ДОПОВІДІ
НАЦІОНАЛЬНОЇ
АКАДЕМІЇ НАУК
УКРАЇНИ

https://doi.org/10.15407/dopovidi2024.01.003
UDC 517.5

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On the Dirichlet problem for beltrami equations with sources in simply connected domains

In this paper, we present our recent results on the solvability of the Dirichlet problem $\mathrm{Re}\,\omega(z)\to\varphi(\zeta)$ *as* $z\to\zeta$, $z\in D, \zeta\in\partial D$, *with continuous boundary data* $\varphi:\partial D\to R$ *for degenerate Beltrami equations* $\omega_{\bar z}=\mu(z)\omega_z+\sigma(z)$, $|\mu(z)|<1$ *a.e., with sources* $\sigma: D\to C$ *that belong to the class* $L_p(D)$, $p>2$, *and have compact supports in* D. *In the case of locally uniform ellipticity of the equations, we formulate, in arbitrary simply connected domains* D *of the complex plane* C *a series of effective integral criteria of the type of BMO, FMO, Calderon-Zygmund, Lehto and Orlicz on singularities of the equations at the boundary for existence of locally Hölder continuous solutions in the class* $W^{1,2}_{\mathrm{loc}}(D)$ *of the Dirichlet problem with their representation through the so-called generalized analytic functions with sources.*

Keywords: *Dirichlet problem, nonhomogeneous degenerate Beltrami equations, generalized analytic functions with sources, BMO (bounded mean oscillation), FMO (finite mean oscillation), singularities at the boundary.*

1. Introduction. Let D be a domain in the complex plane C. We investigate the Dirichlet problem

$$\lim_{z\to\zeta}\mathrm{Re}\,\omega(z)=\varphi(\zeta) \qquad \forall\,\zeta\in\partial D, \tag{1}$$

see Chapter 4 in [1], with continuous boundary data $\varphi:\partial D\to R$ in arbitrary bounded simply connected domains D for the *Beltrami equation*

$$\omega_{\bar z}=\mu(z)\cdot\omega_z+\sigma(z), \quad z\in D, \tag{2}$$

Citation: Gutlyanskiĭ V.Ya., Nesmelova O.V., Ryazanov V.I., Yakubov E. On the Dirichlet problem for Beltrami equations with sources in simply connected domains. *Dopov. nac. akad. nauk Ukr.* 2024. No 1. P. 3—12. https://doi.org/10.15407/dopovidi2024.01.003

with a source $\sigma: D \to C$ in L_p, $p > 2$, where $\mu: D \to C$ is a measurable function with $|\mu(z)| < 1$ a.e., $\omega_{\bar z} = (\omega_x + i\omega_y)/2$, $\omega_z = (\omega_x - i\omega_y)/2$, $z = x + iy$, ω_x and ω_y are partial derivatives of the function ω in x and y, respectively.

For the case $\|\mu\|_\infty < 1$, (2) was initially introduced by L. Ahlfors and L. Bers in the paper [2]. Here, we examine the case of locally uniform ellipticity of the equation (2) when its *dilatation quotient* K_μ is bounded only locally in D,

$$K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|}, \tag{3}$$

i.e., if $K_\mu \in L_\infty$ on each compact set in D but admits singularities at the boundary. A point $\zeta \in \partial D$ is called a *singular point of the equation* (2) if $K_\mu \notin L_\infty$ on each neighborhood of the point.

Here we present the corresponding results in the subject, as proven in our recent paper [3]. Specifically, we demonstrate that if D is an arbitrary bounded simply connected domain in C, then the Dirichlet problem (1) for the equation (2) has a locally Hölder continuous solution ω in class $W^{1,2}_{\text{loc}}(D)$ for a broad range of singularities of (2) at the boundary. Futhermore, this solution is unique up to an additive pure imaginary constant, and it can be expressed through suitable generalized analytic functions with sources.

