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M. M. FYS[1], A. M. BRYDUN[2], M. I. YURKIV[3]

[1,2,3] Department of Cartography and geospatial modeling of Lviv Polytechnic National University, 12, S. Bandery Str., Lviv, 79013, Ukraine, e-mail:
[1]Mykhailo.M.Fys@lpnu.ua, [2]Andrii.M.Brydun@lpnu.ua, [3]Mariana.I.Yurkiv@lpnu.ua

RESEARCHING THE INFLUENCE OF THE MASS DISTRIBUTION INHOMOGENEITY OF THE ELLIPSOIDAL PLANET'S INTERIOR ON ITS STOKES CONSTANTS

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Purpose. Parameters of Earth's gravitational field ($C_{n,k}$, $S_{n,k}$) are determinated by its figure and internal filling (mass distribution) that have a different influence on their formation. Using a well-known representation of the planet masses distribution functions in the biorthogonal series form it is necessary to establish the Stokes constants $C_{n,k}$, $S_{n,k}$ presentation through the planet potential expansion coefficients b_{mnk} and liner combinations of ellipsoid geometric parameters. Based on these formulas, it is the objective to investigate the possible influence of the inhomogeneity of the mass distribution function of the Earth's interior and the representation of its shape with an ellipsoid of rotation onto the values of the Stokes constants and to explore the contribution of the radial distribution of the Earth's mass density to these constants. **Methodology**. The presentation of the planet's interior density function as a sum of the Legendre polynomials of three variables and the approximation of its surface by an ellipsoid, as well as the representation of internal spherical functions in a rectangular coordinate system, makes it possible to integrate expressions for Stokes constant $C_{n,k}$, $S_{n,k}$ and obtain the relation between these values of different orders and the linear combination of the planet potential expansion coefficients b_{pqs} and geometric parameters of ellipsoid α,β,γ . Numerical data obtained from the derived relationships and the constructed graphs make it possible to analyze the influence of the inhomogeneity of the mass's interior distribution of an ellipsoidal planet onto the value of the Stokes constants and determine the intervals of maximum impact. **Results.** The general relations between the expansion coefficients b_{mnk} of the distribution function and the integrals from spherical functions on an ellipsoidal surface that determine Stokes constants of a definite order are established. Herewith Stokes constants of *n* order are expressed in terms of values $C_{n,k}$, $S_{n,k}$. of lower orders. The presented calculations give a procedure for the formation of Stokes constant values, which clearly implies the conclusion about the small effect of the planet's ellipsoidal form on the magnitude and three-dimensionality of the Earth's gravitational field as a result of the inhomogeneous of its interior masses distribution. Also known dependence of the values $C_{2m,0}$ on the geometric compression of the biaxial Earth ellipsoid of constant density is confirmed. **Scientific novelty**. The formulas for the relation between Stokes constants of different orders and linear combinations of parameters α,β,γ are determined. The calculations and verification of the obtained relations for different sets of potential expansion coefficients b_{pqs} allow us to conclude that the three-dimensional gravity field of the Earth predominantly contributes to the Stokes constants, except $C_{2,0}$, and the constructed graphs determine its maximum contribution to the mass distribution in depth. **Practical significance.** The obtained dependences allow us to check the approximation degree of the constructed density model of ellipsoidal planet by comparing Stokes constants which are calculated using model and are obtained from the observations. In addition, it is possible to optimally reconcile the geometric characteristics of the planet's ellipsoid with its gravitational field.

Key words: planet potential, masses distribution model, Stokes constant, ellipsoid, spherical function.

Introduction

Characteristics of external gravitational field with seismology data are important elements to study the Earth's internal structure especially in researching mass distribution function of the Earth. Since difference from zero of Stokes constants is an indicator of planet inhomogeneity and its three-dimensionality, the establishment area of possible formation values $C_{n,k}$, $S_{n,k}$ is an important element in this study. It should be noted that such problem is not considered for the first time. For example, in papers (Tarakanov,1979; Vinnik, 1978) proposed to place abnormal masses at a depth of 600-800 km with interpretation of Stokes' constants of 2–6th order, and for constants of 2–4th order to shift the center of occurrence down to 1000 km. Similar studies are proposed in (Ostach & Ageeva, 1982) and associated with choosing the placement of point masses for the best approximation of the potential. The theoretical aspects of this problem are discussed in detail in the monography (Antonov, Timoshkova & Kholshevnikov 1988). The continuation of these studies was carried out in (Kholshevnikov & Shaidulin, 2015), and a partial case is considered in (Kholshevnikov, Milanov & Shaidulin, 2017). Further detailing of the integrand function (specifically representing a sum of the Legendre polynomials of three variables) will allow us to represent formulas for Stokes constants as a linear combination of coefficients b_{pqs} and the geometric parameters of the ellipsoid, and to investigate their features.

Purpose

Using well-known a representation of the planet masses distribution functions in the biorthogonal series form it is necessary to establish the Stokes constants $C_{n,k}$, $S_{n,k}$ presentation through the planet potential expansion coefficients b_{mnk}. Based on these formulas, the objective is to investigate the possible influence of the inhomogeneity of the mass distribution function of the Earth's interior and the representation of its shape with an ellipsoid of rotation onto the values of Stokes constants and to explore the contribution of the radial distribution of the Earth's mass density to these constants.

Methodology

Stokes constants of the planet σ which figure is limited by the surface Ω, are completely determined by the integral formula

$$C_{n,k}+iS_{n,k}=\frac{1}{Ma_e^n}\int_\sigma \delta\left(U_{nk}+iV_{nk}\right)d\tau, n,k=0,1,2,... \quad (1)$$

where M, a_e − mass and equatorial radius of the planet respectively; δ − mass distribution function of the planet's interior; U_{nk}, V_{nk} – internal spherical functions and

$$U_{nk}+iV_{nk}=\sum_{p+q+s=n}\left(\alpha_{pqs}^n+i\beta_{pqs}^n\right)x_1^p x_2^q x_3^s\ . \quad (2)$$

Analysis of the planet's gravitational field parameters (1) indicate the inhomogeneity of the masses distribution and the deviation from spherical shape. If the body surface is homogeneous in the three axial ellipsoid then the Stokes constants is $C_{2n2k}\neq 0$ (in the case two axial ellipsoid just $C_{2n0}\neq 0$). This fact is one of the conditions of the hydrostatic state of the planet.

