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O. A. TADYEYEV

Department of Geodesy and Cartography, National University of Water and Environmental Engineering, 11, Soborna str., Rivne, Ukraine, 33028, tel. +38(096)7488449, e-mail: oleksandrtad@gmail.com

EVALUATION OF THREE-DIMENSIONAL DEFORMATION FIELDS OF THE EARTH BY METHODS OF THE PROJECTIVE DIFFERENTIAL GEOMETRY. THE MAIN LINEAR DEFORMATIONS

Aim. The aim is to solve the problem of evaluating the Earth's topographic surface deformations using projective differential geometry methods as an expression of the space metric tensor and the group of main linear deformation parameters in the spatial geocentric coordinate system. **Methodology**. Solving the problem is based on using the homeomorphism transformation (mapping) properties of the three-dimensional continuous and closed domains of the space with the hypothesis that this transformation has a geophysical origin and was caused by the deformation. If the base functions meet homeomorphism requirements, the functional model transformation is capable of transmitting the change of metric properties of the domain by different characteristics that, in the accepted hypotheses, are its deformation parameters. The main carrier of these characteristics is the metric tensor of three-dimensional Euclidean space. A tensor is formed by the metric form of the transformed domain of space as the square of the linear element length, which is expressed by differentials of the transformation domain coordinates and then full differentials of base functions e are taken into account. **Results**. Solving the task is carried out on the condition that the transformation domain of space is outlined by the Earth's topographic surface and coordinated on a three-dimensional rectangular geocentric system. The solution results are working formulas for calculating the main spatial linear deformations, which are expressed by coefficients of elongation, compression, and shear of the topographic surface. Directions of these parameters are defined in the geocentric polar system. Various coefficients of elongation and their directions are expressed in metric tensor components. Formulas are obtained for calculating the parameters in any given direction, along the directions of coordinate axis, on projections to coordinate planes, and for the extreme values triad with the respective spatial orientation. **Scientific novelty and practical significance.** It is grounded that studies of the Earth's deformation fields by methods of the projective differential geometry has greater potential capabilities when compared to methods of linear continuum mechanics and also provides generalized solutions. The homeomorphic functional model as the basis for the formation of the tensor allows the expression of the deformation of any character. Formulas for expressing the main linear deformations are obtained. Results are suitable for evaluation of three-dimensional deformation fields of any scale. Deformation parameters are attributed directly to the topographic surface of the Earth. The sufficient coverage of the Earth by GNSS stations and representational observational data that defines the completeness of functional model construction, together with the obtained results are able to provide the evaluation and interpretation of the real deformations, but not within the traditional model surfaces.

Key words: spatial deformations of the Earth; topographic surface; space mapping; space metric form; space metric tensor; coefficient of linear distortion.

Introduction

Research of the Earths deformation fields is an actual problem of modern geodynamics, which comprehensively solved on the basis of interdisciplinary cooperation a wide range of natural sciences. The purpose and content of research using the geodetic branch of knowledge is defined by resolutions of the International Association of Geodesy (IAG) in framework of the activities of Sub-Commission 3.2 "Crystal Deformation" of Commission 3 "Earth Rotation and Geodynamics". Among other, the objectives of their work is "to study the deformation of the crust all scales from global plate tectonics to local deformation, ... the development and coordination of international programs of the observation, analysis and interpretation of deformation fields" [International Association…].

Applied aspects of researches are the basis of activities for the working group 6.1 "Measurement and analysis of deformation" of Commission 6 "Engineering surveys" of International Federation of Surveyors (FIG) [International Federation …].

As a consequence of close cooperation between the two most important international geodetic organizations in area of researches the problem was the creation of a joint working group IAG/FIG 0.2.1 "New technology to deformations monitoring and responding to natural disasters". In its activities, a working group provides the coordination of interdisciplinary approaches to monitoring of deformation fields of natural and engineering objects, development of methods of processing and analyzing the time series of observations on the theoretical level, innovative data processing algorithms, static and dynamic modeling of deformations and so on. The main source of quantitative information for problem solving is defined the data of observations in networks of permanent GNSS-stations. The theoretical basis of researches already is traditionally used a linear homogeneous model of continuum mechanics. Results of the working group activities as well as scientific and applied aspects of researches by all geodetic community discussed at the standing joint (IAG/ FIG) International Symposium on deformations monitoring, for example [Joint …].

Analysis of the research and unresolved parts of the general problem

Repeated geodetic measurements that are performed on physical surface of the Earth are able to provide the discrete character information only about movements of points or their velocities, while evaluation of surface deformation is a result of any modeling of input survey data. Taking into account the specific of methods of constructing classical geodetic networks, but realizing that the deformation is a continuous spatial phenomenon, the data is traditionally divided into horizontal and vertical components. This division is also substantiated by geophysical point of view: tectonic processes that affect on horizontal movements of surface are different as compared to those that affect on vertical. The division into components also defines the area of application of geodetic data in the modeling of deformation fields of the Earth: planes of two-dimensional or three-dimensional space, geosphere, ellipsoid, topographic (physical) surface of the Earth. By calculating the deformation parameters, they belong to one of these surfaces. Mostly the object of research is the horizontal component of the deformation in projection onto the plane or curvilinear spherical or ellipsoidal surfaces. Based on the theory of an infinitesimal locally homogeneous linear deformation of the continuum, all used methods estimate only the linear approximation of deformation tensor on a plane with a rectangular coordinate system or on a plane that is tangent to the curvilinear surface with a spherical or ellipsoidal parameterization. Modeled on that basis a horizontal approximation is justified at the evaluation of the local scale deformations when errors of projections of physical or curvilinear surfaces onto the plane are small.