In this connection, recall that the Vekua monograph [1] was devoted to *generalized analytic functions*, i.e., continuous complex valued functions $H(z)$ of one complex variable $z = x + iy$ of class $W^{1,1}_{\text{loc}}$ in a domain D satisfying the equations

$$\partial_{\bar z} H + aH + b\overline{H} = S, \quad \partial_{\bar z} := (\partial_x + i\partial_y)/2 \tag{4}$$

with complex valued coefficients $a, b, S \in L_p(D)$, $p > 2$.

The paper [4] was devoted to boundary value problems with measurable data for the spacial case of *generalized analytic functions H with sources* $S: D \to C$, when $a \equiv 0 \equiv b$,

$$\partial_{\bar z} H(z) = S(z), \quad z \in D, \tag{5}$$

in regular enough domains D.

2. The Main Existence Lemma. It is well known that the homogeneous Beltrami equation

$$f_{\bar z} = \mu(z) f_z \tag{6}$$

is the basic equation in analytic theory of quasiconformal and quasiregular mappings in the plane with numerous applications in nonlinear elasticity, gas flow, hydrodynamics and other sections of natural sciences.

The equation (6) is termed *degenerate* if $\operatorname{ess\,sup} K_\mu(z) = \infty$. It is known that if K_μ is bounded, then the equation has homeomorphic solutions in $W^{1,2}_{\text{loc}}$ called *quasiconformal mappings* (see historic comments in [5]). Recently, criteria for existence of homeomorphic solutions in $W^{1,1}_{\text{loc}}$ were also established for degenerate Beltrami equations; refer to papers [6]—[9] and monographs [10, 11].

These criteria were formulated both in terms of K_μ and the quantity

$$K^T_\mu(z, z_0) := \frac{\left|1 - \frac{\overline{z - z_0}}{z - z_0}\mu(z)\right|^2}{1 - |\mu(z)|^2} \tag{7}$$

called *tangent dilatation quotient* of the Beltrami equations with respect to a point $z_0 \in C$. Note that

$$K_\mu^{-1}(z) \leqslant K_\mu^T(z,z_0) \leqslant K_\mu(z) \qquad \forall\, z \in D, z_0 \in C. \tag{8}$$

Let D be a domain in the complex plane C. A function $f: D \to C$ in the Sobolev class $W_{\mathrm{loc}}^{1,1}$ is called a *regular solution* of the Beltrami equation (6) if f satisfies (6) a.e., and its Jacobian $J_f(z) = |f_z|^2 - |f_{\bar z}|^2 > 0$ a.e. By Lemma 3 and Remark 2 in [8], we have the following statement on the existence of regular homeomorphic solutions f in C for the Beltrami equation (6).

Proposition 1. *Let $\mu: C \to C$ be a measurable function with $|\mu(z)| < 1$ a.e. and $K_\mu \in L_{1,\mathrm{loc}}(C)$. Suppose that, for each $z_0 \in C$ with some $\varepsilon_0 = \varepsilon(z_0) > 0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\cdot \psi_{z_0,\varepsilon}^2(|z-z_0|)\,dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \to 0 \tag{9}$$

for a family of measurable functions $\psi_{z_0,\varepsilon}: (0,\varepsilon_0) \to (0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, with

$$I_{z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{z_0,\varepsilon}(t)\,dt < \infty \qquad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{10}$$

Then the Beltrami equation (6) has a regular homeomorphic solution f^μ.

Here $dm(z)$ corresponds to the Lebesgue measure in C and by (8) K_μ^T can be replaced by K_μ. We call such solutions f^μ of (6) $\mu-$*conformal mappings*. It is assumed here and further that the dilatation quotients $K_\mu^T(z,z_0)$ and $K_\mu(z)$ are extended by 1 outside of the domain D.

