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Dirichlet problem for general *A*-harmonic equations in simply connected domains

Presented by Academician of the NAS of Ukraine I.I. Skrypnik

The article is devoted to theorems on the existence, representation, and regularity of solutions to the Dirichlet problem with continuous data for general A-harmonic equation div *A* grad *U* = 0 *in the real plane with matrix valued coefficients A. The equation is one of the main equations of the hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media. Here we present a number of effective integral solvability criteria for this problem of the type of Calderon—Zygmund, Dini—Lavrentiev—Lehto, Orlicz, BMO, FMO and VMO in arbitrary bounded simple connected domains including all Jordan domains. These results are based on the theory of the so-called Beltrami equations with two characteristics in the complex plane and formulated in terms of the corresponding two complex parameters associated with A.*

Keywords: *Dirichlet problem, A-harmonic equation, simply connected domains, Beltrami equations.*

1. Introduction. Let us denote by $\mathbb{S}^{2\times 2}$ the collection of all 2 × 2 matrices with real entries,

$$A(z)=\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix} \tag{1}$$

and the *ellipticity condition* $\det(I+A)>0$, where I is the unit 2 × 2 matrix, i.e., $(1+a_{11})(1+a_{22})>a_{12}a_{21}$ (see Section 16.1.5 in [1]).

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By default, we often use here the natural isomorphism of the real plane $\mathbb{R}^2$ and the complex plane $\mathbb{C}$, implicating the natural one-to-one correspondence $Z:=(x,y)\leftrightarrow z:=x+iy$. Given a domain D in $\mathbb{C}$, let us consider the A-harmonic equation in standard notation

$$\nabla A(Z)\nabla U(Z)=0 \tag{2}$$

with a measurable matrix valued coefficient $A:D\to\mathbb{S}^{2\times 2}$.

A continuous function $U:D\to\mathbb{R}$ is called *A-harmonic function* (see e.g. [2]), if U is a weak solution of the equation (2), i.e., if $U\in W^{1,1}_{\mathrm{loc}}(D)$ and

$$\int_D \langle A(Z)\nabla U(Z),\nabla\Psi(Z)\rangle\, dL(Z)=0 \quad \forall\,\Psi\in C_0^\infty(D). \tag{3}$$

Hereafter $dL(Z)$, $Z:=(x,y)\in\mathbb{R}^2$, corresponds to the Lebesgue measure in $\mathbb{R}^2$.

A function $V:D\to\mathbb{R}$ in the Sobolev class $W^{1,1}_{\mathrm{loc}}(D)$ is further called *A-harmonic conjugate* of the *potential function* U (*stream function*) if

$$\nabla V(Z)=\mathbb{H}A(Z)\nabla U(Z) \tag{4}$$

with the *Hodge operator*

$$\mathbb{H}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}. \tag{5}$$

Since $\mathbb{H}^2=-I$, i.e., $\mathbb{H}$ plays the role of imaginary unit in the space of matrices $\mathbb{S}^{2\times 2}$, the functions U and V satisfy *generalized Cauchy—Riemann system*

$$\begin{aligned} V_y &= a_{11}U_x + a_{12}U_y,\\ -V_x &= a_{21}U_x + a_{22}U_y. \end{aligned} \tag{6}$$

By means of (rather lengthy) purely algebraic calculation (see Theorem 16.1.6 in [1]), it follows that the function $f:=U+iV$ satisfies the general Beltrami equation with two characteristics

$$f_{\bar z}=\mu(z)\cdot f_z+\nu(z)\cdot\overline{f_z}, \tag{7}$$

where $f_{\bar z}=(f_x+if_y)/2$, $f_z=(f_x-if_y)/2$, $z=x+iy$, and, with $\mathrm{Tr}A:=a_{11}+a_{22}$,

$$\mu=\mu_A=\frac{a_{22}-a_{11}-i(a_{12}+a_{21})}{1+\mathrm{Tr}A+\det A},\quad \nu=\nu_A=\frac{1-\det A+i(a_{12}-a_{21})}{1+\mathrm{Tr}A+\det A}. \tag{8}$$