Further the Earth's figure is taken as an ellipsoid $\tau:\left\{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\le 1\right\}$ with semi axis a_1,a_2,a_3. We complement the piecewise continuous function of mass distribution $\delta(x_1,x_2,x_3)$ as

$$\delta^*(P)=\begin{cases}\delta(P), & P(x_1,x_2,x_3)\in\sigma,\\ 0, & P\in\tau/\sigma.\end{cases} \quad (3)$$

In this interpretation, all the integral characteristics associated with the masses distribution of the planet's interior stay unchanged, that is,

$$\int_\sigma \delta\ (P)f(P)d\sigma=\int_\tau \delta^*(P)f(P)d\tau.$$

In this regard, we consider the task: to analyze the influence of a three-dimensional structure of the planet's interior mass distribution and its figure approximation by the ellipsoid to the values of Stokes constant (1).

Under the assumption (3), the piecewise continuity of the investigated function allows it to be presented as a series decomposition (Meshcheriakov, 1991)

$$\delta=\sum_{N=m+n+k=0}^{\infty} b_{mnk}W_{mnk}+\delta^0(\rho), \quad (4)$$

where $\delta^0(\rho)$ – one of the generally accepted radial (spherical) models of density, W_{mnk} – generalized Legendre polynomials of three variables (Bateman, 1953; Meshcheriakov, 1991), b_{mnk} − expansion coefficients, $m+n+k=N$ and

$$W_{nmk}=$$

$$=\frac{1}{a_1^m a_2^n a_3^k 2^N m!n!k!}\frac{\partial^N}{\partial x_1^m\partial x_2^n\partial x_3^k}\left(\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}-1\right)^N, \quad (5)$$

$$b_{mnk}=\frac{\int_\tau W_{mnk}\delta d\tau}{\int_\tau W_{mnk}\omega_{mnk}d\tau}. \quad (6)$$

For further research, it is convenient to use the representation of the internal spherical functions in a rectangular coordinate system (Fys, Zazuliak & Zajats', 2004) using complex variables

$$U_{nk}+iV_{nk}=$$

$$=\frac{RR(n-k)!}{Ma_e^n 2^k}\sum_{m=0}^{\left[\frac{n-k}{2}\right]}\frac{x_3^{n-k-2m}(-1)^m(x_1^2+x_2^2)^m}{(n-k-2m)!(k+m)!m!}(x_1+ix_2)^k=$$

$$= \sum_{p+q+s=n} (\alpha_{pqs}^{n} + i\beta_{pqs}^{n}) x_1^p x_2^q x_3^s \, . \qquad (7)$$

Substituting (4), (5) and (7) in (1) gives

$$C_{n,k} + iS_{n,k} = C_{n,k}^{pr} + iS_{n,k}^{pr} + \sum_{t=0}^{n} \left(C_{n,k}^{t} + iS_{n,k}^{t} \right), \quad (8)$$

where $C_{n,k}, S_{n,k}$ – given Stokes constant; $C_{n,k}^{pr}, S_{n,k}^{pr}, C_{n,k}^{t}, S_{n,k}^{t}$ – Stokes constant, calculated using the PREM model and the coefficients b_{mnk} of t order and

$$C_{n,k}^{t} + iS_{n,k}^{t} = \frac{RR(n-k)!}{2^k Ma_e^n} \sum_{p+q=s=t} \int_{\tau} \frac{b_{pqs}}{2^t \, p!q!s!} \frac{\partial^t \left(\rho^2 - 1\right)^t}{a_1^m a_2^n a_3^k \partial x_1^p \partial x_2^q \partial x_3^s} \left(\sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^m x_3^{n-k-2m} (x_1^2 + x_2^2)^m}{\left(n-k-2m\right)! m! (k+m)!} (x_1 + ix_2)^k \right) d\tau \, , \quad (9)$$

$$C_{n,k}^{pr} + iS_{n,k}^{pr} = \frac{RR(n-k)!}{2^k Ma_e^n} \int_{\tau} \delta^0(\rho) \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \int_{\tau} \frac{(-1)^m x_3^{n-k-2m} (x_1^2 + x_2^2)^m}{\left(n-k-2m\right)! m! (k+m)!} (x_1 + ix_2)^k \, d\tau . \qquad (10)$$

Transition to a generalized spherical coordinate system using equality

$$\begin{cases} x_1 = a_1 \rho \sin\vartheta \cos\lambda, \\ x_2 = a_2 \rho \sin\vartheta \sin\lambda, \\ x_3 = a_3 \rho \cos\vartheta, \end{cases} \qquad (11)$$

and integrating give us the next formula where $\alpha = \frac{a_1}{a_e}, \beta = \frac{a_2}{a_e}, \gamma = \frac{a_3}{a_e}$:

$$C_{n,k}^{pr} + iS_{n,k}^{pr} = \frac{3RR(n-k)!k!\delta_c}{2^k (n+1)!!} \int_0^1 \delta^0(\rho) \rho^{n+2} d\rho \frac{1}{(n+1)!} \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^k (n-k-2m-1)!!k!}{2^{2m} (n-k-2m)!(k+m)!} \sum_{l=0}^{n-k-2m} \alpha^{2m+k-l} \beta^l \times$$

$$\times \sum_{i+j=l} \frac{(-1)^{\frac{j}{2}} (2m+k-l-1)!!(l-1)!!}{(m-i)!i!(k-j)!j!} i^{j-2k} \qquad (12)$$