A special place in research are occupied a methods of three-dimensional modeling of deformations. Based on the same theory and using measurement results on points of spatial geodetic networks the first task solutions was achieved by three-dimensional finite element method. The method was implemented in a rectangular coordinate system on simplexes of the three-dimensional space – on tetrahedrons [Esikov, 1979; Brunner, 1979], on elements of the quadrangular form [Kiamehr, Sjoberg, 2005], and also by using a least squares method, on finite elements of arbitrary geometric forms [Reilly, 1987; Pietrantonio, Riguzzi, 2004]. Listed references represent only a tiny fraction midst of total this kind of researches. Deformation invariants as the final result of data processing and the basis of phenomenon interpretations refers to barycenters of selected spatial geometric shapes, but not to topographic or any modeled surface. The three-dimensional finite element method has perspective in evaluation of local deformations of the upper crust horizons in conditions of the broken terrain or, hypothetically, provided carrying out of displacements measurements at a certain depth relatively the topographic surface or extrapolation of survey data deep into the Earth. In latest cases the task is devoid of the logical content.

A weighty motive for the rethinking of theoretical foundations of the deformation analysis was the introduction in geodetic practice the modern satellite navigation technologies based on the use of global satellite positioning and implemented in networks of permanent GNSS-stations. The results of spatial coordinate monitoring of stations allowed to increase the effectiveness of the solution of many problems of modern geodynamics. At the same time, their use has given rise the problems associated with the need to create new models of deformation fields and methods of data processing.

The original solution of the problem presented in the article [Savage et al., 2001]. Assuming the sphericity of the Earth, as input data is used coordinates l, j, r (east, north and zenith) in the local spherical system. They obtained by transformations of geocentric spatial coordinates of stations. Components of the three-dimensional deformation tensor are expressed in spherical coordinates based on the mathematical theory of elasticity [Love, 1944] by Taylor series approximation of displacements using the least squares method. A tensor describes only horizontal deformations and refers to the part of a spherical surface, for which is set the local coordinate system. The authors consciously neglected the vertical movements of the Earth, but consider as the indisputable positive of solution the prospect of evaluation the rotation vector of the local surface around conditional pole which fixes the r coordinate. As the third component of a three-dimensional tensor he is associated with a rigid rotation of the Earth around the Euler vector. This method has found practical application in evaluation of local deformation fields as it is presented, for example, in articles [Savage et al., 2004; Hammond, Thatcher, 2004, 2007; Kreemer et al., 2009]. Though on this basis is estimated a three-dimensional tensor, however, he does not express the spatial deformation of the Earth.

The spatial coordinates of GNSS-stations provides a description of the physical surface of the Earth by the curvilinear surface that is embedded into a three-dimensional Euclidean space with a conditional beginning in the geocenter. Such data caused the development of methodological approaches to solving the problem that can provide the evaluation not only horizontal but also vertical components of deformation fields and in the future also spatial deformations of the Earth, that are referred to its topographic surface.

The first substantial researches of problem in this direction is presented in articles, such as [Xu, Grafarend, 1996; Altiner, 1999; Voosoghi, 2000; Grafarend, Voosoghi, 2003]. Researches based on differential surfaces representation in the plates and shells theory within continuum mechanics. Solving the problem considered from two perspectives. The first is an external modeling of deformations. This approach involves the modeling of the topographic curvilinear surface of the Earth into a three-dimensional space with the following evaluation of three-dimensional deformation tensor and related invariants. It could provide an ideal, in terms of cognition of phenomena the interpretation of deformation fields, but because of the difficult differential wording does not have at present due mathematical solutions. The second view – the internal modeling of deformations of the Earth's surface as graded two-dimensional curvilinear surface, which is embedded in the three-dimensional space, with appropriate evaluation of two-dimensional tensors. Authors believe that this approach is able to provide the estimates of spatial deformations which referred to the topographic surface of the Earth separately in the horizontal and vertical components. Argumentations of such decision are as follow. Presentation of the horizontal component provides the method of geometric modeling the metric changes of the surface, based on the expression of two-dimensional Euler-Lagrange deformation tensor of the first kind. Such method is traditionally used in the interpretation of deformation fields. A vertical deformation describes the associated invariants of the rotation tensor and Euler-Lagrange tensor of the second kind, which expresses the changes of the Gaussian curvature along the normal to the surface. Such innovative solving of the problem greatly expands the informative capabilities of geodetic methods of deformation fields monitoring. Implementations of task solutions are constantly improved and in last years actively introduced in the research practice. Some of optimization mathematical solutions presented, for example, in articles [Moghtased-Azar, Grafarend, 2009; Hossainali et al., 2011a, 2011b; Grafarend, 2012].