In terms of the characteristics μ and ν, the ellipticity condition is written as

$$|\mu(z)|+|\nu(z)|<1 \text{ a.e. in } D. \tag{9}$$

Vice versa, given measurable complex valued functions μ and ν, satisfying (9), one can invert the algebraic system (8) and obtain the matrix valued function

$$A=\begin{bmatrix}\dfrac{|1-\mu|^2-|\nu|^2}{|1+\nu|^2-|\mu|^2} & \dfrac{2\operatorname{Im}(\nu-\mu)}{|1+\nu|^2-|\mu|^2}\\ \dfrac{-2\operatorname{Im}(\nu+\mu)}{|1+\nu|^2-|\mu|^2} & \dfrac{|1+\mu|^2-|\nu|^2}{|1+\nu|^2-|\mu|^2}\end{bmatrix}. \tag{10}$$

Thus, (7) is in fact a complex form of one of the main equations of mathematical physics (2) in anisotropic and inhomogeneous media. We will call the *dilatation quotient* of equation (2) the function

$$K_A(Z):=\frac{1+|\mu_A(Z)|+|\nu_A(Z)|}{1-|\mu_A(Z)|-|\nu_A(Z)|}. \tag{11}$$

2. Setting the task. The purpose of this paper is to study the Dirichlet problem

$$\lim_{Z\to\zeta} U(Z)=\Phi(\zeta) \quad \forall\zeta\in\partial D \tag{12}$$

in arbitrary bounded simply connected domains D with continuous boundary data $\Phi:\partial D\to\mathbb{R}$ to locally uniformly elliptic A-harmonic equations (2), i.e., with locally bounded dilatation quotients $K_A\in L^1(D)$, that allow singularities at the boundary.

A point $\zeta\in\partial D$ is hereinafter referred to as a *singular point* of equation (2) if the dilatation quotient K_A is not essentially bounded at each neighborhood of the point ζ. Here we give a series of integral conditions on K_A at neighborhoods of boundary points that guarantee the existence of solutions for the Dirichlet problem (12) to (2).

It is clear that if $\Phi(\zeta)\equiv\text{const}$, then we have a trivial constant solution to the Dirichlet problem (12) to (2). However, in the case $\Phi(\zeta)\not\equiv\text{const}$, we will seek solutions U with its A-harmonic conjugate V such that the function $f:=U+iV$ is a continuous, discrete and open mapping $f:D\to\mathbb{C}$ of the Sobolev class $W^{1,2}_{\text{loc}}$, for which the Jacobian

$$J_f(z)=|f_z|^2-|f_{\bar z}|^2=\det\begin{bmatrix}U_x & U_y\\ V_x & V_y\end{bmatrix}\neq 0 \text{ a.e. in } D,\ z=x=iy, \tag{13}$$

that are called *regular solutions* of the Dirichlet problem (12) to (2). Note that (13) implies, in particular, that $\operatorname{grad} U\neq 0$ and $\operatorname{grad} V\neq 0$ a.e. in D.

Recall that a mapping $f:D\to\mathbb{C}$ is called *discrete* if the preimage $f^{-1}(w)$ consists of isolated points for every $w\in\mathbb{C}$, and *open* if f maps every open set to an open set in $\mathbb{C}$.

The advanced theory of Beltrami equations (7) (see e.g. articles [3—9]) makes it possible to obtain a number of useful consequences for A-harmonic equations (2). The theory of Beltrami equations with $\nu\equiv 0$ is much more advanced (see e.g. mononographs [10—13] as well as further references therein). By remarks 1 and 2 to Theorem 16.1.6 on the page 413 in [1], they correspond to the A-harmonic equations (2) with symmetric A, i.e. with $a_{12}=a_{21}$, and $\det A=1$. The consequences for the Dirichlet problem to such A-harmonic equations (2) can be found in articles [14, 15].

We refer the reader for a brief summary of the theory of BMO (bounded mean oscillation), FMO (finite mean oscillation), and VMO (vanishing mean oscillation) function classes, the relevant auxiliary statements and historic comments to our last article [9].