Lemma 1. *Let D be a bounded simply connected domain in C, $\sigma \in L_p(D)$, $p > 2$, with compact support in D, $\mu: D \to C$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ be locally bounded in D, $K_\mu \in L_1(D)$ and conditions (9) and (10) hold for all $z_0 \in \partial D$.*

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $W_{\mathrm{loc}}^{1,2}$ of the Dirichlet problem (1) in D for each continuous function $\varphi: \partial D \to R$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega = h \circ f$, where $f: C \to C$ is a $\mu-$conformal mapping with μ extended by zero outside D and $h: D_* \to C$ is a generalized analytic function in $D_* := f(D)$ with the source S of the class $L_{p_*}(D_*)$ for some $p_* \in (2, p)$,

$$S := \sigma \cdot \frac{f_z}{J} \circ f^{-1}, \tag{11}$$

where $J = |f_z|^2 - |f_{\bar z}|^2$ is the Jacobian of f, that satisfies the Dirichlet condition

$$\lim_{w\to\zeta} \operatorname{Re} h(w) = \varphi_*(\zeta) \quad \forall\, \zeta \in \partial D_*, \quad \text{with } \varphi_* := \varphi \circ f^{-1}|_{\partial D_*}. \tag{12}$$

Remark 1. In turn, the generalized analytic function h with the source S by Theorem 1.16 in [1] has the representation $h = A + H$, where

$$H(w) = -\frac{1}{\pi}\int\limits_{D_*} \frac{S(\zeta)}{\zeta - w}\,dm(\zeta), \quad w \in C, \tag{13}$$

with $H_{\overline{w}} = S$, and A is a holomorphic function in D_* with the Dirichlet condition

$$\lim_{w\to\zeta} \operatorname{Re} A(w) = \varphi^*(\zeta) \quad \forall\, \zeta \in \partial D_*, \text{ with } \varphi^* := \varphi_* - \operatorname{Re} H|_{\partial D_*}. \tag{14}$$

Note that H is α_*-Hölder continuous in D_* with $\alpha_* = 1 - 2/p_*$ by Theorem 1.19 and $H|_{D_*} \in W^{1,p_*}(D_*)$ by Theorems 1.36 and 1.37 in [1]. Also note that f and f^{-1} are locally quasiconformal mappings in D and D_*, respectively.

The proof of Lemma 1 is based on known results about the existence of a harmonic function $u: D_* \to R$, satisfying the Dirichlet condition

$$\lim_{w\to\zeta} u(w) = \varphi^*(\zeta) \quad \forall\, \zeta \in \partial D_*, \tag{15}$$

see Corollary 4.1.8 and Theorem 4.2.1 in [12], the existence of a holomorphic function $A := u + iv : D_* \to C$ (unique up to an additive pure imaginary constant) in an arbitrary bounded simple connected domain D and the factorization $\omega = h \circ f$ of solutions of (2) in terms of suitable generalized analytic functions with sources, see Lemma 1 and Remark 2 in [13] for the uniformly elliptic case. The existence of the given $\mu-$conformal mapping f follows from Proposition 1; see further details of the proof in [3].

Remark 2. Note that if the family of the functions $\psi_{z_0,\varepsilon}(t) \equiv \psi_{z_0}(t)$, $z_0 \in \partial D$, in Lemma 1 is independent on the parameter ε, then the condition (9) implies that $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. This follows immediately from arguments by contradiction, apply for it (8) and the condition $K_\mu \in L_1(D)$. Note also that (9) holds, in particular, if, for some $\varepsilon_0 = \varepsilon(z_0)$,

$$\int\limits_{|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0) \cdot \psi_{z_0}^2(|z-z_0|)\, dm(z) < \infty \qquad \forall\, z_0 \in \partial D \tag{16}$$

and $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the solvability of the Dirichlet problem (1) in D for the Beltrami equations with sources (2) for all continuous boundary functions φ, it is sufficient that the integral in (16) converges for some nonnegative function $\psi_{z_0}(t)$ that is locally integrable over $(0, \varepsilon_0]$ but has a nonintegrable singularity at 0. The functions $\log^\lambda(e/|z-z_0|)$, $\lambda \in (0,1)$, $z \in D$, $z_0 \in D$, and $\psi(t) = 1/(t \log(e/t))$, $t \in (0,1)$, show that the condition (16) is compatible with the condition $I_{z_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Furthermore, the condition (9) shows that it is sufficient for the solvability of the Dirichlet problem even that the integral in (16) is divergent but in a controlled way.