In particular, for constant density δ_0 we obtain

$$C_{n,k}^{pr} + iS_{n,k}^{pr} = \frac{3RR(n-k)!k!\delta_0}{2^k (n+3)!!\delta_c} \sum_{\text{ь}=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^m \gamma^{n-k-2m}}{2^{2m} (n-k-2m)!!(k+m)!} \sum_{l=0}^{2m+k} \times$$

$$\times \left(\sum_{i+j=l} \frac{(-1)^{\frac{j}{2}}}{i!(m-i)!(k-j)!j!} i^{j-2\left[\frac{j}{2}\right]} \right) \frac{\alpha^{2m+k-l} \beta^l (2m+k-l-1)^k}{(l-1)!!} , \qquad (13)$$

and for spheroid ($\alpha_1 = a_2$), respectively

$$C_{n0}^{p} = \frac{3\delta_0}{\delta_C} \frac{n!}{(n+3)!!} \sum_{m=0}^{\frac{m}{2}} \frac{(-1)^m \gamma^{n-2m} (2m)!!}{(n-2m)!!(m)!2^{2m}} \alpha^{2m} =$$

$$= \frac{3\delta_0}{\delta_C} \frac{\left(\frac{n}{2}\right)!(n-1)!!2^{\frac{n}{2}}}{(n+3)!!} \sum_{m=0}^{\frac{m}{2}} \frac{(-1)^m \gamma^{n-2m} \left(\alpha^2\right)^m 2^m}{\left(\frac{n}{2} - m\right)!(m)!2^{3m} 2^{\frac{n}{2}-m}}, \quad (14)$$

or, finally,

$$C_{n0}^{p} = \frac{3\delta_0}{\delta_C} \frac{\left(\gamma^2 - 1\right)^{\frac{n}{2}}}{\left(n+1\right)\left(n+3\right)}, \qquad (15)$$

where δ_C – average density.

This expression is a well-known relation between Stokes constants and the geometric compression for a homogeneous biaxial ellipsoidal planet (Cunningham,1970), and equality (13) determines the contribution of the radial masses distribution and is one of the conditions of the planet hydrostatic state. Stokes constants values, except $C_{2,0}$, do not correlate with the expression (15), that means there is a deviation of the Earth's state from a hydrostatically equilibrium state.

We continue to study contribution of heterogeneity of the masses distribution function and the planet's figure approximation by ellipsoid to the value of Stokes constants. Firstly, we note that three-dimensionality is determined by the presence of series elements (4) in the expression (9), and indices t and n affect Stokes constants formation. First of all, all values in (9) are converted to zero if $t > n$ (t-order derivative of a polynomial of lower degree). Therefore, in the sum (8), the decomposition of function δ in series (4) is limited to the coefficients b_{pqs} up to the n degree. For the case $n = t$, the term (9) is as follows

$$C_{nk}^{n} + iS_{nk}^{n} =$$

$$= 3V_e \sum_{p+q+s=n} \frac{b_{pqs}\left(\alpha_{pqs}^{n} + i\beta_{pqs}^{n}\right)}{2^n} \int_{\tau}^{1} \left(\rho^2 - 1\right)^n \rho^2 d\rho =$$

$$= \frac{3}{\delta_C} \frac{n!}{(2n+3)!!} \sum_{p+q+s=n} \left(\alpha_{pqs}^{n} + i\beta_{pqs}^{n}\right) b_{pqs} \quad (16)$$

or

$$C_{nk}^{n} + iS_{nk}^{n} =$$

$$= \sum_{p+q+s=n} \frac{\left(\mu_{nk}^{n} + i\nu_{nk}^{n}\right)}{\delta_C} \int_0^1 \rho^{n-t+2} \left(\left(\rho^2 - 1\right)^t\right) d\rho, \quad (17)$$

where

$$\mu_{nk}^{n} + i\nu_{nk}^{n} = \frac{3}{\delta_C} \frac{n!}{(2n+3)!!} \sum_{p+q+s=n} \left(\alpha_{pqs}^{k} + i\beta_{pqs}^{k}\right) b_{pqs}.$$

Thus, the coefficients for the values of b_{pqs} in the Stokes constants of n order are the same as in the combinations of variables x_1, x_2, x_3 in the corresponding internal spherical functions U_{nk}, V_{nk}, which further allows us to determine their linear combinations through given Stokes constants.

Let's analyze the structure of the terms in expression (9) when $t < n$. For this we submit (9) as follows:

$$C_{n,k}^{t} + iS_{n,k}^{t} = \frac{RR(n-k)!}{2^k M a_e^n} \sum_{t=p+q+s} \frac{b_{pqs}}{2^t p!q!s!} \left(\int_{\tau} \frac{\partial^t}{\partial x_1^p \partial x_1^q \partial x_1^s} \left(\rho^2 - 1\right)^t \times \right.$$

$$\left. \times \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^m x_3^{n-k-2m} m!k!}{2^{2m} m!(n-k-2m)!(m+k)!} \sum_{l=0}^{2m+k} \sum_{2r+j=l} \frac{(-1)^{\frac{j}{2}} (i)^{j-2\left[\frac{j}{2}\right]}}{(m-r)!r!(k-j)!j!} x_1^{2m+k-l} x_2^{l} d\tau \right) =$$

$$= \frac{k!\int_0^1 (\rho^2 - 1)^t \rho^{n-t+2} d\rho}{(n-t+1)!!} \sum_{p+q+s=t} \frac{2^{-t} b_{pqs}}{p!q!s!} \sum_{m=0}^{\left[\frac{n-k}{2}\right]} \frac{(-1)^m \gamma^{n-k-s-2m} (2m+k)!}{2^{2m} (m+k)!(n-k-2m-s)!!} \sum_{l=0}^{2m+k} \times$$

$$\times \sum_{2r+j=l} \frac{(-1)^{\frac{j}{2}} (i)^{j-2\left[\frac{j}{2}\right]} l!(2m+k-l)! \alpha^{2m+k-l} \beta^l}{(m-r)!r!(k-j)!j!(2m+k-l-p)!!(l-q)!!}. \quad (18)$$