3. The main lemma. Combining Lemma 3.1 from our article [9] and Theorem 16.1.6 from monograph [1], we obtain the following lemma.

Lemma 1. *Let D be a bounded simply connected domain and $A: D \to \mathbb{S}^{2\times 2}$ be a measurable matrix valued function with the locally bounded $K_A \in L^1(D)$ and, for each $Z_0 \in \partial D$,*

$$\int\limits_{\varepsilon<|Z-Z_0|<\varepsilon_0} K_A(Z)\cdot \psi^2_{Z_0,\varepsilon}(|Z-Z_0|)\, dL(Z) = o(I^2_{Z_0}(\varepsilon)) \ \text{ as } \varepsilon \to 0 \tag{14}$$

for measurable functions $\psi_{Z_0,\varepsilon}: (0,\infty)\to(0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$, $\varepsilon_0 = \varepsilon(Z_0)$, with

$$0 < I_{Z_0}(\varepsilon) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi_{Z_0,\varepsilon}(t)\, dt < \infty . \tag{15}$$

Then the Dirichlet problem (12) *for the A-harmonic equation* (2) *has a regular solution U for each inconstant continuous function $\Phi: \partial D \to \mathbb{R}$.*

Hereinafter we assume that K_A is extended by 1 outside the domain D.

Corollary 1. *Let D be a bounded simply connected domain and let $A: D \to \mathbb{S}^{2\times 2}$ be a measurable matrix valued function with the locally bounded $K_A \in L^1(D)$ and, for each $Z_0 \in \partial D$,*

$$\int\limits_{\varepsilon<|z-z_0|<\varepsilon_0} K_A(Z)\cdot \psi^2(|Z-Z_0|)\, dL(Z) \leqslant O\left(\int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt\right) \ \text{ as } \varepsilon \to 0 \tag{16}$$

for some $\varepsilon_0 > 0$ and a measurable function $\psi: (0,\infty)\to(0,\infty)$ such that

$$\int\limits_0^{\varepsilon_0} \psi(t)\, dt = \infty, \quad 0 < \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty \quad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{17}$$

Then the Dirichlet problem (12) *for the equation* (2) *has a regular solution U for each inconstant continuous function $\Phi: \partial D \to \mathbb{R}$.*

Remark 1. By Remark 3.10 in [9], each regular solution u to the Dirichlet problem (12) for the A-harmonic equation (2) in Lemma 1 and Corollary 1 is locally Hölder in D and has the representation

$$U = H \circ h, \tag{18}$$

where $h: D \to \mathbb{D}$ is a locally quasiconformal homeomorphism of D onto the unit disk $\mathbb{D}$, which extends to a homeomorphism $\tilde{h}$ of $\overline{D_P}$ (the completion of the domain D by prime ends of Caratheodory) onto $\overline{\mathbb{D}}$, and $H: D \to \mathbb{R}$ is the unique harmonic solution in $\mathbb{D}$ of the Dirichlet problem with

$$\lim_{w\to\xi} H(w) = \varphi(\xi) := \Phi(\tilde{h}^{-1}(\xi)) \quad \forall \xi \in \partial\mathbb{D}, \tag{19}$$

and where the same boundary values $\Phi = \varphi \circ \tilde{h}$ under the mapping $\tilde{h} : \overline{D_P} \to \overline{\mathbb{D}}$ correspond to the prime ends of the domain D associated with the same points ς in ∂D (cf. Theorems 4.1.8 and 4.2.1 from the classic potential theory in [16]).