Author's argumentation concerning referring the tensors and their invariants to the topographic surface of the Earth is questionable. It confirms the content of the term "the internal modeling of deformations of the Earth's surface as graded two-dimensional curvilinear surface", and some examples of practical implementation of the method. So, the graded presentation of the surface causes the need of its division into finite elements. The result of such division is called as grid. Hypothetically the grid on a topographic surface could be implemented, but it is hardly can be achieved into the near future. Beforehand the surface should be parameterized. From a geometrical point of view, a topographical surface is extremely complicated and is not subject to the two-dimensional parameterization by traditional methods even within such model as geoid. In implementing of the method, such as is done in studies [Voosoghi, 2000; Altiner et al., 2006], as the parameterized surface authors have used the ellipsoid. Calculated on ellipsoidal triangles [Voosoghi, 2000] or quadrilateral [Altiner et al., 2006] tensors and their invariants are referred to barycenters of these geometric shapes, but not to the topographic surface of the Earth. Such a shortcoming is caused by the theoretical basis of the problem solving.

Methods of solving the problem on such theoretical basis have another shortcoming – most of solutions are able to express only linear component of deformation fields. It is well known that the spatial-time regularities of Earth deformations having more complicated nature compared to linear. This fact is confirmed by empirical calculations which are presented in article [Tadyeyeva et al., 2012]. Others, formal, confirmation is as follows. If the displacements field is approximated by the arbitrary function and members of the second and higher orders during the decomposition of the corresponding empirical formula in the series will be significant compared to the member of the first (linear) order, thereby will also be confirmed the fact of non-linear spatial distribution law of displacements. However, the ascertained fact mostly by used methods does not count: which would not have been the functional filing of displacement field, the deformation tensor describes only him linear component. In deterministic relation "function-tensor" based on classical linear theory of the continuum deformation does not take part the actual function that expresses the deformation, but only her local linear approximation into the infinitely small scale. When is formed the tensor, actual functions are subject to linearization and only obtained approximation defines the structure of tensor and associated invariants. In this connection, arises the problem of expediency a nonlinear functional filing of displacement fields for the needs of the following deformation analysis on such theoretical basis. Obviously, the use of nonlinear functional models of displacement fields is endowed the logical content only for their spatial interpolation.

Presents an analysis became the motivation for finding the alternative ways and development methods of problem solving which have not been burdened by ascertained shortcoming. In the article [Tadyeyev et al., 2013] we substantiated the expediency of solving a general problem by methods of projective differential geometry. In the article [Tadyeyev, 2013c] presented the ways of its solution on this basis. Based on the theory of surfaces mapping are obtained solutions of the problem of evaluation the horizontal deformations of the Earth's surface assigned to the plane [Tadyeyev, 2013a], geosphere [Tadyeyev, 2013b] and the Earth's ellipsoid of revolution [Tadyeyev, 2015]. If you compare the obtained solutions with similar, which are used in research practice (even listed in Analysis), from the point of view the formulation of the problem the differences between them does not exist. By following the terminology of [Grafarend, Voosoghi, 2003], they are a means of "internal modeling" of horizontal deformations of the earth's surface on according parameterized two-dimensional model surfaces with evaluation belonging to them two-dimensional tensors after simple transformations of geocentric spatial coordinates. But if to take into account that the problem solutions based on the theory of surfaces mapping are not limited only linear functional model, they have obvious advantages.

Aim

Based on the projective differential geometry methods, try to summarize the obtained earlier solutions on a case the evaluation of spatial (three-dimensional) Earths surface deformations. Solutions must be adapted to the direct use of data in the geocentric spatial coordinate system and referred to the topographic surface of the Earth.

Methodology

Problem statement should be considered in the context of "external modeling" [Grafarend, Voosoghi, 2003] spatial deformations of the Earths topographic surface in three-dimensional Euclidean space. Generally, methods of projective-differential geometry make it possible to describe the mapping of any two-dimensional surfaces which are parameterized in one or another coordinate lines system. But the topographic surface is such a kind of them, which is not subject to parameterization by accepted into geodesy methods.

We define the content of some basic provisions and terms of projective-differential (metric) geometry [Kagan, 1947; Norden, 1956; Finikov, 1937], which are directly related to solving the problem. We formulate them compared with relevant conventional wording of the deformation analysis.

The general theory solves the problem into the triply orthogonal system of con-focal surfaces with arbitrary curvature. The geocentric spatial system (x, y, z) in which, as input data, defined the coordinates of GNSS-stations, is a partial case of a orthogonal Cartesian coordinate system in three-dimensional Euclidean space E_3 with the right-hand orientation and the coordinate zero curvature surfaces such as planes xOy, yOz, xOz. From this perspective, problem solving can be considered as trivial relatively to the total.

A mapping (or transformation) of the space is a process where for each point M of space is put in correspondence a certain point M'. The point M' is a mapping (or a projection) of M. A mapping is an unambiguous (or mutually an unambiguous) if for the point M is corresponds the one and only one point M'. The totality of points M_i ($i = \overline{1,n}$) of a certain part or even the whole space, which are subject to unambiguous mapping (or transformation) forms the domain of transformation Δ. The totality of points M_i' that corresponds to points M_i forms the domain of mapping Δ' (or transformed domain). If in the three-dimensional Euclidean space is installed the system of geocentric coordinates (x, y, z) and the domain Δ is closed and continuous, the points $M_i(x_i, y_i, z_i)$ is completely defines (or delineates) the domain Δ. If, due to unambiguous transformation of space the domain Δ mapped on Δ' and latter retained properties of the closed and continuous domain, the points $M_i'(x_i', y_i', z_i')$ completely define the domain Δ'. Under such conditions the mapping of Δ on Δ' can be expressed analytically by equations

$$\left.\begin{aligned} x' &= u(x, y, z) \\ y' &= v(x, y, z) \\ z' &= w(x, y, z) \end{aligned}\right\}. \qquad (1)$$

According to the general theory of mappings, functions u, v, w as base functions of unambiguous mapping should be unambiguous and continuous, respectively, and thus differentiated (with partial derivatives to the second order inclusive). These requirements express the homeomorphism property of mapping: the homeomorphic mapping is mutually unambiguous and mutually continuous. To solve the problem is fundamentally important that the homeomorphism property does not limit the class of base functions (1), and only imposes upon them these requirements. But in terms of the adaptation of such property to determine the functions that implements the mapping and development of the optimal mathematical problem solving arises certain problems.