3. The main existence integral criteria. Lemma 1 enables us to derive several effective integral criteria for the solvability of the Dirichlet problem for Beltrami equations with sources.

Firstly, recall that a real-valued function u in a domain D in C is said to be of *bounded mean oscillation* in D, abbr. $u \in \mathrm{BMO}(D)$, if $u \in L^1_{\mathrm{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B|\, dm(z) < \infty, \tag{17}$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|} \int_B u(z)\, dm(z).$$

We write $u \in \mathrm{BMO}_{\mathrm{loc}}(D)$ if $u \in \mathrm{BMO}(U)$ for each relatively compact subdomain U of D. We also write sometimes for short BMO and $\mathrm{BMO}_{\mathrm{loc}}$, respectively.

The class BMO was introduced by John and Nirenberg (1961) in the paper [14] and soon became an important concept in harmonic analysis, partial differential equations, and related areas.

Following [15], we say that a function $u: D \to R$ has *finite mean oscillation* at the point $z_0 \in D$, abbr. $u \in \mathrm{FMO}(z_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \; \fint_{B(z_0,\varepsilon)} |u(z) - \tilde{u}_\varepsilon(z_0)| \, dm(z) < \infty, \tag{18}$$

where

$$\tilde{u}_\varepsilon(z_0) = \fint_{B(z_0,\varepsilon)} u(z) \, dm(z) \tag{19}$$

is the mean value of the function $u(z)$ over disk $B(z_0, \varepsilon) := \{z \in C : |z - z_0| < \varepsilon\}$. Note that the condition (18) includes the assumption that u is integrable in some neighborhood of the point z_0. We say also that a function $u: D \to R$ is of *finite mean oscillation in* D, abbr. $u \in \mathrm{FMO}(D)$ or simply $u \in \mathrm{FMO}$, if $u \in \mathrm{FMO}(z_0)$ for all points $z_0 \in D$.

The following statement is obvious by the triangle inequality.

Proposition 2. *If, for a collection of numbers* $u_\varepsilon \in R$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim_{\varepsilon \to 0}} \; \fint_{B(z_0,\varepsilon)} |u(z) - u_\varepsilon| \, dm(z) < \infty, \tag{20}$$

then u *is of finite mean oscillation at* z_0.

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $u: D \to R$ if u is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \to 0} \fint_{B(z_0,\varepsilon)} |u(z) - u(z_0)| \, dm(z) = 0. \tag{21}$$

Thus, we have by Proposition 2 the next corollary.

Corollary 1. *Every locally integrable function* $u: D \to R$ *has a finite mean oscillation at almost every point in* D.

Remark 4. The latter shows that the FMO condition is very weak. Clearly, $\mathrm{BMO}(D) \subset \mathrm{BMO}_{\mathrm{loc}}(D) \subset \mathrm{FMO}(D)$ and as well-known $\mathrm{BMO}_{\mathrm{loc}} \subset L^p_{\mathrm{loc}}$ for all $p \in [1, \infty)$, see, e.g., [14]. However, FMO is not a subclass of L^p_{loc} for any $p > 1$ but only of L^1_{loc}, see Examples 2.3.1 in [10]. Thus, the class FMO is much more wider than $\mathrm{BMO}_{\mathrm{loc}}$.

Versions of the next lemma have been first proved for the class BMO in [7]. For the FMO case, see the paper [15] and the monographs [10] and [11].

Lemma 2. *Let* D *be a domain in* C *and let* $u: D \to R$ *be a non-negative function of the class* $\mathrm{FMO}(z_0)$ *for some* $z_0 \in D$. *Then*

$$\int\limits_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{u(z) \, dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = O\left(\log\log \frac{1}{\varepsilon}\right) \quad \text{as} \quad \varepsilon \to 0 \tag{22}$$

for some $\varepsilon_0 \in (0, \delta_0)$ *where* $\delta_0 = \min(e^{-e}, d_0)$, $d_0 = \sup_{z \in D} |z - z_0|$.

We assume further that the dilatation quotients $K_\mu^T(z,z_0)$ and $K_\mu(z)$ are extended by 1 outside the domain D.

