doi:https://doi.org/10.15407/emodel.40.05.041

UDC 519.21

**A.V. Makarichev,** Dr Sc. (Phys.-Math.), **A.A. Kud**
Kharkiv National Automobile and Highway University
(Yaroslav Mudry St. 25, Kharkiv, 61002, Ukraine,
e-mail: amsol12904@gmail.com),
**A.B. Shchukin**
N.Ye. Zhukovsky National Aerospace University «Kharkiv Aviation Institute»
(Chkalov St. 17, Kharkiv, 61070, Ukraine)

# Assessment of the Probability of System Failure with Maximum Service Accumulation Elements

A single-line queuing systems is considered. Incoming stream of requirements is the Poisson flow. The requirement service is performed in the order of arrival. The maintenance of the requirements consists of two independent non-negative random variables. In the course of time, distributed as the first of two quantities, there appears another Poisson flow of requirements with independent identically distributed lengths. Of these, the maximum length requirement is chosen. This maximum length is the second term of the service requirement time. The number of requirements in the system as a function of time forms a regenerating process. The moments of the absence of requirements in the system are the moments of regeneration. At the moment of transition of the random process from the state $n$ to the state $n+1$, a failure occurs ($n = 1, 2, \dots$). Two-way estimates for the probability of failure at the regeneration period are found. Moreover, the upper and lower bounds for ordinary duplication ($n = 1$) coincide.

*K e y w o r d s: the probability of system failure, maximum service accumulation elements.*

In the present work, the direction of the studies of A.D. Soloviev and his students is continued in the field of obtaining estimates for the probability of failure of the restored redundant systems during the regeneration period. One such upper bound for another method can be improved. This is the subject of this study.

The repair body, which is a single-line queuing system, receives a Poisson flow with the parameter  of requirements for restoring the elements. Requirements are serviced in the order in which they are received. The service times of requirements are independent in the aggregate and are equally distributed with the distribution function $H(x)$. We denote

$$h(s) = \int_{x>0} \exp(-sx)\, dH(x)$$

the Laplace transform for the service time of the requirements. During the maintenance of a requirement with intensity $v$ a simple flow of service accumulation arises. Depending on the number of $k$ points of this Poisson flow, during the requirement service, an additional set of independent equally distributed distributions $F(x)$ of random times $X_1, X_2, ..., X_k$ from which the conditional maximum is chosen $X_{\max}^{(k)} = \max\{X_1, ..., X_k\}$ with distribution function

$$F^{(k)}(x) = P\{X_{\max}^{(k)} \le x\} = \prod_{j=1}^{k} P\{X_j \le x\} = F^k(x)$$

and it is added to the maintenance time. Let

$$R(x) = \int_{t>0} \sum_{k\ge 0} \frac{(vt)^k}{k!} \exp(-vt)\,[F(x)]^k\, dH(t) = h\{-v\,[1-F(x)]\}$$

be the distribution function of the unconditional maximum $X_{\max}$ there may be additional time that has arisen by the time when the requirement is serviced,

$$G(x) = P\{\eta + X_{\max} \le x\} = \int_0^x R(x-t)\, dH(t)$$

is the distribution function of the total service time of requirement.

After the recovery, the item returns to where it came from. The random service process at $t$ is given by the number of elements that are restored in the repair body. This random process is regenerative. The moments of the regeneration are the moments of the transition of the random process to the state $\{0\}$. In the moment of the transition of the random process from the state $\{n\}$ to the state $\{n+1\}$ there is the failure, $n = 1, 2, \dots$. Denote by $q$ the probability of failure during the regeneration period of this random service process. Let $\overline{G}(x) = 1 - G(x)$ and

$$b_{n-1} = \int_0^\infty \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} \exp(-\lambda x)\, \overline{G}(x)\, dx.$$

**Lemma 1.** Suppose that for the numbers $a_{ij} \ge 0,\ b_i > 0,\ x_j > 0,\ i = 1, 2, ..., n,$ $j = 1, 2, ..., n,$ it is known that $x_i \le b_i + \sum_{j=1}^{n} a_{ij} x_j$ for all $i = 1, 2, ..., n.$ Then for all $i = 1, 2, ..., n$ the inequality $x_i \le \frac{b_i}{1-\alpha}$ holds, where

