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V.P. Sylovanyuk, https://orcid.org/0000-0002-3765-5962
N.A. Ivantyshyn, https://orcid.org/0000-0003-1835-0466
M.V. Filipov, https://orcid.org/0009-0009-6940-9940

Karpenko Physico-Mechanical Institute of the NAS of Ukraine, Lviv, Ukraine
E-mail: vsylovanyuk@gmail.com, n.ivantyshyn@gmail.com, maksym.filipov@gmail.com

Torsional strength of a transversally isotropic cylinder containing a system of healed cracks

Presented by Corresponding Member of the NAS of Ukraine O.Ye. Andreikiv

The strength of a transversally isotropic cylinder with a system of healed torsional cracks is investigated in this paper. Degradation of service characteristics of the reinforced-concrete materials in APP structures, mine shafts, collectors, and hydraulic works delved into the necessity of serviceability renewal, repair, and the restoration of carrying ability in such structural elements. A mathematical model has been developed to evaluate the effect of cylinder strengthening using injection technologies for healing defects. The calculations are based on the concepts of fracture mechanics and linear theory of elasticity. The model of thin elastic inclusions and the theory of harmonic functions are applied, which enables reduction to the problem of solving a system of integral equations. The parameters that affect the efficiency of crack healing are identified. Analytical solutions for particular cases were obtained, which allowed for optimizing the choice of injection material. It is shown that the correct selection of the injection material stiffness can ensure the complete restoration of the strength of a cracked cylinder.

Keywords: *transversally isotropic body, crack, crack healing, strength.*

Technologies for injecting defective zones in long-term structures as a restoring method of their load-bearing capacity are widely used in practice [1, 2]. To optimize such technologies and predict the service life of the restored elements of structures, calculation models and methods for assessing their performance are needed. It is proposed to base such approaches on the concepts of fracture mechanics, the science of the strength of materials and structures with defects such as cracks, especially those filled with another material. Problems on the limit equilibrium of bodies with filled cracks form a separate class of boundary value problems of the elastic continuum. Unlike cracks whose surfaces are free of stresses or whose stresses are known in advance, for filled

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defects, additional conditions are necessary for the interaction of the filler material with the base material. This interaction, as the transfer of a part of the load carried by the body with the crack and the optimal choice of the filler material, is the essence of the idea of "healing" structural damage using injection technology. Several studies have been performed previously in this direction [3—5]. A mathematical model is proposed below to calculate the effect of injection strengthening of a transversally isotropic circular cylinder containing a system of parallel cracks on the isotropy axis under torsional load.

Relations of the theory of elasticity of transversally isotropic bodies. The solution of boundary value problems of the linear theory of elasticity of transversally isotropic bodies in the absence of bulk forces is reduced to the establishment of three functions φ_j (j = 1, 2, 3) satisfying the following equation [6]

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \Theta^2}+n_j\frac{\partial^2}{\partial z^2}\right)\varphi_j=0. \quad (1)$$

The cylindrical coordinate system r, θ, z is chosen so that the z-axis coincides with the isotropy axis of the elastic properties. Here, n_1 and n_2 are the roots of the equation:

$$A_{11}A_{44}n^2+[A_{13}(A_{13}+2A_{44})-A_{11}A_{33}]n+A_{33}A_{44}=0 \quad (2)$$

and

$$n_3=\frac{2A_{44}}{A_{11}-A_{12}};$$

A_{11}, A_{22}, A_{44} — elastic constants of a transversally isotropic body.

The components of the displacement vector are determined on the basis of the following relations:

$$u_r(r,\theta,z)=\frac{\partial}{\partial r}(\phi_1+\phi_2)-\frac{1}{r}\frac{\partial\phi_3}{\partial\theta},$$

$$u_\theta(r,\theta,z)=\frac{1}{r}\frac{\partial}{\partial\theta}(\phi_1+\phi_2)+\frac{\partial\phi_3}{\partial r}, \quad (3)$$

$$u_z(r,\theta,z)=\frac{\partial}{\partial z}(m_1\phi_1+m_2\phi_2)$$

and the components of the stress tensor in the plane z = const

$$\sigma_{zz}=A_{44}\frac{\partial^2}{\partial z^2}[(1+m_1)n_1\phi_1+(1+m_2)n_2\phi_2],$$

$$\sigma_{rz}=A_{44}\left(\frac{\partial^2}{\partial r\partial z}((1+m_1)\phi_1+(1+m_2)\phi_2)-\frac{1}{r}\frac{\partial^2\phi_3}{\partial\theta\partial z}\right), \quad (4)$$

$$\sigma_{\theta z} = A_{44}\left(\frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}((1+m_1)\phi_1+(1+m_2)\phi_2)+\frac{\partial^2\phi_3}{\partial r\partial z}\right),$$

$$m_i = \frac{A_{11}n_i - A_{44}}{A_{13}+A_{44}} = \frac{(A_{13}+A_{44})n_i}{A_{33}-A_{44}n_i}, \quad i = 1, 2.$$