We can assume that equality (18) is a set of expressions (9) of corresponding orders and quantities α, β, γ, which are surface integrals over an ellipsoid from internal spherical functions. The verification of this hypothesis is based on concrete examples in the paper (Fys, 1982). Given this, the expression (18) can be written as follows:

$$C_{nk}^{t} + iS_{nk}^{t} =$$

$$= \frac{t!}{(t+n+3)!!\delta_C} \sum_{l,s} \left(C_{tl}^{t} + iS_{tl}^{t}\right) \int_{\tau} u_{n-t,s} d\tau, \; t \le n \quad (19)$$

and, taking into account (8) and (16), we get

$$\sum_{p+q+s=n} \left(\alpha_{pqs}^{n} + i\beta_{pqs}^{n}\right) b_{pqs} =$$

$$= \frac{\delta_C (2n+3)!!}{2^n n! 3} \left(C_{nk} + iS_{nk} - \frac{1}{\delta_C} \sum_{t=0}^{n-1} \left(C_{nk}^{t} + iS_{nk}^{t}\right) \right). \quad (20)$$

The right-hand side of (20) is expressed in terms of the given Stokes constants of n order. The value of the sum is calculated using combinations of coefficients b_{pqs}, which are calculated using Stokes constants lower orders. Therefore, in the final result, the right-hand side (20) is the sum of Stokes constants to n order inclusive, and is therefore similarly constant with the right-hand side (8).

Equality (20) can be represented as follows

$$C_{n,k} + iS_{n,k} =$$

$$= \frac{1}{\delta_C} \sum_{t=0}^{n} \int_0^1 \left[\sum_{l,s} \left(\mu_{tl}^{t} + i\nu_{tl}^{t}\right) \int_{\Omega} u_{n-t,s} d\Omega \right] \left(\rho^2 - 1\right)^t \rho^{n-t+2} d\rho \quad (21)$$

Expression (19) is a linear combination of coefficients b_{pqs} and parameters α, β, γ. This is explained by the fact that the derivatives of spherical functions are the sum again of spherical functions of lower orders. It is extremely difficult to establish the general form of such dependence, since it requires laborious and complex transformations, but this is not necessary for solving our problem, since due to the ambiguity of the potential representation, it is sufficient to establish only some of the coefficients that ensure the equalities (linear combinations in (16)).

First of all, in all relations (19), except for the case of $k = n$, there is a term $C_{n-2,k}^{n-2}\int_{\Omega} u_{2,0} d\Omega$ (or after transformations $\left(\gamma^2 - 1\right) C_{n-2,k}^{n-2}$), which means there is inconsistency between the fall in the values of the Stokes constants of these orders and the power law. The values $C_{n,k}, S_{n,k}$ are obtained mainly due to the inhomogeneity of the mass distribution, which is quite expected, because for the spherical planet in (19) the sum is absent, since everything $\int_{\tau} u_{nk} d\tau = 0$ or $\int_{\Omega} u_{nk} d\Omega = 0$, except for the case $n = k = 0$.

Thus, a short algorithm for the implementation of the above methodology is as follows:

1. We determine the coefficients b_{000}, b_{100}, b_{010}, b_{001} of zero and first order by the given Stokes constants.

2. We calculate, using second-order Stokes constants and formulas (16), one of the variants of the coefficients $b_{pqs}(p + q + s = 2)$.

3. Using the already known second order coefficients b_{pqs}, we calculate the third order coefficients according to (20).

4. The iterative process is continued with the established order N.

5. At each step, using the already calculated values b_{pqs}, we return to the definition of the coefficients of the polynomials μ_{nk}^t, ν_{nk}^t by which we further construct graphs of integrand functions (21), and also calculate the sequences $\sum_{t=0}^{\lambda+4} C_{n,k}^t$, $\sum_{t=0}^{\lambda+2} S_{n,k}^t$ that determine the Stokes constants $C_{n,k}$, $S_{n,k}$ by the formula (20).

Results

According to this algorithm, we performed calculations using one of the models of the Earth's gravitational field (EGM2008) and in Table 1 we gave the values C_{nk}^{λ}, S_{nk}^{λ} obtained by formula (20), where $\lambda = \begin{cases} 0, & (n-k) - \text{even}, \\ 1 & (n-k) - \text{odd}. \end{cases}$ We constructed graphs of the dependence of the mass distribution contribution (21) along the radius (depth) in Stokes constants (see Fig. 1.2) and analyzed the results.

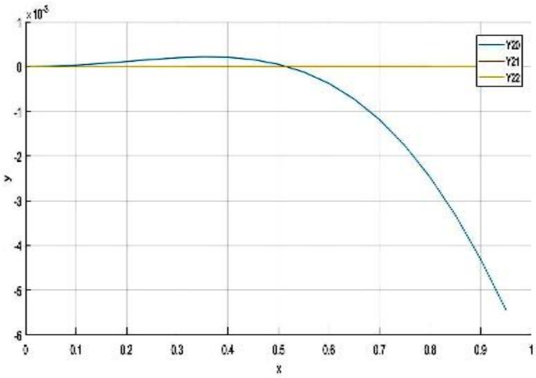
Table 1

The values of given Stokes constants (model EGM2008) and calculated values for different orders