Considering the discrete structure of geodetic data, the only available means of definition the functions is empirical. It involves the approximation of unknown functions by the known discrete distribution. The task of derive the empirical formulas that meet approximation functions does not have an unambiguous strict solution. This violates the homeomorphism conditions. Therefore, solution of the problem needs to motivate from the standpoint of the correctness of its setting. The choice of analytical form of base functions can be substantiated by the content of the task by a priori information about the character of transformation or formally from a mathematical point of view by the criteria of approximation accuracy. The latest motivation seems as a more promising due to the possibility of evaluation the degree of approximation of the final solution to a strict with regard to homeomorphism conditions. In the processing of geodetic data for solving of this kind of tasks are most often used the least squares method. He is able to provide the determination of empirical formulas and evaluation of their accuracy. However, this kind of problem has place also in traditional problem solving at the stage of construction a functional models of the Earths displacement fields.

The homeomorphism property allows describe the transformation domain by metric forms. As linear elements of measure, they describe not only the projection (mapping), but also the internal geometry of space transformation which is caused by a change of metric properties. Solving the problem can be achieved with the hypothesis that the change of metric properties of space has a geophysical origin and caused by its deformation. Then the geometric parameters of homeomorphic mapping of Δ on Δ' that describe such change and transfer the distortion of the projection, this is, in fact, the deformation characteristics of the domain Δ. Formulated hypothesis is a fundamental from the perspective of the use of mappings theory for evaluation of three-dimensional deformation fields of the Earth.

A metric form of transformation domain Δ describes a linear element ds which by differentials of coordinates expresses the formula

$$ds^2 = dx^2 + dy^2 + dz^2 . \qquad (2)$$

Fig. 1 shows the element ds in the system of right orientation trihedron. The direction of the element ds defines the polar geocentric coordinates

$$l = arctg\frac{dy}{dx},$$

$$j = arctg\left(\frac{dz}{dy}\sin l\right) = arctg\left(\frac{dz}{dx}\cos l\right).$$

In projections ds_{yz} and ds_{xz} on coordinate planes yOz and xOz (l lies in the plane xOy)

$$j_{yz} = arctg\frac{dz}{dy}, \quad j_{xz} = arctg\frac{dz}{dx}.$$

For mapping ds' of the linear element ds that meets to the transformed domain Δ', we have the following formula:

$$ds'^2 = dx'^2 + dy'^2 + dz'^2 . \qquad (3)$$

Let us assume empirical formulas that meet to the base functions (1) for whatever considerations are established. Their full differentials

$$dx' = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz ,$$

$$dy' = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz ,$$

$$dz' = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

allow to express the quadratic form (3) by differentials of the domain Δ coordinates:

$$ds'^2 = e_{xx}dx^2 + e_{yy}dy^2 + e_{zz}dz^2 + $$
$$+2e_{xy}dxdy + 2e_{yz}dydz + 2e_{xz}dxdz . \qquad (4)$$

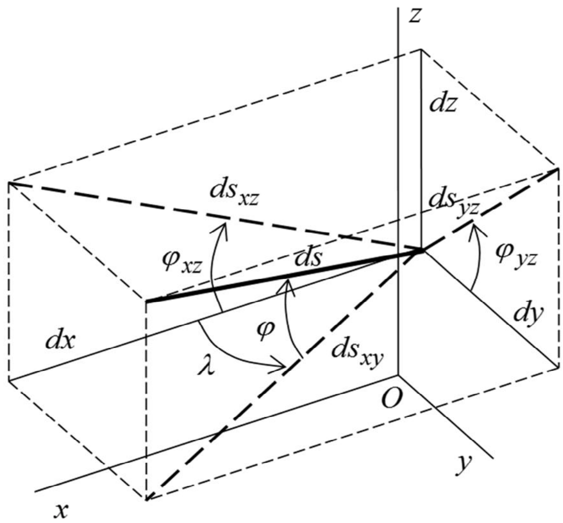

***Fig. 1.** The linear element ds in the right orientation trihedron*

Coefficients of the metric form (4) reveals as follows:

$$e_{xx} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2;$$

$$e_{yy} = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2;$$

$$e_{zz} = \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2;$$

$$e_{xy} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y};$$

$$e_{yz} = \frac{\partial u}{\partial y}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial z};$$

$$e_{xz} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z}.$$

They form a symmetrical matrix

$$\begin{pmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{pmatrix}. \qquad (5)$$

Quadratic form (4) is a main metric form and the matrix (5) – the main metric tensor of space transformations. A tensor (5) fully defined by functions that express the concrete realization of mapping (1): tensor components are the partial derivatives of base functions of the mapping.