Choosing $\psi(t)=1/(t\log(1/t))$ in Lemma 1, see also Remark 1, we obtain by Lemma 2 the following result with the FMO type criterion.

Theorem 1. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu: D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$, $K_\mu^T(z,z_0)\leqslant Q_{z_0}(z)$ a.e. in U_{z_0} for each point $z_0\in\partial D$, a neighborhood U_{z_0} of z_0, a function $Q_{z_0}:U_{z_0}\to[0,\infty]$ in the class $\mathrm{FMO}(z_0)$.*

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{\mathrm{loc}}$ of the Dirichlet problem (1) in D for each continuous function $\varphi:\partial D\to R$ that is unique up to an additive pure imaginary constant.

Furthermore, $\omega=h\circ f$, $h:=A+H$, where $f:C\to C$ is a $\mu-$conformal mapping with μ extended by zero outside of D, $H:D_*\to C$ is a generalized analytic function in $D_*:=f(D)$ with the source S calculated in (11) and A is a holomorphic function in D_* with the Dirichlet condition (14).

In particular, choosing $\varphi_\varepsilon\equiv0$, $\varepsilon\in(0,\varepsilon_0]$ in Proposition 2, we obtain:

Corollary 2. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu: D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and, for each point $z_0\in\partial D$,*

$$\overline{\lim_{\varepsilon\to0}}\ \fint_{B(z_0,\varepsilon)} K_\mu^T(z,z_0)\,dm(z)<\infty. \tag{23}$$

Then all the conclusions of Theorem 1 on solutions for the Dirichlet problem (1) with arbitrary continuous boundary data $\varphi:\partial D\to R$ to the Beltrami equation (2) with the source σ hold.

Since $K_\mu^T(z,z_0)\leqslant K_\mu(z)$ for all z and $z_0\in C$, we also obtain the following consequences of Theorem 1 with the BMO-type criterion.

Corollary 3. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu: D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D and K_μ have a dominant $Q\in$ $\mathrm{BMO}_{\mathrm{loc}}$ in a neighborhood of ∂D. Then the conclusions of Theorem 1 hold.*

Corollary 4. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu: D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D and K_μ have a dominant $Q\in$ FMO in a neighborhood of ∂D. Then the conclusions of Theorem 1 hold.*

Similarly, choosing in Lemma 1 the function $\psi(t)=1/t$, see also Remark 1, we come to the next statement with the Calderon-Zygmund type criterion.

Theorem 2. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu: D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and, for each point $z_0\in\partial D$ and $\varepsilon_0=\varepsilon(z_0)>0$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_\mu^T(z,z_0)\frac{dm(z)}{|z-z_0|^2}=o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right)\quad \text{as } \varepsilon\to0. \tag{24}$$

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $W^{1,2}_{\mathrm{loc}}$ of the Dirichlet problem (1) in D for each continuous function $\varphi:\partial D\to R$ that is unique up to an additive pure imaginary constant.

Furthermore, $\omega = h \circ f$, $h := A + H$, *where* $f : C \to C$ *is a* $\mu-$*conformal mapping with* μ *extended by zero outside* D, $H : D_* \to C$ *is a generalized analytic function in* $D_* := f(D)$ *with the source* S *calculated in (11) and* A *is a holomorphic function in* D_* *with the Dirichlet condition (14).*

Remark 5. Choosing in Lemma 1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, we are able to replace (24) by the conditions

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} \frac{K_\mu^T(z, z_0)\, dm(z)}{\left(|z - z_0| \log \frac{1}{|z-z_0|}\right)^2} = o\left(\left[\log\log \frac{1}{\varepsilon}\right]^2\right) \qquad \forall\, z_0 \in \partial D \tag{25}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(z_0) > 0$. More generally, we would be able to give here the whole scale of the corresponding conditions in $\log$ using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log\log 1/$ $/t \cdot \ldots \cdot \log \ldots \log 1/t)$.

Choosing in Lemma 1 the functional parameter $\psi_{z_0}(t) := 1/[t k_\mu^T(z_0, t)]$, where $k_\mu^T(z_0, r)$ is the integral mean of $K_\mu^T(z, z_0)$ over the circle $S(z_0, r) := \{z \in C : |z - z_0| = r\}$, we obtain the Lehto type criterion.