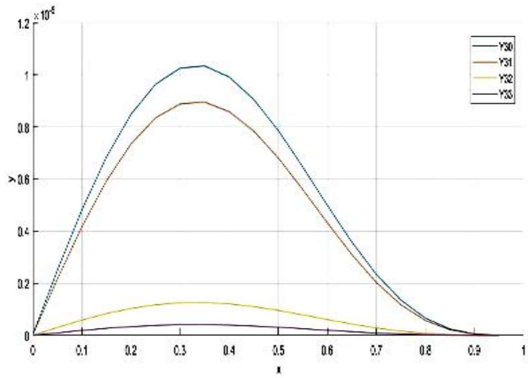
n	k	$C_{n,k}$	$\sum_{t=0}^{\lambda} C_{n,k}^t$	$\sum_{t=0}^{\lambda+2} C_{n,k}^t$	$\sum_{t=0}^{\lambda+4} C_{n,k}^t$	$\sum_{t=0}^{\lambda+6} C_{n,k}^t$	$\sum_{t=0}^{\lambda+8} C_{n,k}^t$	$S_{n,k}$	$\sum_{t=0}^{\lambda+2} S_{n,k}^t$	$\sum_{t=0}^{\lambda+4} S_{n,k}^t$	$\sum_{t=0}^{\lambda+6} S_{n,k}^t$	$\sum_{t=0}^{\lambda+8} S_{n,k}^t$
1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	1.0E+00	1.0E+00	0.0E+00	–	–	–	–	–	–	–	–
1	0	0.0E+00	0.0E+00	0.0E+00	–	–	–	0.0E+00	–	–	–	–
1	1	0.0E+00	0.0E+00	0.0E+00	–	–	–	0.0E+00	–	–	–	–
2	0	-1.1E-03	-1.3E-03	-1.1E-03	–	–	–	0.0E+00	–	–	–	–
2	1	-2.8E-10	0.0E+00	-2.8E-10	–	–	–	1.9E-09	1.9E-09	–	–	–
2	2	1.6E-06	0.0E+00	1.6E-06	–	–	–	-9.0E-07	-9.0E-07	–	–	–
3	0	2.5E-06	0.0E+00	0.0E+00	2.5E-06	–	–	0.0E+00	0.0E+00	–	–	–
3	1	2.2E-06	0.0E+00	0.0E+00	2.2E-06	–	–	2.7E-07	0.0E+00	2.7E-07	–	–
3	2	3.1E-07	0.0E+00	0.0E+00	3.1E-07	–	–	-2.1E-07	0.0E+00	-2.1E-07	–	–
3	3	1.0E-07	0.0E+00	0.0E+00	1.0E-07	–	–	2.0E-07	0.0E+00	2.0E-07	–	–
4	0	1.6E-06	3.8E-06	2.7E-06	1.6E-06	–	–	0.0E+00	0.0E+00	–	–	–
4	1	-5.1E-07	0.0E+00	6.3E-13	-5.1E-07	–	–	-4.5E-07	-4.2E-12	-4.5E-07	–	–

Continuation of Table 1

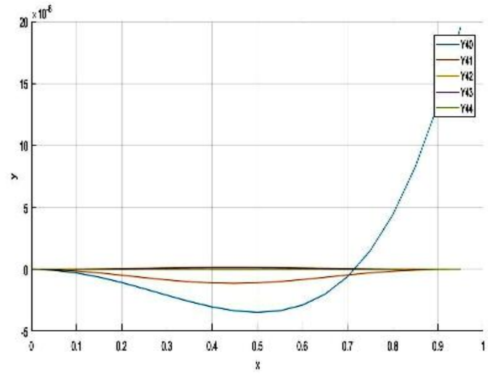
1	2	3	4	5	6	7	8	9	10	11	12	13
4	2	7.8E-08	0.0E+00	-1.2E-09	7.8E-08	–	–	1.5E-07	6.7E-10	1.5E-07	–	–
4	3	5.9E-08	0.0E+00	0.0E+00	5.9E-08	–	–	-1.2E-08	0.0E+00	-1.2E-08	–	–
4	4	-4.0E-09	-2.2E-15	-2.2E-15	-4.0E-09	–	–	6.5E-09	0.0E+00	6.5E-09	–	–
5	0	2.3E-07	0.0E+00	0.0E+00	-1.5E-08	2.3E-07	–	0.0E+00	0.0E+00	0.0E+00	0.0E+00	–
5	1	-5.4E-08	0.0E+00	0.0E+00	-8.0E-09	-5.4E-08	–	-8.1E-08	0.0E+00	-9.8E-10	-8.1E-08	–
5	2	1.1E-07	0.0E+00	0.0E+00	-5.6E-10	1.1E-07	–	-5.2E-08	0.0E+00	3.9E-10	-5.2E-08	–
5	3	-1.5E-08	0.0E+00	0.0E+00	-6.1E-11	-1.5E-08	–	-7.1E-09	0.0E+00	-1.2E-10	-7.1E-09	–
5	4	-2.3E-09	0.0E+00	0.0E+00	8.5E-21	-2.3E-09	–	3.9E-10	0.0E+00	0.0E+00	3.9E-10	–
5	5	4.3E-10	0.0E+00	0.0E+00	-1.4E-21	4.3E-10	–	-1.6E-09	0.0E+00	8.2E-21	-1.6E-09	–
6	0	-5.4E-07	-1.4E-08	-9.1E-09	-7.4E-10	-5.4E-07	–	0.0E+00	0.0E+00	0.0E+00	0.0E+00	–
6	1	-6.0E-08	0.0E+00	-1.9E-15	2.6E-09	-6.0E-08	–	2.1E-08	1.3E-14	2.3E-09	2.1E-08	–