The equilibrium equations are automatically satisfied, and the harmonic functions (ϕ_j, θ, z) in different coordinate systems are determined from the boundary conditions.

In the torsional theory of transversally isotropic bodies of revolution with the axis of isotropy coinciding with the axis of rotation, it is assumed that the cross-sections are not distorted and there are no displacements in the radial directions, i. e.

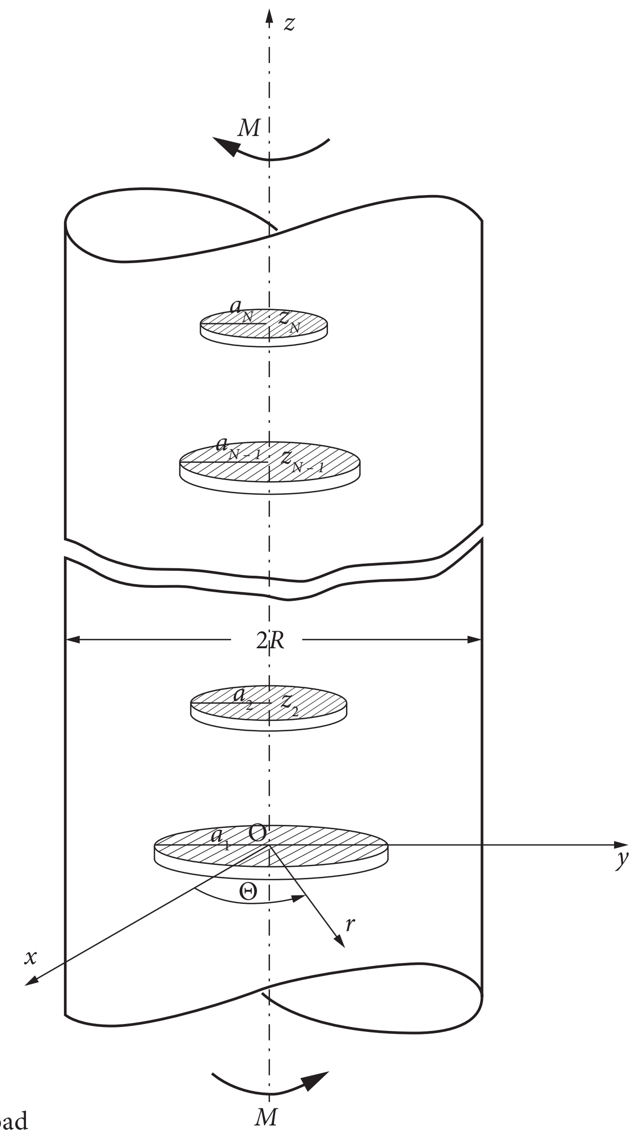
$$u_r = u_z = 0;\ u_\theta = u_r(r,z). \tag{5}$$

Then it is obvious that

$$\varphi_1 = 0, \quad \varphi_2 = 0,\ u_\theta = \frac{\partial\varphi_3}{\partial r},$$

$$\sigma_{\theta z} = A_{44}\frac{\partial^2\varphi_3}{\partial r\partial z}. \tag{6}$$

Problem statement. Consider a body in the form of a circular cylinder of radius R containing a system of N parallel circular cracks with radii a_i, $i = 1, \ldots, N$ on the axis of rotation (Figure). The material of the cylinder is assumed to be transversally isotropic, with the isotropy axis aligned with the cylinder's longitudinal (z) axis. The cylindrical coordinate system (ρ, φ, z) is defined in such a way that the O_z axis coincides with the isotropy axis of the elastic properties of the body. As a result of the injection hardening technology, the cracks are filled with a liquid material that can form strong adhesive bonds after hardening. The cylinder is subject to a torsional load with moment M applied to the ends. This problem is solved by dividing it into two sub-problems.