4. Criteria in terms of BMO, FMO and VMO. Recall that a real-valued function Φ in a domain D of $\mathbb{R}^2$ is called a *bounded mean oscillation* in D, abbr. $\Phi \in \mathrm{BMO}\,(D)$, if

$$\| \Phi \|_* := \sup_B \frac{1}{|B|} \int_B |\Phi(Z) - \Phi_B| \, dL(Z) < \infty , \tag{20}$$

where $\Phi \in L^1_{\mathrm{loc}}(D)$, the supremum is taken over all discs B in D and

$$\Phi_B := \frac{1}{|B|} \int_B \Phi(Z)\, dL(Z).$$

We also write $\Phi \in \mathrm{BMO}(\overline{D})$ if $\Phi_* \in \mathrm{BMO}(D_*)$ for some extension Φ_* of the function Φ into a domain D_* containing $\overline{D}$.

The class BMO was introduced by John and Nirenberg (1961) in the paper [17] and soon became an important concept in harmonic analysis, partial differential equations and related areas (see e.g. monographs [2] and [18]).

Theorem 1. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a measurable matrix valued function with locally bounded K_A. Suppose also that K_A has a dominant $Q : \mathbb{R}^2 \to [1, \infty)$ in the class* $\mathrm{BMO}\,(\overline{D})$. *Then the Dirichlet problem* (12) *for the A-harmonic equation* (2) *has a regular locally Hölder continuous solution U with the representation* (18) *for each inconstant continuous function $\Phi : \partial D \to \mathbb{R}$.*

A function Φ in BMO is considered to have a *vanishing mean oscillation*, abbr. $\Phi \in \mathrm{VMO}$, if the supremum in (20) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \to 0$. VMO has been introduced by Sarason in [19]. There are a number of papers devoted to the study of PDEs with coefficients of the class VMO. Note that $W^{1,2}(D) \subset \mathrm{VMO}(D)$ (see e.g. [20]).

Corollary 2. *In particular, the conclusion of Theorem 1 on existence of a regular solution for the Dirichlet problem* (12) *for the A-harmonic equation* (2) *holds if the dominant Q of K_A belongs to the class $W^{1,2}(\overline{D})$.*

Following [21], we say that a locally integrable function $\Phi : D \to \mathbb{R}$ *has finite mean oscillation* at a point $Z_0 \in D$, abbr. $\Phi \in \mathrm{FMO}(Z_0)$, if

$$\overline{\lim_{\varepsilon \to 0}} \frac{1}{|B(Z_0, \varepsilon)|} \int_{B(Z_0,\varepsilon)} \left| \Phi(Z) - \tilde{\Phi}_\varepsilon(Z_0) \right| dL(Z) < \infty, \tag{21}$$

where

$$\tilde{\Phi}_\varepsilon(Z_0) = \frac{1}{|B(Z_0, \varepsilon)|} \int_{B(Z_0,\varepsilon)} \Phi(Z)\, dL(Z) < \infty, \tag{22}$$

is the mean integral value of the function $\Phi(Z)$ over disk $B(Z_0, \varepsilon) := \{Z \in \mathbb{R}^2 : |Z - Z_0| < \varepsilon\}$.

Theorem 2. *Let D be a bounded simply connected domain in $\mathbb{R}^2$ and let $A : D \to \mathbb{S}^{2\times 2}$ be a measurable matrix valued function with locally bounded K_A. Suppose also that $K_A(Z) \leqslant Q_{Z_0}(Z)$*

a.e. in U_{Z_0} *for every point* $Z_0 \in \partial D$, *a neighborhood* U_{Z_0} *of* Z_0 *and a function* $Q_{Z_0} : U_{Z_0} \to [0,\infty]$ *in the class* $\mathrm{FMO}(Z_0)$. *Then the Dirichlet problem* (12) *for the A-harmonic equation* (2) *has a regular locally Hölder continuous solution* U *with the representation* (18) *for each inconstant continuous function* $\Phi : \partial D \to \mathbb{R}$.

Corollary 3. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and* $A : D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* K_A. *Suppose also that*

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{|B(Z_0,\varepsilon)|} \int\limits_{B(Z_0,\varepsilon)} K_A(Z)\, dL(Z) < \infty \qquad \forall\, Z_0 \in \partial D. \tag{23}$$

Then the conclusion of Theorem 2 holds.