$$\alpha = \max_{1\le i\le n} \sum_{j=1}^{n} \frac{a_{ij} b_j}{b_i}.$$

P r o o f. From the condition of the lemma $0 < x_j \le b_j$ for all $j = 1, 2, \ldots, n$. Hence, for all $j = 1, 2, ..., n$ it is true $x_j = \frac{b_j}{1-\alpha_j}$, where $\alpha_j = 1 - \frac{b_j}{x_j}$. Let $\alpha_k = \max_{1\le j\le n} \alpha_j$. Then from the condition and last equality the chain of relations is true

$$b_k\left(1+\frac{\alpha_k}{1-\alpha_k}\right) = \frac{b_k}{1-\alpha_k} = x_k \le b_k + \sum_{j=1}^{n} a_{kj} x_j = b_k\left(1+\frac{\sum_{j=1}^{n} a_{kj} x_j}{b_k}\right) =$$

$$= b_k\left(1+\frac{\sum_{j=1}^{n} a_{kj}\frac{b_j}{(1-\alpha_j)}}{b_k}\right) \le b_k\left(1+\frac{\sum_{j=1}^{n} a_{kj} b_j}{b_k(1-\alpha_k)}\right).$$

Comparing the left and right parts of these relations, we see that

$$b_k\left(1+\frac{\alpha_k}{1-\alpha_k}\right) \le b_k\left(1+\frac{\sum_{j=1}^{n} a_{kj} b_j}{b_k(1-\alpha_k)}\right),$$

and hence the inequalities

$$\alpha_j \le \alpha_k \le \sum_{j=1}^{n} \frac{a_{kj} b_j}{b_k} \le \alpha = \max_{1\le i\le n} \sum_{j=1}^{n} \frac{a_{ij} b_j}{b_i}.$$

Consequently for all $j = 1, 2, ..., n$ are true

$$x_j = \frac{b_j}{1-\alpha_j} \le \frac{b_j}{1-\alpha}.$$

**Lemma 2.** For any nonnegative integers $i$ and $j$ the inequality $b_i b_j \le C_{i+j}^{i} b_0 b_{i+j}$ is true.

P r o o f. Let us

$$f(x) = \frac{\lambda \exp(-\lambda x)\,\overline{G}(x)}{\int_0^\infty \lambda \exp(-\lambda x)\,\overline{G}(x)\,dx} \quad \text{and} \quad M_i = \int_0^\infty x^i f(x)\,dx.$$

Note that $M_i = \dfrac{i! b_i}{\lambda^i b_0}$. Hence the inequality for moments [1] $M_i M_j \le M_{i+j}$ the chain of ratios follows

$$b_i b_j = \frac{b_0^2}{i!\,j!}\,\frac{i! b_i}{\lambda^i b_0}\,\frac{j! b_j}{\lambda^j b_0}\lambda^{i+j} = \lambda^{i+j}\frac{b_0^2}{i!\,j!} M_i M_j \le \lambda^{i+j}\frac{b_0^2}{i!\,j!} M_{i+j} =$$

$$= \lambda^{i+j}\frac{b_0^2}{i!\,j!}\,\frac{(i+j)! b_{i+j}}{\lambda^{i+j} b_0} = C^i_{j+j} b_0 b_{i+j},$$

that is $b_i b_j \le C^i_{i+j} b_0 b_{i+j}$.

Let us denote $q_r(n+1)$ the conditional probability of failure of the regeneration period, provided that in its beginning in repair body there are exactly $r$ complete requirements for the restoration of elements, $r = 1, 2, \ldots, n-1$.

**Theorem.** For all natural numbers $n$ the following inequality is true

$$q = q_1(n+1) \le \frac{b_{n-1}}{1 - b_0(2^{n-1} - 1)}.$$

P r o o f. Let us denote by $j$ the number of failed elements, during the recovery of the first failed item for the employment period in the repair body. By the formula of total probability record

$$q_1(n+1) = b_{n-1} + \sum_{j=1}^{n-1} a_j q_j(n+1).$$

Using the formula of total probability, we write down the expression for the probability of failure $q_r(n+1)$, when at the beginning of the period of employment in the repair body are exactly $r \ge 2$ (no less than two) full requirements

$$q_r(n+1) = b_{n-r} + \sum_{j=1}^{n-r} a_j q_{r-1+j}(n+1), \;\; 2 \le r \le n.$$