Geometry of the cylinder with cracks and torsional load

These problems assume that the crack sizes are small compared to the diameter of the cylinder, i. e., $R >> a_i$.

Note that the function ϕ_3 in relation (6) is harmonic in the coordinate system (r, θ, z_3), $z_3 = z n_3^{-1/2}$.

This aspect makes it possible to write the function ϕ_3 in the form of the Hankel integral representation

$$\varphi_3(r, z_3) = \int_0^\infty A(\xi)\xi^{-1}\exp(-\xi z_3) J_0(r\xi) d\xi\,, \tag{7}$$

$J_0(r\xi)$ is a Bessel function of the first kind; $A(\xi)$ is an unknown function that is determined from the boundary conditions.

Based on the principle of superposition in the linear theory of elasticity, the problem is divided into two sub-problems: torsion by a moment M of a cylinder without cracks and a cylinder with cracks, to the surfaces of which forces are applied

$$\sigma_{z\theta}(r, z_i) = -\frac{2Mr}{\pi R^4} + \frac{[u^*_{0i}(r)]G}{2h_i(r)},$$

$$\sigma_{zr}(r, z_i) = \sigma_{zz}(r, z_i) = 0, \quad 0 \le r \le a_i, \quad i = 1, \ldots, N. \tag{8}$$

Here, thin layers formed as a result of the hardening of the injected material in the cracks, modeled according to the Winkler shear model [2]; G is the shear modulus of the injected material after hardening; $u^*_{\theta i}$ is the displacement of the points of the layer surface, which is taken as the sum of $u^*_{\theta i} = u_{\theta i} + u^0_{\theta i}$; $u_{\theta i}$ is the displacement of the surfaces of the i-th mathematical section under the action of the forces specified by the boundary conditions (8); $u^0_{\theta i}(z) = \dfrac{2Mrh_i(r)}{A_{44}\pi R^4}$; $2h_i(r)$ is the thickness of the i-th layer; $[u^*_{\theta i}(r)]$ means the jump of the function $u^*_{\theta i}(r)$; z_i is the coordinates of the center of the i-th crack along the Oz axis.

The solution to the first problem is trivial and is not related to the crack. To solve the second problem, based on the principle of superposition, we represent the stress state in the body as a sum [7].

$$\hat{\sigma} = \sum_{i=1}^{N} \hat{\sigma}_i\,. \tag{9}$$

Here σ_i is the stress tensor in a cylinder with one i-th crack. The boundary conditions for such cylinders with cracks are as follows:

$$\sigma_{z\theta i}(r, z_i) = -\frac{2Mr}{\pi R^4} + \frac{[u^*_{\theta i}(r)]G}{2h_i(r)} - \sum_{j=1(\ne i)}^{N} \sigma_{z\theta i}(r, z_i), \quad i = 1, \ldots, N. \tag{10}$$

Note that the boundary conditions for the stress tensor components σ_{zr} and σ_{zz} are automatically satisfied, as follows from Eqs (4) and (7). Taking into account the symmetry of the problems, they can be reduced to boundary value problems for half-spaces $z \ge z_i$. To do this, the condition that the displacements $u_{\theta i}(r, z_i)$ outside the crack are zero should be added to the boundary condition (10)

$$u_{\theta i}(r, z_i) = 0, \quad a_i \le r < \infty. \tag{11}$$

From the boundary conditions (10) and (11) and relations (6) and (7), we obtain the systems of dual integral equations

$$\int_0^\infty A_i(\xi) J_1(\xi r) d\xi = 0, \quad a_i \le r < \infty,$$

$$\int_0^\infty \xi A_i(\xi) J_1(\xi r) d\xi = \frac{2Mr(1-\varepsilon)}{\pi R^4 \sqrt{\mu\hat{\mu}}} - \frac{G}{h(r)\sqrt{\mu\hat{\mu}}} \int_0^\infty A_i(\xi) J_1(\xi r) d\xi -$$
$$- \sum_{j=1(\ne i)}^{N} \frac{\operatorname{sgn}(z_i - z_j)}{\sqrt{n_3}} \int_0^\infty \xi A_j(\xi) \exp\left(-\frac{\xi}{\sqrt{n_3}}\left|z_i - z_j\right|\right) J(\xi r) d\xi, \tag{12}$$

$$0 \le r \le a_i, \quad i = 1, \ldots, N.$$

$\mu, \tilde{\mu}$ are the shear modules in the plane of isotropy and in the transverse direction; $\varepsilon = G/A_{44}$.