Theorem 3. *Let* D *be a bounded simply connected domain in* C, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to C$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D, $K_\mu \in L_1(D)$ *and, for each point* $z_0 \in \partial D$ *and* $\varepsilon_0 = \varepsilon(z_0) > 0$,

$$\int\limits_0^{\varepsilon_0} \frac{dr}{r k_\mu^T(z_0, r)} = \infty. \tag{26}$$

Then the Beltrami equation (2) with the source σ *has a locally Hölder continuous solution* ω *in the class* $W_{\text{loc}}^{1,2}$ *of the Dirichlet problem (1) in* D *for each continuous function* $\varphi : \partial D \to R$ *that is unique up to an additive pure imaginary constant.*

Moreover, $\omega = h \circ f$, $h := A + H$, *where* $f : C \to C$ *is a* $\mu-$*conformal mapping with* μ *extended by zero outside* D, $H : D_* \to C$ *is a generalized analytic function in* $D_* := f(D)$ *with the source* S *calculated in (11) and* A *is a holomorphic function in* D_* *with the Dirichlet condition (14).*

Corollary 5. *Let* D *be a bounded simply connected domain in* C, $\sigma \in L_p(D)$, $p > 2$, *with compact support,* $\mu : D \to C$ *be a measurable function with* $|\mu(z)| < 1$ *a.e.,* K_μ *be locally bounded in* D, $K_\mu \in L_1(D)$ *and, for each point* $z_0 \in \partial D$,

$$k_\mu^T(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \qquad \text{as}\ \varepsilon \to 0. \tag{27}$$

Then all conclusions of Theorem 3 on solutions for the Dirichlet problem (1) with arbitrary continuous boundary data $\varphi : \partial D \to R$ *to the Beltrami equation (2) with the source* σ *hold.*

Remark 6. In particular, the conclusions of Theorem 3 hold if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \qquad \text{as} \quad z \to z_0 \quad \forall\, z_0 \in \partial D. \tag{28}$$

Moreover, the condition (27) can be replaced by the series of weaker conditions

$$k_\mu^T(z_0,\varepsilon)=O\left(\left[\log\frac{1}{\varepsilon}\cdot\log\log\frac{1}{\varepsilon}\cdot\ldots\cdot\log\ldots\log\frac{1}{\varepsilon}\right]\right)\qquad\forall\, z_0\in\partial D. \tag{29}$$

Combining Theorems 2.5 and 3.2 in [9] with Theorems 3 we obtain the following significant result with the Orlicz type criterion.

Theorem 4. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and, for each point $z_0\in\partial D$ and a neighborhood U_{z_0} of z_0,*

$$\int_{U_{z_0}}\Phi_{z_0}(K_\mu^T(z,z_0))\,dm(z)<\infty, \tag{30}$$

where $\Phi_{z_0}:(0,\infty]\to(0,\infty]$ is a convex non-decreasing function such that

$$\int_{\Delta(z_0)}^{\infty}\log\Phi_{z_0}(t)\frac{dt}{t^2}=+\infty\quad\text{for some }\Delta(z_0)>0. \tag{31}$$

Then the Beltrami equation (2) with the source σ has a locally Hölder continuous solution ω in the class $W_{\text{loc}}^{1,2}$ of the Dirichlet problem (1) in D for each continuous function $\varphi:\partial D\to R$ that is unique up to an additive pure imaginary constant.

Moreover, $\omega=h\circ f$, $h:=A+H$, where $f:C\to C$ is a $\mu-$conformal mapping with μ extended by zero outside D, $H:D_*\to C$ is a generalized analytic function in $D_*:=f(D)$ with the source S calculated in (11) and A is a holomorphic function in D_* with the Dirichlet condition (14).