Note that $a_0 = 1 - b_0$ и $a_j = b_{j-1} - b_j$, $j \ge 1$. These inequalities and the Abel transformation make it possible to write down and estimate from above the second terms in the right-hand sides of the last two series of expressions, respectively, in the form

$$\sum_{j=1}^{n-1} a_j q_j(n+1) = \sum_{j=1}^{n-1} [b_{j-1} - b_j]\, q_j(n+1) =$$

$$= \sum_{j=1}^{n-1} b_{j-1} [\, q_j(n+1) - q_{j-1}(n+1)] - b_{n-1}\, q_{n-1}(n+1) \le$$

$$\le \sum_{j=1}^{n-1} b_{j-1}[\, q_j(n+1) - q_{j-1}(n+1)],$$

here by definition we assume $q_0(n+1)=0$,

$$\sum_{j=0}^{n-r} a_j q_{r-1+j}(n+1) = \sum_{j=1}^{n-r}[b_{j-1}-b_j]\, q_{r-1+j}(n+1) + [1-b_0]\, q_{r-1}(n+1) =$$

$$= \sum_{j=1}^{n-r} b_{j-1}[\, q_{r-1+j}(n+1) - q_{r-2+j}(n+1)] - b_{n-r}\; q_{n-1}(n+1) + q_{r-1}(n+1) \le$$

$$\le \sum_{j=1}^{n-r} b_{j-1}[\, q_{r-1+j}(n+1) - q_{r-2+j}(n+1)] + q_{r-1}(n+1)$$

for $2 \le r \le n$. Denote by $\gamma_{n-j+1}(n+1) = q_j(n+1) - q_{j-1}(n+1),\ 2 \le j \le n-1$. By definition $\gamma_n(n+1) = q_1(n+1)$.

Substituting instead of the second summands in the right-hand sides of the above-mentioned expressions the upper bounds obtained for them, transposing the last terms (for $r \ge 2$ no less than two) from right to left, we transform our formulas for the failure probability in the form of equalities in the inequalities for them and their differences, respectively

$$\gamma_n(n+1) \le b_{n-1} \sum_{j=1}^{n-1} b_{j-1} \gamma_{n-(j-1)}(n+1)$$

for $r=1$ and

$$\gamma_{n+1-r}(n+1) \le b_{n-r} \sum_{j=1}^{n-r} b_{j-1} \gamma_{n+1-r-(j-1)}(n+1)$$

for $2 \le r \le n$. For this system of inequalities, under the conditions of Lemma 1 the matrix $A = \{a_{ij}\}$ has the form

$$A = \begin{pmatrix} b_{n-1} b_{n-2} b_{n-3} \dots b_1 b_0 \\ b_{n-2} b_{n-3} b_{n-4} \dots b_0 0 \\ \dots\dots\dots\dots\dots\dots\dots \\ b_2 b_1 b_0 \dots\dots\dots 00 \\ b_1 b_0 0 \dots\dots\dots 00 \\ b_0 00 \dots\dots\dots 00 \end{pmatrix}.$$

For the above-mentioned system of inequalities we use Lemma 1, taking $x_i = \gamma_{n+1-i}(n+1), i = 1, 2, ..., n$. From this system of inequalities, according to Lemma 1

$$\gamma_n(n+1) = q_1(n+1) \le \frac{b_{n-1}}{1-\alpha},$$

where

$$\alpha = \max_{1 \le i \le n} \sum_{j=1}^{n-i} \frac{b_{j-1} b_{n-j}}{b_{n-1}}.$$

According to Lemma 2 for all integers $1 \le j \le n$ the inequality $b_{j-1} b_{n-j} \le C_{n-1}^{j-1} b_0 b_{n-1}$ is true and

$$\alpha = \max_{1 \le i \le n} \sum_{j=1}^{n-i} \frac{b_{j-1} b_{n-j}}{b_{n-1}} \le \max_{1 \le i \le n} = b_0 \max_{1 \le i \le n} \sum_{j=1}^{n-i} C_{n-1}^{j-1} =$$

$$= b_0 \max_{1 \le i \le n} \sum_{j=1}^{n-i} C_{n-1}^{j-1} = b_0 \sum_{j=1}^{n-i} C_{n-1}^{j-1} = b_0 (2^{n-1} - 1).$$

Having obtained the upper bound for the quantity $\alpha$, $\alpha \le b_0 (2^{n-1} - 1)$.

In so doing we estimated the probability of failure of the system during the regeneration period of the random process in backup models with recovery

$$q = q_1(n+1) \le \frac{b_{n-1}}{1 - b_0 (2^{n-1} - 1)}.$$

For the probability of failure $q$ the bilateral valuation

$$b_{n-1} \le q = q_1(n+1) \le \frac{b_{n-1}}{1 - b_0 (2^{n-1} - 1)}, \quad n = 1, 2, \dots,$$

is correct.

