+1}}{b_i} \leqslant m_{i+1}, \qquad m_i^+ \cdot \frac{b_{i+1}}{b_i} \leqslant m_{i+1}^+.$$

Let $e_i = (\underbrace{1, 1, \ldots, 1}_{m_i \cdot \frac{b_{i+1}}{b_i}}, 0, 0, \ldots, 0) \in \mathbf{St}_{m_{i+1}}$, $e_i^+ = (\underbrace{1, 1, \ldots, 1}_{m_i^+ \cdot \frac{b_{i+1}}{b_i}}, 0, 0, \ldots, 0) \in \mathbf{St}_{m_{i+1}^+}$. Then

$$\mathbf{St}_{m_i \frac{b_{i+1}}{b_i}} \cong e_i \mathbf{St}_{m_{i+1}} e_i, \qquad \mathbf{St}_{m_i^+ \frac{b_{i+1}}{b_i}} \cong e_i^+ \mathbf{St}_{m_{i+1}^+} e_i^+,$$

and, therefore, the measure algebra M_i (resp. M_i^+) is isomorphic to the corner

$$(e_i \otimes 1) M_{i+1} (e_i \otimes 1) \qquad (\text{resp. } (e_i^+ \otimes 1) M_{i+1}^+ (e_i^+ \otimes 1)).$$

Let

$$H(r,s) = \bigcup_{i \geqslant 1} M_i, \qquad H^+(r,s) = \bigcup_{i \geqslant 1} M_i^+.$$

Theorem 2. *Any countable locally standard measure algebra is scalar equivalent to one of the following measure algebras:* $\mathbf{St}_n$; $H(\infty) \otimes \mathcal{H}(s)$, $s \in \mathbf{S}N$; $H(r, s)$, $H^+(r, s)$, $r \in [0, \infty)$, $s \in \mathbf{S}N \setminus N$.

Remark 2. Unital measure algebra $\mathcal{H}(s)$ appear as $H(r, s)$, where $s \in \mathbf{S}N \setminus N$, $r = u/v$, $u, v \in N$, and v divides s.

3. Sofic measure algebras.

Definition 7. A group is called *sofic* if it is embeddable in a metric ultraproduct of symmetric groups with normalized Hamming distances.

Equivalently, it is sofic if it is locally ε-embeddable in symmetric groups (see [11, 12]). We formulate a similar concept for measure algebras.

Let (H, μ) be a unital measure algebra; and let $X \subset H$ be a finite subset containing 1. Let $\varepsilon > 0$. As above, $(\mathbf{St}_n, \mu_n)$ is the standard measure algebra with normalized Hamming distance $d_n(a, b) = \mu_n(a - b)$.

Definition 8. A mapping $\varphi : X \to \mathbf{St}_n$ is called an *ε-embedding* if

(1) $d_n(\varphi(a)+\varphi(b), \varphi(a+b)) \leqslant \varepsilon$ as long as $a, b, a+b \in X$;

(2) $d_n(\varphi(a)\varphi(b), \varphi(ab)) \leqslant \varepsilon$ as long as $a, b, ab \in X$;

(3) $\varphi(1) = 1$;

(4) $d_n(\varphi(a), \varphi(b)) \geqslant 1/4$ for all distinct elements $a, b \in X$.

Let I be an infinite set, and let $\mathcal{F}$ be a non-principle ultrafilter on I (see [13]). Let $(\mathbf{St}_{n_i}, \mu_{n_i})_{i\in I}$ be a family of standard measure algebras. Consider the ultraproduct

$$U = \prod_{i\in I} \mathbf{St}_{n_i} / \mathcal{F}.$$

For an element $\mathbf{a} = (a_i)_{i\in I} / \mathcal{F}$, $a_i \in \mathbf{St}_{n_i}$, define

$$\tilde{\mu}(\mathrm{a}) = \lim_{\mathcal{F}} \mu_{n_i}(a_i).$$

Then

$$R = \{\mathbf{a} \in \prod_{i\in I} \mathbf{St}_{n_i} / \mathcal{F} \mid \tilde{\mu}(\mathbf{a}) = 0\}$$

is an ideal in the Boolean algebra U and $\mu(\mathbf{a} + R) = \tilde{\mu}(\mathbf{a})$ is a measure. We call the Boolean algebra U with the measure μ a *metric ultraproduct of standard measure algebras* $\mathbf{St}_{n_i}$.

Proposition 1. The following two conditions on a unital measure algebra (H, μ) are equivalent:

(a) for an arbitrary finite subset $1 \in X \subset H$ and an arbitrary $\varepsilon > 0$ there exists an ε-embedding of X in a standard measure algebra,

(b) (H, μ) is embeddable in a metric ultraproduct of standard measure algebras.

The problem is if all groups are sofic is still open. The general expectation is that there is a counterexample. The answer for measure algebras, however, is positive.

Theorem 3. *All unital measure algebras are sofic.*

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АЛГЕБРАЇЧНА ТЕОРІЯ АЛГЕБР З МІРОЮ

Абстрактна теорія алгебр з мірою була започаткована А. Хорном і А. Тарським. Незалежно від них В. Сущанський, Б. Олійник і П. Камерон досліджували прямі границі просторів Хемінга. У цій статті наведено нові приклади локально стандартних алгебр з мірою та завершено класифікацію зліченних локально стандартних алгебр з мірою. Зліченні унітальні локально стандартні алгебри з мірою знаходяться у взаємно однозначній відповідності з числами Стейніца. Для даного числа Стейніца s така алгебра з мірою ізоморфна булевій алгебрі s-періодичних послідовностей із 0 та 1. Неунітальні локально стандартні алгебри з мірою параметризуються парами (s, r), де s — число Стейніца, а r — дійсне число, яке більше або дорівнює 1. Також показано, що довільна (не обов'язково локально стандартна) алгебра з мірою занурюється в метричний ультрадобуток стандартних алгебр з мірою. Іншими словами, довільна алгебра з мірою є софічною.

Ключові слова: *алгебра з мірою, локально матрична алгебра, булева алгебра, простір Хемінга, число Стейніца.*