Corollary 4. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and* $A : D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* K_A. *Suppose also that* $K_A(z) \leqslant Q(z)$ *a.e. in* D *with a function* Q *of the class* $\mathrm{FMO}(\overline{D})$. *Then the conclusion holds.*

5. Criteria of the Calderon—Zygmund type.

Theorem 3. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and let* $A : D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* $K_A \in L^1(D)$. *Suppose also that*

$$\int\limits_{\varepsilon<|Z-Z_0|<\varepsilon_0} K_A(Z) \frac{dL(Z)}{|Z-Z_0|^2} = o\left(\left[\log\frac{1}{\varepsilon}\right]^2\right) \qquad \forall Z_0 \in \partial D \tag{24}$$

as $\varepsilon \to 0$ for some $\varepsilon_0 = \varepsilon(Z_0) > 0$. *Then the Dirichlet problem* (12) *for the A-harmonic equation* (2) *has a regular locally Hölder continuous solution* U *with the representation* (18) *for each inconstant continuous function* $\Phi : \partial D \to \mathbb{R}$.

Remark 2. We are also able here to replace (24) by

$$\int\limits_{\varepsilon<|Z-Z_0|<\varepsilon_0} \frac{K_A(Z)dL(Z)}{\left(|Z-Z_0|\log\frac{1}{|Z-Z_0|}\right)^2} = o\left(\left[\log\log\frac{1}{\varepsilon}\right]^2\right). \tag{25}$$

In general, we can present the entire scale of the corresponding conditions in terms of iterated logarithms, i.e., in terms of functions of the form $1/(t\log 1/t \cdot \log\log 1/t \cdot \ldots \cdot \log\ldots\log 1/t)$.

6. Criteria of the Dini—Lavrentiev—Lehto type. *Further* $k_A(Z_0,r)$ *denotes the integral mean of* $K_A(Z)$ *over the circle* $S(Z_0,r) := \{Z \in \mathbb{R}^2 : |Z-Z_0| = r\}$.

Theorem 4. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and let* $A : D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* $K_A \in L^1(D)$. *Suppose also that*

$$\int\limits_0^{\varepsilon} \frac{dr}{rk_A(Z_0,r)} = \infty \qquad \forall Z_0 \in \partial D, \quad \varepsilon > 0. \tag{26}$$

Then the Dirichlet problem (12) *for the A-harmonic equation* (2) *has a regular locally Hölder continuous solution* U *with the representation* (18) *for each inconstant continuous function* $\Phi : \partial D \to \mathbb{R}$.

Corollary 5. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and* $A: D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* $K_A \in L^1(D)$. *Suppose also that*

$$k_A(Z_0, \varepsilon) = O\left(\log\frac{1}{\varepsilon}\right) \qquad \text{as } \varepsilon \to 0 \qquad \forall\, Z_0 \in \partial D. \tag{27}$$

Then the conclusion of Theorem 4 holds.

Remark 3. In particular, the conclusion of Theorem 4 holds if

$$K_A(Z) = O\left(\log\frac{1}{|Z - Z_0|}\right) \qquad \text{as} \quad Z \to Z_0 \quad \forall\, Z_0 \in \partial D. \tag{28}$$

Moreover, the condition (27) can be replaced by the whole series of more weak conditions

$$k_A(Z_0, \varepsilon) = O\left(\left[\log\frac{1}{\varepsilon}\cdot \log\log\frac{1}{\varepsilon}\cdot \ldots \cdot \log\ldots\log\frac{1}{\varepsilon}\right]\right) \qquad \forall\, Z_0 \in \partial D. \tag{29}$$

7. Criteria of the Orlicz type.

Theorem 5. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and let* $A: D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* K_A. *Suppose also that*

$$\int_D \Phi(K_A(Z))\, dL(Z) < \infty \tag{30}$$

for a convex non-decreasing function $\Phi: [0, \infty] \to [0, \infty]$ *such that, for* $\Delta > 0$,

$$\int_\Delta^\infty \log\, \Phi(t)\, \frac{dt}{t^2} \quad = \quad +\infty\,. \tag{31}$$

Then the Dirichlet problem (12) *for the A-harmonic equation* (2) *has a regular locally Hölder continuous solution* U *with the representation* (18) *for each inconstant continuous function* $\Phi: \partial D \to \mathbb{R}$.