According to the hypothesis that changing metric properties of space is caused by the deformation, tensor (5) should be recognized as the deformation tensor of space. As the main carrier of information about such a change, the tensor able to transmit her signs by various numeral characteristics of different geometric content – by parameters of the space deformation in the generally accepted interpretation of the deformation analysis. A tensor (5), equally as the deformation tensors in continuum mechanics, has the differential origin. Therefore, should expect the identity of content and quantify of deformation parameters based on both fundamental theories: the space mapping theory and the continuum deformation theory.

Results

Try to express the linear distortions of mapping (1) – linear deformations of the space domain Δ when it is transformed into the appropriate space domain Δ'. Such a sign of the space deformation expresses by a coefficient of elongation in a specified direction. It is also called as the mapping scale or elongation module. For this we use the metric forms, that correspond to Δ and Δ', and the deformation tensor (5).

A distortion in any given direction (l, j) expresses by the coefficient $m = \frac{ds'}{ds}$. Considering the quadratic forms (2) and (4) and relations $dx = ds \cos j \cos l$, $dy = ds \cos j \sin l$ and $dz = ds \sin j$ we obtain the following:

$$m^2 = e_{xx} \cos^2 j \cos^2 l + e_{yy} \cos^2 j \sin^2 l + e_{zz} \sin^2 j + e_{xy} \cos^2 j \sin 2l + e_{yz} \sin 2j \sin l + e_{xz} \sin 2j \cos l . \quad (6)$$

Thus, space domain elongation in a direction (l, j) fully defined by the tensor (5).

If you set values of (l, j) that correspond to directions of the coordinate system axes then from formula (6) are follows:

$$m_x^2 = e_{xx} ; \quad (7)$$

$$m_y^2 = e_{yy} ; \quad (8)$$

$$m_z^2 = e_{zz} . \quad (9)$$

Thus, space domain elongations in directions of the coordinate axes alone defined by diagonal components of the deformation tensor.

Particular importances for the deformation analysis have parameters of extreme (principal) elongations. Corresponding to them coefficients m_{ext} and principal directions (l_0, j_0) and $(l_0 + 90^0, j_0 + 90^0)$ provides the solution of the equations system

$$\left. \begin{array}{l} \dfrac{d\left(m^2\right)}{dl} = 0 \\ \dfrac{d\left(m^2\right)}{dj} = 0 \end{array} \right\} .$$

Such derivations of the formula (6) provides a system of nonlinear equations

$$\left. \begin{array}{l} 2tg j_0 \left(e_{yz} \cos l_0 - e_{xz} \sin l_0\right) + \\ \quad + \left(e_{yy} - e_{xx}\right) \sin 2l_0 + 2e_{xy} \cos 2l_0 = 0 \\ tg 2j_0 \left(e_{zz} - e_{xx} \cos^2 l_0 - e_{yy} \sin^2 l_0 - \right. \\ \left. -e_{xy} \sin 2l_0\right) + 2\left(e_{yz} \sin l_0 + e_{xz} \cos l_0\right) = 0 \end{array} \right\} . \quad 10)$$

Solving of the system defines the principal directions and corresponding to them parameters of extreme elongations.

Solving of equations (10) relatively to the coordinate planes of the system (x, y, z) provides the following results.

On the coordinate plane xOy the value $j = const = 0^0$ and $tg j = 0$. Then from the first equation of the system (10) for the principal direction l_0 we obtain the formula

$$tg 2l_0 = \frac{2e_{xy}}{e_{xx} - e_{yy}} . \quad (11)$$

The maximum elongation of the space domain in this direction expresses the distortion coefficient

$$m_{xy\max}^2 = \frac{1}{2}\left(e_{xx} + e_{yy} + \sqrt{\left(e_{xx} - e_{yy}\right)^2 + 4e_{xy}^2}\right). \quad (12)$$

The formula (12) is obtained from (6) based on the results of simple transformations of the formula (11). The minimum elongation (compression) expresses the distortion coefficient in the direction $l_0 + 90^0$:

$$m_{xy\min}^2 = \frac{1}{2}\left(e_{xx} + e_{yy} - \sqrt{\left(e_{xx} - e_{yy}\right)^2 + 4e_{xy}^2}\right). \quad (13)$$

The formula (13) is obtained on the same basis as the formula (12). Directions of extreme elongations are shown in Fig. 2a.

Extreme elongations of the space domain in principal directions on coordinate planes yOz and xOz can be obtained from the solution of the second equation of (10):

$$tg 2j_0 = \frac{2e_{yz} \sin l_0 + 2e_{xz} \cos l_0}{e_{xx} \cos^2 l_0 + e_{yy} \sin^2 l_0 + e_{xy} \sin 2l_0 - e_{zz}} . \quad (14)$$

On the coordinate plane yOz at a value $l = const = 90^0$ from the equation (14) follows:

$$tg 2j_{yz0} = \frac{2e_{yz}}{e_{yy} - e_{zz}} . \quad (15)$$

In the direction j_{yz0} (Fig. 2b) the maximum elongation expresses the coefficient

$$m_{yz\max}^2 = \frac{1}{2}\left(e_{yy} + e_{zz} + \sqrt{\left(e_{yy} - e_{zz}\right)^2 + 4e_{yz}^2}\right). \quad (16)$$

In the direction $j_{yz\mathbf{0}} + 90^{\mathbf{0}}$

$$m^2_{yz\min} = \frac{1}{2}\left(e_{yy} + e_{zz} - \sqrt{\left(e_{yy} - e_{zz}\right)^2 + 4e^2_{yz}} \right). \quad (17)$$