Corollary 6. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D, $K_\mu\in L_1(D)$ and, for each point $z_0\in\partial D$, a neighborhood U_{z_0} of z_0 and $\alpha(z_0)>0$,*

$$\int_{U_{z_0}}e^{\alpha(z_0)K_\mu^T(z,z_0)}\,dm(z)<\infty. \tag{32}$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1) with continuous data $\varphi:\partial D\to R$ to the Beltrami equation (2) with the source σ hold.

Corollary 7. *Let D be a bounded simply connected domain in C, $\sigma\in L_p(D)$, $p>2$, with compact support, $\mu:D\to C$ be a measurable function with $|\mu(z)|<1$ a.e., K_μ be locally bounded in D and, for a neighborhood U of ∂D,*

$$\int_U\Phi(K_\mu(z))\,dm(z)<\infty, \tag{33}$$

where $\Phi:(0,\infty]\to(0,\infty]$ is a convex non-decreasing function with, for $\delta>0$,

$$\int_\delta^\infty\log\Phi(t)\frac{dt}{t^2}=+\infty. \tag{34}$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1) with continuous data $\varphi:\partial D\to R$ *to the Beltrami equation (2) with the source* σ *hold.*

Remark 7. By Theorems 2.5 and 5.1 in [9], condition (34) is not only sufficient but also necessary to have the regular solutions of the Dirichlet problem (1) in D for arbitrary Beltrami equations with sources (2), satisfying the integral constraints (33), for all continuous functions $\varphi:\partial D\to R$ because such solutions have the representation through regular homeomorphic solutions $f=f^{\mu}$ of the homogeneous Beltrami equation (6) from Proposition 1.

Corollary 8. *Let* D *be a bounded simply connected domain in* C, $\sigma\in L_p(D)$, $p>2$, *with compact support,* $\mu:D\to C$ *be a measurable function with* $|\mu(z)|<1$ *a.e.,* K_{μ} *be locally bounded in* D *and, for a neighborhood* U *of* ∂D *and* $\alpha>0$,

$$\int_U e^{\alpha K_{\mu}(z)}dm(z)<\infty. \tag{35}$$

Then all conclusions of Theorem 4 on solutions for the Dirichlet problem (1) with continuous data $\varphi:\partial D\to R$ *to the Beltrami equation (2) with the source* σ *hold.*

Acknowledgements. The first 3 authors are partially supported by the Grant EFDS-FL2-08 of the found of the European Federation of Academies of Sciences and Humanities (ALLEA).

REFERENCES

1. Vekua, I. N. (1962). Generalized analytic functions. Pergamon Press. London-Paris-Frankfurt: Addison-Wesley Publishing Co., Inc., Reading, Mass.
2. Ahlfors, L. V. & Bers, L. (1960). Riemann's mapping theorem for variable metrics. Ann. Math., 2, No. 72, pp. 385-404. https://doi.org/10.2307/1970141
3. Gutlyanskii, V., Nesmelova, O., Ryazanov, V. & Yakubov, E. (2023). The Dirichle problem for the Beltrami equations with sources. Ukr. Mat. Visn., 20, No. 1, pp. 24-59; translated in (2023). J. Math. Sci. (N.Y.), 273, No. 3, pp. 351—376; see also arXiv:2305.16331v2 [math.CV]. https://doi.org/10.1007/s10958-023-06503-0
4. Gutlyanskii, V., Nesmelova, O., Ryazanov, V. & Yefimushkin, A. (2021). Logarithmic potential and generalized analytic functions. Ukr. Mat. Visn., 18, No. 1, pp. 12-36; translated in (2021). J. Math. Sci. (N.Y.), 256, No. 6, pp. 735-752. https://doi.org/10.1007/s10958-021-05457-5
5. Bojarski, B., Gutlyanskii, V., Martio, O. & Ryazanov, V. (2013). Infinitesimal geometry of quasiconformal and bi-Lipschitz mappings in the plane. EMS Tracts in Mathematics, (Vol. 19). Zürich: European Mathematical Society (EMS). https://doi.org/10.4171/122
6. Gutlyanskii, V., Martio, O., Sugawa, T. & Vuorinen, M. (2005). On the degenerate Beltrami equation. Trans. Amer. Math. Soc., 357, No. 3, pp. 875-900. https://doi.org/10.2307/3845154
7. Ryazanov, V., Srebro, U. & Yakubov, E. (2001). BMO-quasiconformal mappings. J. d'Anal. Math., 83, pp. 1-20. https://doi.org/10.1007/BF02790254
8. Ryazanov, V., Srebro, U. & Yakubov, E. (2006). On the theory of the Beltrami equation. Ukr. Math. J., 58, No. 11, pp. 1786-1798. https://doi.org/10.1007/s11253-006-0168-4
9. Ryazanov, V., Srebro, U. & Yakubov, E. (2012). Integral conditions in the theory of the Beltrami equations. Complex Var. Elliptic Equ., 57, No. 12, pp. 1247-1270. https://doi.org/ 10.1080/17476933.2010.534790
10. Gutlyanskii, V., Ryazanov, V., Srebro, U. & Yakubov, E. (2012). The Beltrami Equation: A Geometric Approach. Developments in Mathematics, (Vol. 26). Berlin: Springer. https://doi.org/10.1007/978-1-4614-3191-6
11. Martio, O., Ryazanov, V., Srebro, U. & Yakubov, E. (2009). Moduli in modern mapping theory. Springer Monographs in Mathematics. New York: Springer. https://doi.org/10.1007/978-0-387-85588-2
12. Ransford, Th. (1995). Potential theory in the complex plane. London Mathematical Society Student Texts, (Vol. 28). Cambridge: Univ. Press. https://doi.org/10.1017/CBO9780511623776