To solve the systems of dual integral equations (12), the functions $A_i(\xi)$ are represented in the form of a finite Fourier transform

$$A_i(\xi) = \int_0^{a_i} \Psi_i(t) \cos \xi t dt, \tag{13}$$

where $\Psi_i(t)$ are continuous functions with derivatives in the interval $[0, a_i]$. Then, the first equation (12) is satisfied if the condition with the requirement that

$$\int_0^{a_i} \Psi_i(t) dt = 0. \tag{14}$$

From the second equation (12), taking into account (13) and (14), after some transformations, we obtain a system of integral equations for determining the unknown functions $\Psi_i(t)$

$$\Psi_i(t) = \frac{2}{\pi} \frac{2M(1-\varepsilon)(a_i^2/3 - t^2)}{\pi R^4 \sqrt{\mu\tilde{\mu}}} +$$
$$+ \frac{2\varepsilon}{\pi}\left(-\int_0^t \frac{dr}{\sqrt{t^2 - r^2}\, h_i(r) r} \int_r^{a_i} \frac{\Psi_i(x) x}{\sqrt{x^2 - r^2}} dx + \frac{1}{a_i} \int_0^{a_i} \frac{r dr}{\sqrt{a_i^2 - r^2}} \int_0^r \frac{dr}{h_i(r) r} \int_0^{a_i} \frac{x \Psi_i(x) dx}{\sqrt{x^2 - r^2}} \right) -$$
$$- \frac{2}{\pi\sqrt{n_3}} \sum_{j=1(\ne 1)}^{N} \int_0^{a_i} \Psi_j(x) \left(\frac{(z_i - z_j) n_3^{-1/2} ((z_i - z_j)^2 n_3^{-1} + t^2 + x^2)}{((z_i - z_j)^2 n_3^{-1} + (t-x)^2)(z_i - z_j)^2 n_3^{-1} + (t+x)^2} - \right.$$

$$-\frac{1}{2a_i}\left(\operatorname{arctg}\frac{(a_i-x)\sqrt{n_3}}{z_i-z_j}+\operatorname{arctg}\frac{(a_i+x)\sqrt{n_3}}{z_i-z_j}\right)dx\,,$$

$$i=1,\ldots,N\,. \tag{15}$$

If a solution to the system of integral equations (15) is obtained by one of the methods, then the asymptotic expression for the stresses in the vicinity of the crack at $r \to a_i$ is as follows:

$$\sigma_{z\theta i}=\frac{-A_{44}\Psi_i(a_i)a_i}{r\sqrt{r^2-a_i^2}}+O(1)\,. \tag{16}$$

The stress intensity factors calculated by the formula $K_{\mathrm{III}i}=\lim\limits_{r\to a_i}\sqrt{2\pi(r-a_i)}\sigma_{z\theta i}(r)$ are determined from the expression

$$K_{\mathrm{III}i}=-\sqrt{\frac{\mu\tilde{\mu}}{\pi a_i}}\frac{2M\chi_i(a_i)}{R^4},\;\;\chi_i(a_i)=\frac{\pi R^4\Psi_i(a_i)}{2M}\,. \tag{17}$$

The limit torsional load M_* of the cylinder with healed cracks is set based on the force criterion, according to which fracture will begin in the vicinity of the crack with the highest stress intensity factor, provided that $\max(K_{\mathrm{III}i}) = K_{\mathrm{III}c}$.

Given the relation (17), a cylinder with N healed cracks will be able to withstand a maximum torsion with a moment equal to

$$M_*=-\frac{K_{\mathrm{III}c}\sqrt{\pi a_i}R^4}{2\sqrt{\mu\tilde{\mu}}\chi_i(a_i)}\,, \tag{18}$$

where the index “i” corresponds to the crack with the maximum stress intensity factor.