On the coordinate plane xOz at a value $l = const = 0^{\mathbf{0}}$ the solution

$$tg\,2j_{xz\mathbf{0}} = \frac{2e_{xz}}{e_{xx} - e_{zz}} \quad (18)$$

of the equation (14) defines two principal directions $j_{xz\mathbf{0}}$ and $j_{xz\mathbf{0}} + 90^{\mathbf{0}}$ (see. Fig. 2c) with extreme elongations, which are expressed by coefficients

$$m^2_{xz\max} = \frac{1}{2}\left(e_{xx} + e_{zz} + \sqrt{\left(e_{xx} - e_{zz}\right)^2 + 4e^2_{xz}} \right), \quad (19)$$

$$m^2_{xz\min} = \frac{1}{2}\left(e_{xx} + e_{zz} - \sqrt{\left(e_{xx} - e_{zz}\right)^2 + 4e^2_{xz}} \right). \quad (20)$$

Formulas for calculating of domain elongations which belong to coordinate planes are identical to similar ones, which was obtained on the same basis for the evaluation of horizontal deformations. Latest is presented in article [Tadyeyev, 2013a]. Assuming that base functions of mapping are the linear, the obtained formulas expresses elongations, as is customary for the traditional approach to the deformation analysis. Such deformations then referred to as shear (or sliding). If to follow such tradition, obtained with respect to the coordinate planes results should be interpreted in this way.

On the plane xOy

$$m^2_{xy\max} = r_{xy} + \frac{g_{xy_m}}{2}; \quad m^2_{xy\min} = r_{xy} - \frac{g_{xy_m}}{2}.$$

$$r_{xy} = \frac{e_{xx} + e_{yy}}{2}; \quad g_{xy_m} = \sqrt{g^2_{xy1} + g^2_{xy2}}; \quad (21)$$

$$g_{xy1} = e_{xx} - e_{yy}; \quad g_{xy2} = 2e_{xy}.$$

The values $m^2_{xy\max}$, $m^2_{xy\min}$ and r_{xy} are called the maximum, minimum and average elongations respectively. The value g_{xy_m} is the maximum shear in the plane xOy; and g_{xy1}, g_{xy2} – its components. These values in relation to coordinate planes yOz and xOz have the following expression:

$$m^2_{yz\max} = r_{yz} + \frac{g_{yz_m}}{2}; \quad m^2_{yz\min} = r_{yz} - \frac{g_{yz_m}}{2};$$

$$r_{yz} = \frac{e_{yy} + e_{zz}}{2}; \quad g_{yz_m} = \sqrt{g^2_{yz1} + g^2_{yz2}}; \quad (22)$$

$$g_{yz1} = e_{yy} - e_{zz}; \quad g_{yz2} = 2e_{yz};$$

$$m^2_{xz\max} = r_{xz} + \frac{g_{xz_m}}{2}; \quad m^2_{xz\min} = r_{xz} - \frac{g_{xz_m}}{2};$$

$$r_{xz} = \frac{e_{xx} + e_{zz}}{2}; \quad g_{xz_m} = \sqrt{g^2_{xz1} + g^2_{xz2}}; \quad (23)$$

$$g_{xz1} = e_{xx} - e_{zz}; \quad g_{xz2} = 2e_{xz}.$$

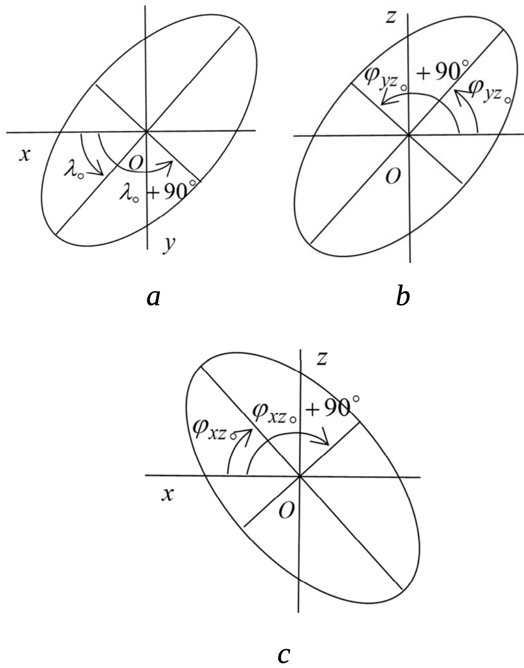

***Fig. 2.** Directions of extreme elongations in coordinate planes of the (x, y, z) system*

The spatial orientation of extreme elongations sets the triad of orthogonal directions ($l_{\mathbf{0}} + 90^{\mathbf{0}}, j_{\mathbf{0}}, j_{\mathbf{0}} + 90^{\mathbf{0}}$).