Remark 4. Note that condition (31) is not only sufficient but also necessary the existence of ordinary solutions to the Dirichlet problem (12) for the *A*-harmonic equations (2) with the integral constraints (30) for all continuous inconstant data $\Phi: \partial D \to \mathbb{R}$.

Corollary 6. *Let* D *be a bounded simply connected domain in* $\mathbb{R}^2$ *and* $A: D \to \mathbb{S}^{2\times 2}$ *be a measurable matrix valued function with locally bounded* K_A. *Suppose also that for some* $\alpha > 0$,

$$\int_D \exp\, \alpha K_A(Z)\, dL(Z) < \infty\,. \tag{32}$$

Then the conclusion of Theorem 5 holds.

Conclusions. The main result of this work is a series of effective integral criteria of the type of BMO (Bounded Mean Oscillation by John—Nirenberg), Dini—Lavrentiev—Lehto, Zygmund—Calderon and Orlich for the existence, representation and regularity of solutions to the Dirichlet problem with continuous data for the general equation of hydrodynamics in anisotropic and inhomogeneous media. These results are formulated in arbitrary bounded simply connected domains in the real plane, which include, in particular, all Jordan domains, and based on the authors' recently developed theory of the Dirichlet problem for the so-called Beltrami equations with two characteristics in the complex plane, that admit singularities at the boundary. The results of this article can be applied for mathematical modeling of the filtration of incompressible fluids under development of oil fields and aquifers.

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REFERENCES

1. Astala, K., Iwaniec, T. & Martin, G. (2009). Elliptic partial differential equations and quasiconformal mappings in the plane. Princeton Mathematical. Series 48. Princeton, NJ: Princeton University Press.
2. Heinonen, J., Kilpelainen, T. & Martio, O. (1993). Nonlinear potential theory of degenerate elliptic equations. Oxford: Oxford University Press.
3. Bojarski, B. V (2009). Generalized solutions of a system of differential equations of the first order and elliptic type with discontinuous coefficients. University of Jyväskylä. Department of Mathematics and Statistics, Report 118 (pp. 1-64). Jyväskylä: University Printing House.
4. Bojarski, B., Gutlyanskii, V. & Ryazanov, V. (2014). On existence and representation of solutions for general degenerate Beltrami equations. Complex Var. Elliptic Equ., 59, No. 1, pp. 67-75. https://doi.org/10.1080/17476933.2013.795955
5. Bojarski, B., Gutlyanskii, V. & Ryazanov, V. (2013). Dirichlet problem for the general Beltrami equation in Jordan domains. J. Math. Sci., 190, No. 4, pp. 525-538. https://doi.org/10.1007/s10958-013-1269-x
6. Bojarski, B., Gutlyanskii, V. & Ryazanov, V. (2012). On the Dirichlet problem for general degenerate Beltrami equations. Bull. Soc. Sci. Lett. Lodz, Ser. Rech. Deform., 62, No. 2, pp. 29-43.
7. Bojarski, B., Gutlyanskii, V. & Ryazanov, V. (2011). On integral conditions for the general Beltrami equations. Complex Anal. Oper. Theory, 5, No. 3, pp. 835-845. https://doi.org/10.1007/s11785-010-0088-z
8. Bojarski, B., Gutlyanskii, V. & Ryazanov, V. (2009). On the Beltrami equations with two characteristics. Complex Var. Elliptic Equ., 54, No. 10, pp. 935-950. https://doi.org/10.1080/17476930903030069
9. Gulyanskii, V., Ryazanov, V., Salimov, R. & Sevost'yanov, E. (2025). Dirichlet problem for general Beltrami equations in simply connected domains. Ukr. Math. Bull., 22, No. 4, pp. 477-498.
10. Bojarski, B., Gutlyanskii, V., Martio, O. & Ryazanov, V. (2013). Infinitesimal geometry of quasiconformal and bi-Lipschitz mappings in the plane. EMS Tracts in Mathematics, Vol. 19. Zürich: European Mathematical Society. https://doi.org/10.4171/122
11. Gutlyanskii, V., Ryazanov, V., Srebro, U. & Yakubov, E. (2012). The Beltrami equation: A geometric approach. Developments in Mathematics, Vol. 26. Berlin: Springer. https://doi.org/10.1007/978-1-4614-3191-6
12. Martio, O., Ryazanov, V., Srebro, U. & Yakubov, E. (2009). Moduli in modern mapping theory. Springer Monographs in Mathematics. New York: Springer. https://doi.org/10.1007/978-0-387-85588-2
13. Hencl S. & Koskela P. (2014). Lectures on mappings of finite distortion. Lecture Notes in Mathematics, Vol. 2096. Cham: Springer. https://doi.org/10.1007/978-3-319-03173-6
14. Gutlyanskii, V., Ryazanov, V., Sevost'yanov, E. & Yakubov, E. (2022). BMO and Dirichlet problem for degenerate Beltrami equation. J. Math. Sci., 268, No. 2, pp. 157-177. https://doi.org/10.1007/s10958-022-06189-w
15. Gutlyanskii, V. Ya., Ryazanov, V. I., Sevost'yanov, E. A. & Yakubov, E. (2023). On the Dirichlet problem for A-harmonic functions. Dopov. Nac. akad. nauk Ukr., No. 4, pp. 11-19. https://doi.org/10.15407/dopovidi2023.04.011
16. Ransford, Th. (1995). Potential theory in the complex plane. London Mathematical Society Student Texts, Vol. 28. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9780511623776