In the case of a single crack in a cylinder, the system of integral equations is reduced to a single equation, which, when $h(r)=\frac{c}{a}\sqrt{a^2-r^2}$, has an exact solution

$$\Psi(t)=-\frac{12M(1-\varepsilon)(t^2-a^2/3)}{\pi R^4\sqrt{\mu\tilde{\mu}}(3\pi+4\varepsilon a/c)}\,. \tag{19}$$

The stress intensity factor, in this case, is

$$K_{\mathrm{III}i}=\frac{8M(1-\varepsilon)a^{3/2}}{\sqrt{\pi}R^4(3\pi+4\varepsilon a/c)} \tag{20}$$

and, accordingly, the limiting value of the moment is

$$M_*=\frac{\sqrt{\pi}K_{\mathrm{III}c}R^4(3\pi+4\varepsilon a/c)}{8(1-\varepsilon)a^{3/2}}\,. \tag{21}$$

The tangential stresses that arise at the interface and, according to the model considered in the injected material, are as follows:

$$\sigma_{r\theta}(r) = \frac{2Mr\mu_*}{\pi R^4 \sqrt{\tilde{\mu}}} \left(\frac{1}{\sqrt{\tilde{\mu}}} + \frac{4(1-\varepsilon)\beta}{\sqrt{\mu}(3\pi + 4\beta\varepsilon)} \right).0 \tag{22}$$

Under the intensity of external loads

$$M \ldots M_*^i = \frac{\pi \tau_B R^4 \sqrt{\tilde{\mu}}}{2\mu_* a} \left(\frac{1}{\sqrt{\tilde{\mu}}} + \frac{4(1-\varepsilon)\beta}{\sqrt{\mu}(3\pi + 4\beta\varepsilon)} \right)^{-1}, \tag{23}$$

failure of the injection material from the crack contour is possible. Here, τ_B is the shear strength of the injected material. In practice, it is important to choose an injection material with a strength that would ensure that the condition $M_*^i > M_*$ is met, i. e., it does not collapse before the main material. The required shear strength of the injected material is obtained from relations (20) and (22):

$$\tau_B \ldots \frac{K_{\mathrm{III}c}(3\pi + 4\beta\varepsilon)\mu_*}{4\sqrt{a}(1-\varepsilon)\sqrt{\pi\tilde{\mu}}} \left(\frac{1}{\sqrt{\tilde{\mu}}} + \frac{4(1-\varepsilon)\beta}{\sqrt{\mu}(3\pi + 4\beta\varepsilon)} \right). \tag{24}$$

Conclusions. The mathematical model of restoring the strength of a cylinder containing a system of parallel circular cracks using injection technologies for healing defects has been constructed. The cylinder material is assumed to be transversally isotropic. Based on the Winkler model of thin elastic layers and the theory of harmonic functions, the problem is reduced to solving a system of integral equations. For the case of a single defect, an exact analytical solution of the corresponding equation is given. The parameters on which the efficiency of crack healing depends are determined. From the obtained results, it follows that by selecting the stiffness of the injection material, it is possible to achieve a complete restoration of the strength of the damaged cylinder.

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В.П. Сylованюк, https://orcid.org/0000-0002-3765-5962
Н.А. Івантишин, https://orcid.org/0000-0003-1835-0466
М.В. Філіпов, https://orcid.org/0009-0009-6940-9940

Фізико-механічний інститут ім. Г.В. Карпенка НАН України, Львів, Україна
E-mail: vsylovanyuk@gmail.com, n.ivantyshyn@gmail.com, maksym.filipov@gmail.com

МІЦНІСТЬ ЗА КРУЧЕННЯ ТРАНСВЕРСАЛЬНО-ІЗОТРОПНОГО ЦИЛІНДРА, ЩО МІСТИТЬ СИСТЕМУ “ЗАЛІКОВАНИХ” ТРІЩИН

Досліджено міцність трансверсально-ізотропного циліндра з системою “залікованих” крутильних тріщин. Погіршення експлуатаційних характеристик залізобетонних матеріалів у конструкціях АЕС, шахтних стволів, колекторів та гідротехнічних споруд зумовило необхідність відновлення працездатності, ремонту та відновлення несучої здатності таких елементів конструкцій. Розроблено математичну модель для визначення ефекту зміцнення циліндрів із застосуванням ін’єкційних технологій для “заліковування” дефектів. Розрахунки ґрунтуються на концепціях механіки руйнування та лінійної теорії пружності. Застосовано модель тонких пружних включень і теорію гармонічних функцій, що дає можливість звести задачу до розв’язання системи інтегральних рівнянь. Визначено параметри, які впливають на ефективність “заліковування” тріщин. Отримано аналітичні розв’язки для окремих випадків, що дало змогу оптимізувати вибір матеріалу для ін’єкції. Показано, що правильний підбір жорсткості ін’єкційного матеріалу може забезпечити повне відновлення міцності циліндра з тріщиною.

***Ключові слова**: трансверсально-ізотропне тіло, тріщина, “заліковування” тріщини, міцність.*