6	2	6.1E-09	0.0E+00	2.1E-12	-2.4E-10	6.1E-09	–	-4.7E-08	-1.2E-12	-4.6E-10	-4.7E-08	–
6	3	1.2E-09	0.0E+00	-8.7E-25	-9.1E-11	1.2E-09	–	1.9E-10	-5.8E-24	1.9E-11	1.9E-10	–
6	4	-2.3E-11	-3.6E-16	-3.6E-16	2.1E-12	-2.3E-11	–	-1.8E-09	0.0E+00	-3.4E-12	-1.8E-09	–
6	5	-2.2E-10	0.0E+00	-1.1E-25	-6.9E-22	-2.2E-10	–	-4.3E-10	-3.6E-25	-4.2E-22	-4.3E-10	–
6	6	2.2E-12	3.0E-17	3.0E-17	3.0E-17	2.2E-12	–	-5.5E-11	-1.7E-22	-4.0E-22	-5.5E-11	–
7	0	3.5E-07	0.0E+00	0.0E+00	8.3E-11	-2.2E-09	3.5E-07	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00
7	1	2.1E-07	0.0E+00	0.0E+00	3.1E-11	3.4E-10	2.1E-07	7.0E-08	0.0E+00	3.8E-12	5.4E-10	7.0E-08
7	2	3.3E-08	0.0E+00	0.0E+00	1.5E-12	-4.7E-10	3.3E-08	9.3E-09	0.0E+00	-9.9E-13	2.3E-10	9.3E-09
7	3	3.5E-09	0.0E+00	0.0E+00	9.5E-14	4.0E-11	3.5E-09	-3.1E-09	0.0E+00	1.9E-13	1.9E-11	-3.1E-09
7	4	-5.8E-10	0.0E+00	0.0E+00	-1.5E-20	3.1E-12	-5.8E-10	-2.6E-10	0.0E+00	0.0E+00	-5.2E-13	-2.6E-10
7	5	5.9E-13	0.0E+00	0.0E+00	4.6E-22	-1.9E-13	5.9E-13	6.3E-12	0.0E+00	-3.1E-22	7.4E-13	6.3E-12
7	6	-2.5E-11	0.0E+00	0.0E+00	1.0E-22	1.0E-22	-2.5E-11	1.1E-11	0.0E+00	-2.8E-23	-4.0E-23	1.1E-11
7	7	2.8E-14	0.0E+00	0.0E+00	9.3E-23	8.0E-23	2.8E-14	4.5E-13	0.0E+00	2.4E-22	2.4E-22	4.5E-13
8	0	2.0E-07	6.3E-11	3.8E-11	-1.4E-11	5.9E-09	2.0E-07	0.0E+00	0.0E+00	0.0E+00	0.0E+00	0.0E+00
8	1	1.6E-08	0.0E+00	6.9E-18	-1.2E-11	5.0E-10	1.6E-08	4.0E-08	-4.6E-17	-1.1E-11	-1.6E-10	4.0E-08
8	2	6.6E-09	1.7E-15	-3.8E-15	8.2E-13	-3.6E-11	6.6E-09	5.4E-09	3.2E-15	1.5E-12	2.7E-10	5.4E-09
8	3	-2.0E-10	0.0E+00	2.5E-24	2.0E-13	-4.8E-12	-2.0E-10	-8.7E-10	1.8E-23	-4.1E-14	-7.0E-13	-8.7E-10
8	4	-3.2E-10	-1.4E-16	-1.4E-16	-2.9E-15	5.6E-14	-3.2E-10	9.1E-11	3.2E-22	4.5E-15	4.2E-12	9.1E-11
8	5	-4.7E-12	0.0E+00	0.0E+00	-1.4E-21	2.5E-13	-4.7E-12	1.6E-11	-2.8E-26	1.9E-22	5.1E-13	1.6E-11
8	6	-1.8E-12	1.0E-18	1.0E-18	1.0E-18	-8.7E-16	-1.8E-12	8.6E-12	-3.3E-23	-5.7E-23	2.2E-14	8.6E-12
8	7	3.4E-13	0.0E+00	-6.5E-29	4.0E-23	4.6E-23	3.4E-13	3.8E-13	1.5E-26	-1.0E-23	-1.0E-23	3.8E-13
8	8	-1.6E-13	-3.6E-19	-3.6E-19	-3.6E-19	-3.6E-19	-1.6E-13	1.5E-13	0.0E+00	1.6E-24	5.0E-24	1.5E-13