17. John, F. & Nirenberg, L. (1961). On functions of bounded mean oscillation. Comm. Pure Appl. Math., 14, pp. 415-426. https://doi.org/10.1002/cpa.3160140317
18. Reimann, H. M. & Rychener, T. (1975). Funktionen beschränkter mittlerer oszillation. Lecture Notes in Mathematics, Vol. 487. Berlin: Springer. https://doi.org/10.1007/BFb0081825
19. Sarason, D. (1975). Functions of vanishing mean oscillation. Trans. Amer. Math. Soc., 207, pp. 391-405. https://doi.org/10.1090/S0002-9947-1975-0377518-3
20. Brezis, H. & Nirenberg, L. (1995). Degree theory and BMO; part I: Compact manifolds without boundaries. Selecta Math., New Ser., 1, No. 2, pp.197-263. https://doi.org/10.1007/BF01671566
21. Ignat'ev, A. A. & Ryazanov, V. I. (2005). Finite mean oscillation in the mapping theory. Ukr. Math. Bull., 2, No. 3, pp. 403-424.

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ЗАДАЧА ДІРІХЛЕ ДЛЯ ЗАГАЛЬНОГО *А*-ГАРМОНІЧНОГО РІВНЯННЯ В ОДНОЗВ'ЯЗНИХ ОБЛАСТЯХ

Розглянуто теореми існування, представлення та регулярності розв'язків задачі Діріхле з неперервними даними для загального A-гармонічного рівняння $\mathrm{div}\, A\, \mathrm{grad}\, U = 0$ на дійсній площині з матричнозначними коефіцієнтами. Зазначене рівняння є одним з основних рівнянь гідромеханіки (механіки рідини) в анізотропних та неоднорідних середовищах. Наведено ряд ефективних інтегральних критеріїв розв'язності цієї задачі типу Кальдерона—Зигмунда, Діні—Лаврентьєва—Лехто, Орліча, BMO, FMO та VMO у довільних обмежених однозв'язних областях, які включають всі жорданові області. Ці результати базуються на теорії так званих рівнянь Бельтрамі з двома характеристиками на комплексній площині і сформульовані через відповідні два комплексні параметри, пов'язані з A.

Ключові слова: *задача Діріхле, A-гармонічне рівняння, однозв'язна область, рівняння Бельтрамі.*