**Conclusion.**

For simple duplication, when $n = 1$, this estimate gives an accurate value for the probability $q_1(2) = b_0$. In a method similar to the upper bound obtained by our great teacher in [2, pp. 98-100], there is no deductible unit in brackets in the denominator. It should also be noted that with the number of reserve elements $n = 2$ the value in parentheses in the denominator in the new estimate is half that of the corresponding multiplier at the upper bound of this probability in a remarkable work [2]. It is very important in applications.

REFERENCES

1. Gnedenko, B.V. (1969), *Kurs teorii veroyatnostei, 5 izd.* [Course of the theory of probability, Ed. 5], Nauka, Moscow, USSR.
2. Barzilovich, E.Yu., Belyaev, Yu.K., Kashtanov, V.A., Kovalenko, I.N., Soloviev, A.D. and Ushakov, I.A. (1983), *Voprosy matematicheskoy teorii nadyozhnosti* [Questions of the mathematical theory of reliability], Ed by B.V.Gnedenko, Radio i svyaz, Moscow, USSR.

Received 25.06.18

*О.В. Макаричев, О.О. Кудь, О.Б. Щукін*

ОЦІНКА ЙМОВІРНОСТІ ВІДМОВИ СИСТЕМИ
З МАКСИМАЛЬНИМ НАКОПИЧЕННЯМ ОБСЛУГОВУВАННЯ ЕЛЕМЕНТІВ

Розглянуто однолінійну систему масового обслуговування. Вхідний потік заявок є потоком Пуассона, обслуговування заявок у порядку надходження. Час обслуговування складається з двох незалежних невід'ємних випадкових величин. За час, розподілений як перша з цих величин, виникає інший потік Пуассона заявок з незалежними однаково розподіленими довжинами. З них обирається заявка максимальної довжини. Ця максимальна довжина є другим складовим часу обслуговування заявки. Число заявок у системі в залежності від часу створює регенеруючий процес. Моменти відсутності заявок у системі є моментами регенерації. В момент переходу випадкового процесу з стану $n$ в стан $n+1$ виникає відмова $(n=1,2,...)$. Знайдені двосторонні оцінки для ймовірності відмови на періоді регенерації. При цьому верхня та нижня оцінки за звичайного дублювання $(n=1)$ співпадають.

*К л ю ч о в і  с л о в а: ймовірність відмови відновлюваної системи на періоді регенерації, максимальне накопичення обслуговування елементів.*

*А.В. Макаричев, А.А. Кудь, А.Б. Щукин*

ОЦЕНКА ВЕРОЯТНОСТИ ОТКАЗА СИСТЕМЫ
С МАКСИМАЛЬНЫМ НАКОПЛЕНИЕМ ОБСЛУЖИВАНИЯ ЭЛЕМЕНТОВ

Рассмотрена однолинейная система массового обслуживания. Входящий поток требований — пуассоновский, обслуживание требований в порядке поступления. Время обслуживания состоит из двух независимых неотрицательных случайных величин. За время, распределенное как первая из этих величин, возникает другой пуассоновский поток требований с независимыми одинаково распределенными длинами. Из них выбирается требование максимальной длины. Эта максимальная длина является вторым слагаемым времени обслуживания требования. Число требований в системе в зависимости от времени образует регенерирующий процесс. Моменты отсутствия требований в системе являются моментами регенерации. В момент перехода случайного процесса из состояния $n$ в состояние $n+1$ наступает отказ $(n=1,2,...)$. Найдены двусторонние оценки для вероятности отказа на периоде регенерации. При этом верхняя и нижняя оценки при обычном дублировании $(n=1)$ совпадают.

*К л ю ч е в ы е  с л о в а: вероятность отказа восстанавливаемой системы на периоде регенерации, максимальное накопление обслуживания элементов.*

*MAKARICHEV Alexandr Vladimirovich, Doctor of sciences (Phys.-Math.), Professor of the Kharkiv National Automobile and Highway University, graduated from the Moscow State University in 1981. The field of scientific interests: streams of random events with variable intensity, evaluation of reliability characteristics of sets of complex recoverable systems, distributions for sums of extremes in random service processes.*

*KUD Alexandr Alexandrovich, Simcord LLC, graduated from the Kharkiv National Automobile and Highway University in 2002. The field of scientific interests: the construction of summation algorithms for determining random elements of binary matrices.*

*SHCHUKIN Alexandr Borisovich, Simcord LLC, graduated from the N.Ye. Zhukovsky National Aerospace University «Kharkiv Aviation Institute» in 1988. The field of scientific interests: transformation of additive functional from random processes, random flows with varying intensity.*