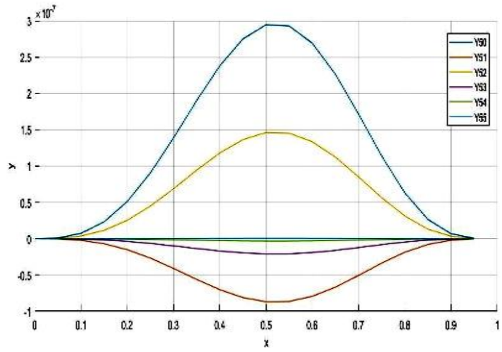
a) C_{20}–C_{22}



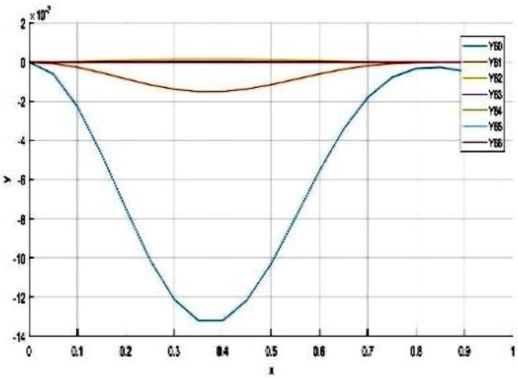
b) C_{30}–C_{33}



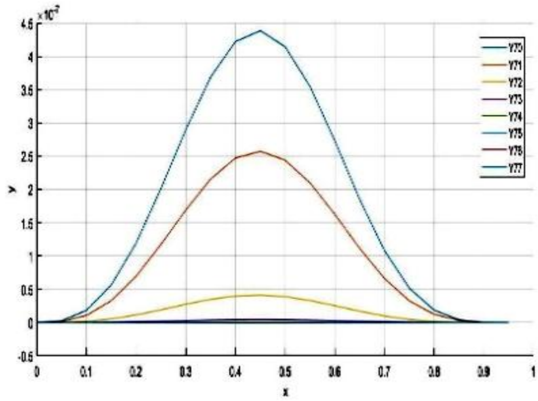
c) C_{40}–C_{44}



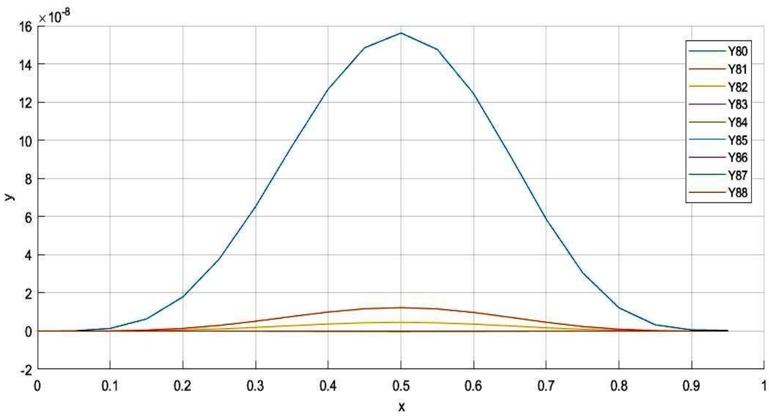
d) C_{50}–C_{55}



e) C_{60}–C_{66}

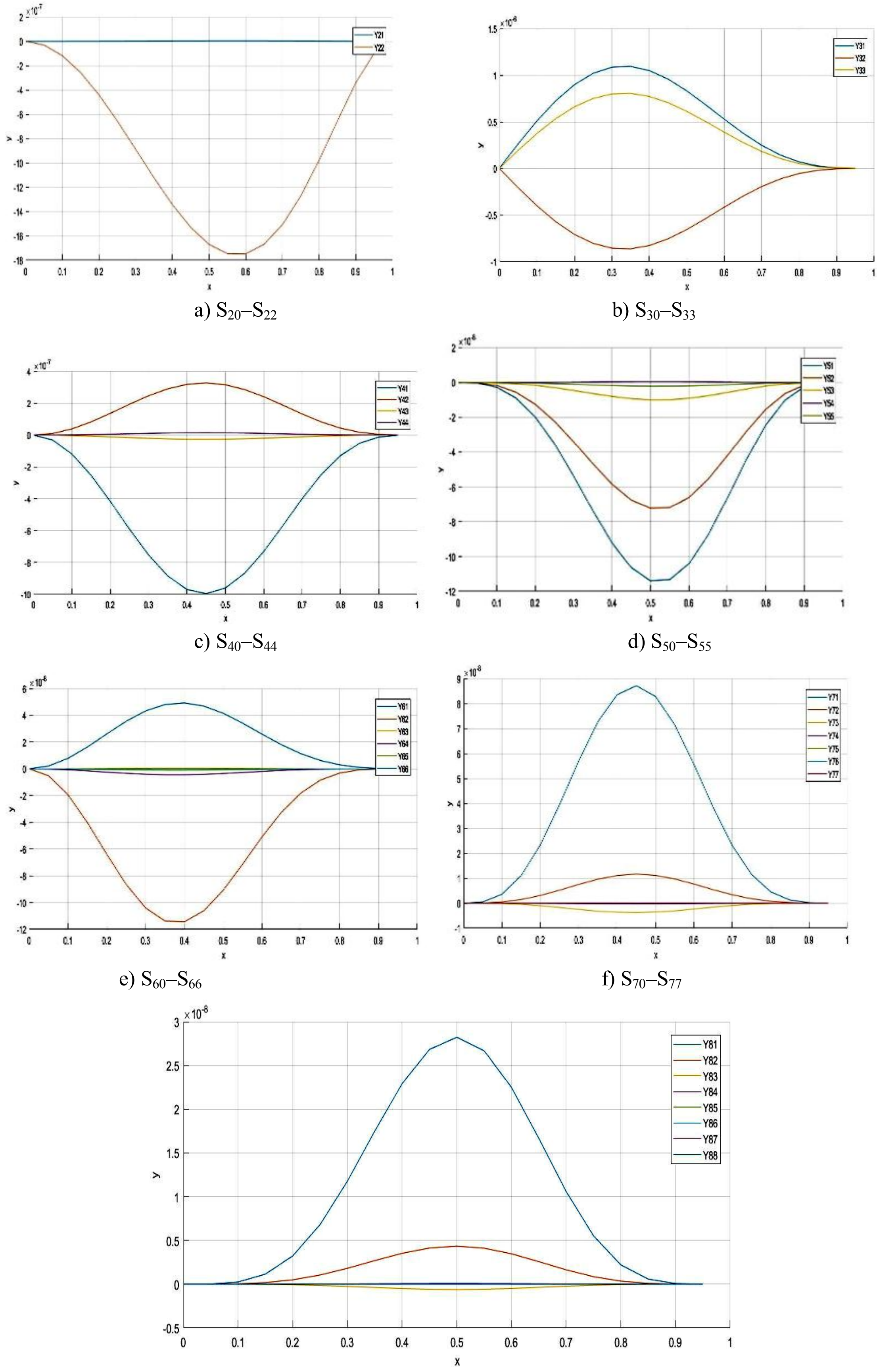


f) C_{70}–C_{77}



g) C_{80}–C_{88}

Fig. 1. Graphs of the dependence of the contribution of mass distribution along a radius in the Stokes constants C_{nk}



a) S_{20}–S_{22} b) S_{30}–S_{33}

c) S_{40}–S_{44} d) S_{50}–S_{55}

e) S_{60}–S_{66} f) S_{70}–S_{77}

g) S_{80}–S_{88}

Fig. 2. Graphs of the dependence of the contribution of mass distribution along a radius in the Stokes constants S_{nk}

Given that the influence of the ellipsoidal shape of the planet is manifested in the values $\int_{\tau} u_{nk} d\tau$, which are multipliers in the sum (20), and the results of calculations with Table 1, one can argue the insignificant influence of ellipsoidalness on the formation of the Stokes constant values. Indeed, the total contribution effect of the computed constants $C_{n,k}^{\lambda}$, $S_{n,k}^{\lambda}$ $(\lambda < n)$ is small and tangible except for values $C_{2,0}, C_{4,0}$. Therefore, we can conclude that the values of the Stokes constants are formed mainly due to the anisotropy of the planet mass distribution, described by the elements of the sum (19) when $t=n$,

and deviations of the figure from the spherical shape don't significantly affect the formation of the values of the Earth's gravitational field parameters.

Treating in relation (21) the integrand function as the average over the unit sphere value

$$\frac{1}{S_{\Omega}}\int_{\Omega} \delta(\vartheta,\lambda,\rho)\,d\Omega = \frac{1}{4\pi\rho^2}\int_0^{\pi}\int_0^{2\pi} \delta(\vartheta,\lambda,\rho)\sin\vartheta\, d\vartheta\, d\lambda,$$

it is possible to construct its graphs (Fig. 1, 2), giving a general idea of the total density contribution along the radius.

From these figures, it can be seen that the maximum effect for the reduced range of Stokes constants is reached mainly for relative radii $0.3 \le \rho \le 0.6$ when the sign of the corresponding Stokes constant is stored. Again, for the value of $C_{2,0}$ its formation is realized at $\rho > 0.5$ in the mantle of the Earth. The value $C_{4,0}$ gets its true value when $\rho > 0.7$, that is, in the upper mantle. Obviously, the results obtained for the above method are connected, first of all, with the extremums of the function $\left(\rho^2 - 1\right)^n \rho^2$ at points $\rho = \frac{1}{\sqrt{2n+1}}$.