The direction $l_{\mathbf{0}} + 90^{\mathbf{0}}$ and the corresponding elongation define formulas (11) and (13). By leaving the main directions on the plane xOy as fixed, now discloses other unknowns. The direction $j_{\mathbf{0}}$ is disclosed by the formula (14). After her transformation, this direction can also be expressed directly by coefficients of the tensor (5) and shear components in coordinate planes, for example:

$$tg2j_{\mathbf{0}} = 2\frac{2e_{yz}\sqrt{\dfrac{g_{xym} - g_{xy1}}{2g_{xym}}} + 2e_{xz}\sqrt{\dfrac{g_{xym} + g_{xy1}}{2g_{xym}}}}{e_{xx} + e_{yy} - 2e_{zz} + g_{xym}}. \quad (24)$$

For the direction $j_{\mathbf{0}}$ corresponds the maximum elongation of the space domain, which expresses the coefficient $m_{\max}$ and for the orthogonal to it direction $j_{\mathbf{0}} + 90^{\mathbf{0}}$ – the minimal elongation with a corresponding coefficient $m_{\min}$:

$$m^2_{\max} = \frac{1}{4}\left(e_{xx} + e_{yy} + 2e_{zz} + g_{xym} + \sqrt{\left(e_{xx} + e_{yy} - 2e_{zz} + g_{xym}\right)^2 + 4\left(2e_{yz}\sqrt{\frac{g_{xym} - g_{xy1}}{2g_{xym}}} + 2e_{xz}\sqrt{\frac{g_{xym} + g_{xy1}}{2g_{xym}}} \right)^2} \right); \quad (25)$$

$$m^2_{\min} = \frac{1}{4}\left(e_{xx} + e_{yy} + 2e_{zz} + g_{xym} - \sqrt{\left(e_{xx} + e_{yy} - 2e_{zz} + g_{xym}\right)^2 + 4\left(2e_{yz}\sqrt{\frac{g_{xym} - g_{xy1}}{2g_{xym}}} + 2e_{xz}\sqrt{\frac{g_{xym} + g_{xy1}}{2g_{xym}}} \right)^2} \right). \quad (26)$$

Considering designations (21)–(23), formulas (24)–(26) can be presented in any other convenient forms, including a more compact, and on their basis the appropriate parameters properly interpret.

Thus, to the triad of directions $(l_0 + 90^0, j_0, j_0 + 90^0)$ corresponds coefficients $\left(m^2_{xy\min}, m^2_{\max}, m^2_{\min}\right)$. Taken together, latest are parameters of extreme linear deformations of the domain Δ at her transformation into the domain Δ'.

Scientific novelty and practical significance

There is grounded that studies of Earths deformation fields by methods of the projective differential geometry has greater potential capabilities as compared to methods of linear continuum mechanics and provides a generalized solutions. The homeomorphic functional model as the basis for the formation of the tensor allows to expressing the deformation of any character. Formulas for expressing the main linear deformations are obtained. Results are suitable for evaluation of three-dimensional deformation fields of any scale. Deformation parameters are attributed directly to the topographic surface of the Earth. The sufficient coverage of the Earth by GNSS stations and representational observational data that defines the completeness of functional model constructing, together with the obtained results are able to provide the evaluation and interpretation the real deformations, but not within the traditional model surfaces.

Conclusions and prospects of further researches

At this stage of the research is grounded the expediency of the use the projective differential geometry methods for modeling a deformations of the Earth's topographic surface in three-dimensional geocentric coordinate system. Statement of the problem is formulated. Analytical expressions for linear deformations in an arbitrary set direction, along the directions of the coordinate axes, in projections on coordinate planes and also for the triad of extreme values with the corresponding spatial orientation are obtained. Formation of the deformation tensor that defines these parameters is not burdened by linearization of the displacements functional model. The obtained results make it possible to evaluate the real, not modeled linear deformations of a topographic surface, far as their can be submit by the homeomorphic functional model.

The topographic surface as a continuous closed domain of space which is the object of researches by methods of the projective differential geometry is not limited by any conditions concerning its size and geometrical forms. In practical terms this means the following. First, there is no need to divide the surface on finite elements. Second, as a consequence, there are no limits to the scale of deformation fields: on this theoretical basis is equally possible to investigate the geodynamic processes of global, regional or local scales. Probably, for the effective interpretation of three-dimensional deformation fields of regional and local scales will be expedient to transform the input data from the geocentric coordinate system in a conditional topocentric system.

Various parameters of the space domain linear deformations are united by a similar geometric content and origin – they are a consequence of relations the linear elements of the space domain into pre- and after its deformation. For this reason, organize them into one group with an appropriate name - the main linear deformations. By following the established traditions of the deformation analysis together with a group of linear deformations for the interpretation of fields are used another two groups of parameters - parameters of relative changes in the volume or area (dilatation) and angular distortion parameters. According to the latest express the "rotation of the domain as an absolutely rigid body". The expression of deformation parameters that are attributed to these two groups by using the projective differential geometry methods, are the subject of further researches.

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ТАДЄЄВ О. А.

Кафедра геодезії та картографії, Національний університет водного господарства та природокористування, вул. Соборна, 11, Рівне, Україна, 33028, тел. +38(096)7488449, ел. пошта: oleksandrtad@gmail.com

ОЦІНЮВАННЯ ТРИВИМІРНИХ ДЕФОРМАЦІЙНИХ ПОЛІВ ЗЕМЛІ МЕТОДАМИ ПРОЕКТИВНО-ДИФЕРЕНЦІАЛЬНОЇ ГЕОМЕТРІЇ. ГОЛОВНІ ЛІНІЙНІ ДЕФОРМАЦІЇ