13. Gutlyanskii, V., Nesmelova, O., Ryazanov, V. & Yakubov, E. (2022). On the Hilbert problem for semi-linear Beltrami equations. Ukr. Mat. Visn., 19, No. 4, pp. 489-516; translated in (2023). J. Math. Sci. (N.Y.), 270, No. 3, pp. 428-448. https://doi.org/10.1007/s10958-023-06356-7
14. John, F. & Nirenberg, L. (1961). On functions of bounded mean oscillation. Comm. Pure Appl. Math., 14, pp. 415-426. https://doi.org/10.1002/cpa.3160140317
15. Ignat'ev, A. A. & Ryazanov, V. I. (2005). Finite mean oscillation in the mapping theory. Ukr. Mat. Visn., 2, No. 3, 395-417, 443; translated in (2006). Ukr. Math. Bull., 2, No. 3, pp. 403-424. https://doi.org/10.1007/BF02771785

Received 13.07.2023

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ПРО ЗАДАЧУ ДІРІХЛЕ ДЛЯ РІВНЯНЬ БЕЛЬТРАМІ З ДЖЕРЕЛАМИ В ОДНОЗВ'ЯЗАНИХ ОБЛАСТЯХ

У цій статті ми представляємо наші нещодавно отримані результати про розв'язність задачі Діріхле $\mathrm{Re}\,\omega(z)\to\varphi(\zeta)$ для $z\to\zeta,\ z\in D,\zeta\in\partial D,$ з неперервними граничними даними $\varphi:\partial D\to R$ для вироджених рівнянь Бельтрамі $\omega_{\bar z}=\mu(z)\omega_z+\sigma(z)$, $|\mu(z)|<1$ м.в., з джерелами $\sigma:D\to C$, що належать до класу $L_p(D)$, $p>2$, з компактними носіями в *D*. У випадку локально рівномірної еліптичності рівнянь сформульовано у довільних однозв'язаних областях *D* комплексної площини *C* низку ефективних інтегральних критеріїв, типу *BMO, FMO*, Кальдерона-Зигмунда, Лехто та Орлича, сингулярності рівнянь на границі для існування локально неперервних за Гельдером розв'язків у класі $W^{1,2}_{\mathrm{loc}}(D)$ задачі Діріхле з представленням їх через так звані узагальнені аналітичні функції з джерелами.

Ключові слова: *задача Діріхле, неоднорідні вироджені рівняння Бельтрамі, узагальнені аналітичні функції з джерелами, BMO, обмежені середні коливання, FMO, скінченні середні коливання, сингулярності на границі.*