Thus, the conducted study requires further developments, primarily in the direction of establishing general relations between the quantities that are determined through the fixed Stokes constants, and the geometric characteristics, represented as integrals of spherical functions. However, the obtained results allow us to draw some conclusions.

Conclusions

1. The general relations between the expansion coefficients b_{mnk} of the masses distribution of the planet's interior and the integrals of spherical functions on the ellipsoidal surface, which determine the Stokes constants of a given order, are obtained.
2. The formation of the parameters of the planet's external gravitational field is mainly influenced by deviation from the radial distribution of the interior planet's masses.
3. The value of lower-order Stokes constants is included in the cumulative effect of the contribution to the values of upper-order Stokes constants.
4. The small contribution of the ellipsoidal form to lower-order Stokes constants values is due to multipliers $\int_{\tau} u_{nk} d\tau$, which are zero for a sphere, while for a biaxial ellipsoid, they are proportional to $\left(\gamma^2 - 1\right)^{\frac{n}{2}}$ $(n = 2m, k = 0)$.
5. The deviation of the decrease of the Stokes constants $C_{n,k}$, $S_{n,k}$ from the potential law can be partially explained by the presence of the terms $\left(\gamma^2 - 1\right)^{\frac{n}{2}} C_{n-2,k-2}, \left(\gamma^2 - 1\right)^{\frac{n}{2}} S_{n-2,k-2}$ for $n,k > 2$.
6. The construction of the dependence of the contribution of the radial mass distribution of the planet's interior to the values of Stokes constants on depth reveals an ambiguous interpretation. It can only be stated that the values of the constants are mainly formed within the relative radius $0.3 \le \rho \le 0.6$.
7. For a more complete study, it is necessary to derive formulas for the dependence of the Stokes constants among themselves and on the geometric characteristics of the planet, in particular, on the semi-axes of the Earth's ellipsoid.
8. Taking into account the geometry (semiaxes) and the parameters of the gravitational field of the planet simultaneously can give a more precise agreement between them when imposing additional conditions, for example, the minimum deviation between the calculated and the given potential of the general Earth ellipsoid.

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М. М. ФИС[1], А. М. БРИДУН[2], М. І. ЮРКІВ[3]

[1,2,3] Кафедра картографії та геопросторового моделювання, Національний університет "Львівська політехніка", вул. С. Бандери, 12, Львів, 79013, Україна, ел. пошта
[1]Mykhailo.M.Fys@lpnu.ua, [2]Andrii.M.Brydun@lpnu.ua, [3]Mariana.I.Yurkiv@lpnu.ua

ДОСЛІДЖЕННЯ ВПЛИВУ НЕОДНОРІДНОСТІ РОЗПОДІЛУ МАС НАДР ПЛАНЕТИ ЕЛІПСОЇДАЛЬНОЇ ФОРМИ НА ЇЇ СТОКСОВІ ПОСТІЙНІ

Мета. Параметри гравітаційного поля Землі ($C_{n,k}$, $S_{n,k}$) визначаються її фігурою та внутрішнім наповненням (розподілом мас), які по-різному впливають на їх формування. Подаючи функцію розподілу мас надр планети у вигляді біортогональних рядів, встановимо зображення стоксових постійних $C_{n,k}$, $S_{n,k}$ через коефіцієнти b_{mnk} розкладу потенціалу планети та лінійні комбінації геометричних характеристик еліпсоїда. На основі отриманих формул вивчити можливий вплив неоднорідності функції розподілу мас надр Землі та подання її фігури еліпсоїдом обертання на значення величин стоксових постійних та дослідити вклад радіального розподілу густини мас Землі у значення цих постійних. **Методика.** Подання функції густини надр планети у вигляді суми многочленів Лежандра трьох змінних і апроксимація її поверхні еліпсоїдом, а також представлення внутрішніх кульових функцій у прямокутній системі координат, роблять можливим інтегрування виразів для стоксових постійних $C_{n,k}$, $S_{n,k}$ та отримання співвідношення між цими величинами різних порядків і лінійною комбінацією коефіцієнтів розкладу b_{pqs} потенціалу планети й геометричних параметрів еліпсоїда α,β,γ. Числові дані, отримані за виведеними співвідношеннями, і побудовані графіки дають можливість провести аналіз впливу неоднорідності розподілу мас надр планети еліпсоїдальної форми на значення стоксових постійних та визначити інтервали максимального впливу. **Результати.** Отримано загальні співвідношення між коефіцієнтами розкладу b_{mnk} функції розподілу та інтегралами від кульових функцій по еліпсоїдальній поверхні, які визначають стоксові постійні заданого порядку. При цьому стоксові постійні n -го порядку виражаються через величини $C_{n,k}$, $S_{n,k}$ нижчих порядків. Проведені обчислення дають загальну картину формування значень стоксових постійних, з якої чітко випливає висновок про невеликий вплив еліпсоїдальної форми планети на їх величину та про тривимірність гравітаційного поля Землі як результату неоднорідного за всіма координатами розподілу мас її надр. Підтверджена залежність значень величини $C_{2m,0}$ від геометричного стиснення двохосьового земного еліпсоїда постійної густини. **Наукова новизна.** Визначені формули зв'язку між стоксовими постійними різних порядків та лінійними комбінаціями параметрів еліпсоїда α, β, γ. Проведені обчислення та перевірка отриманих співвідношень для різних наборів коефіцієнтів b_{pqs} розкладу потенціалу дають можливість зробити висновок про переважний вклад тривимірності гравітаційного поля Землі в значення стоксових постійних, за винятком $C_{2,0}$, а побудовані графіки визначають інтервали її максимального вкладу в розподіл мас за глибиною. **Практична значущість.** Отримані залежності дозволяють перевіряти степінь наближення побудованої моделі густини еліпсоїдальної планети шляхом порівняння обчислених за нею та взятих зі спостережень стоксових постійних. Крім цього, з'являється можливість оптимального узгодження геометричних характеристик еліпсоїда планети з її гравітаційним полем.

Ключові слова: потенціал планети, модель розподілу мас, стоксові постійні, еліпсоїд.

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