Мета. Оцінювання деформацій топографічної поверхні Землі методами проективно-диференціальної геометрії спрямоване на вираження метричного тензора простору і групи параметрів головних лінійних деформацій у геоцентричній просторовій системі координат. **Методика.** Виконання завдання ґрунтується на використанні властивостей гомеоморфізму перетворення (відображення) тривимірної замкненої неперервної області простору за гіпотези, що це перетворення має геофізичне походження і спричинене деформацією. За умови відповідності базових функцій вимогам гомеоморфізму, функціональна модель перетворення здатна передавати різними характеристиками зміну метричних властивостей області, які, за прийнятої гіпотези, є параметрами її деформації. Основним їхнім носієм є метричний тензор тривимірного евклідового простору. Тензор формується метричною формою перетвореної області простору – квадратом довжини лінійного елемента, вираженого за диференціалами координат області перетворення з урахуванням повних диференціалів базових функцій. **Результати.** Виконання завдання здійснено за умови, що область перетворення простору окреслена топографічною поверхнею Землі і координована в геоцентричній тривимірній прямокутній системі. Результатом виконання є робочі формули для обчислення головних просторових лінійних деформацій – коефіцієнтів розширення, стиснення та зсуву топографічної поверхні. Напрями цих показників визначено в геоцентричній полярній системі. Різні коефіцієнти розширень та їхні напрями виражені в компонентах метричного тензора. Одержано формули для обчислення параметрів у довільному заданому напрямі, вздовж напрямів координатних осей, у проекціях на координатні площини, а також для тріади їхніх екстремальних значень з відповідною просторовою орієнтацією. **Наукова новизна і практична значущість.** Обґрунтовано, що під час досліджень деформаційних полів Землі методи проективно-диференціальної геометрії мають більші потенційні можливості порівняно з методами лінійної механіки суцільного середовища і забезпечують узагальнені розв'язки. Гомеоморфна функціональна модель як основа формування тензора дає змогу виражати будь-які деформації. Одержано розрахункові формули для вираження головних лінійних деформацій. Результати придатні для оцінювання тривимірних деформаційних полів будь-яких масштабів. Параметри деформації зараховують безпосередньо до топографічної поверхні Землі. Достатнє покриття Землі GNSS-станціями і репрезентативні дані спостережень, що визначає повноту побудови функціональної моделі, разом з одержаними результатами здатні оцінити та інтерпретувати реальні деформації, а не ті, що належать до традиційних модельних референцних поверхонь.

Ключові слова: просторові деформації Землі; топографічна поверхня; відображення простору; метрична форма простору; метричний тензор простору; коефіцієнт лінійного спотворення

ТАДЕЕВ А. А.

Кафедра геодезии и картографии, Национальный университет водного хозяйства и природопользования, ул. Соборная, 11, Ровно, Украина, 33028, эл. почта: oleksandrtad@gmail.com

ОЦЕНИВАНИЕ ТРЕХМЕРНЫХ ДЕФОРМАЦИОННЫХ ПОЛЕЙ ЗЕМЛИ МЕТОДАМИ ПРОЕКТИВНО-ДИФФЕРЕНЦИАЛЬНОЙ ГЕОМЕТРИИ. ГЛАВНЫЕ ЛИНЕЙНЫЕ ДЕФОРМАЦИИ

Цель. Решение задачи оценивания деформаций топографической поверхности Земли методами проективно-дифференциальной геометрии направленное на выражение метрического тензора пространства и группы параметров главных линейных деформаций в геоцентрической пространственной системе координат. **Методика**. Решение задачи основывается на использовании свойств гомеоморфизма преобразования (отображения) трехмерной замкнутой непрерывной области пространства при гипотезе, что это преобразование имеет геофизическое происхождение и обусловлено деформацией. При условии соответствия базовых функций требованиям гомеоморфизма, функциональная модель преобразования способна передавать различными характеристиками изменения метрических свойств области, которые, в соответствии с гипотезой, являются параметрами ее деформации. Основным их носителем является метрический тензор трехмерного евклидова пространства. Тензор формируется метрической формой преобразованной области пространства – квадратом длины линейного элемента, выраженного дифференциалами координат области преобразования с учетом полных дифференциалов базовых функций. **Результаты**. Решение задачи осуществлено при условии, что область преобразования пространства очерчена топографической поверхностью Земли и координирована в геоцентрической трехмерной прямоугольной системе. Результатом решения есть рабочие формулы для вычисления главных пространственных линейных деформаций – коэффициентов расширения, сжатия и сдвига топографической поверхности. Направления этих показателей определены в геоцентрической полярной системе. Различные коэффициенты расширения и их направления выражены в компонентах метрического тензора. Получены формулы для вычисления параметров в произвольном заданном направлении, вдоль направлений координатных осей, в проекциях на координатные плоскости, а также для триады их экстремальных значений с соответствующей пространственной ориентацией. **Научная новизна и практическая значимость**. Обосновано, что при исследовании деформационных полей Земли методы проективно-дифференциальной геометрии имеют большие потенциальные возможности в сравнении с методами линейной механики сплошной среды и обеспечивают обобщенные решения. Гомеоморфная функциональная модель, как основа формирования тензора, позволяет выражать деформации не только линейного характера. Получены расчетные формулы для выражения главных линейных деформаций. Результаты пригодны для оценивания деформационных полей всех масштабов. Параметры деформации отнесены непосредственно к топографической поверхности Земли. Достаточное покрытие Земли GNSS-станциями и репрезентативные данные наблюдений, определяющие полноту построения функциональной модели, вместе с полученными результатами способны обеспечить оценки и интерпретацию реальных деформаций, а не отнесенных к традиционным модельным референцным поверхностям.

Ключевые слова: пространственные деформации Земли; топографическая поверхность; отображение пространства; метрическая форма пространства; метрический тензор пространства; коэффициент линейного искажения

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