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THE METHODOLOGY OF APPROXIMATE CONSTRUCTION OF THE THREE -DIMENSIONAL MASS DISTRIBUTION FUNCTION AND ITS GRADIENT FOR THE ELLIPSOIDAL PLANET SUBSIDIES

Purpose. To create an algorithm for constructing a three-dimensional masses distribution function of the planet and its derivatives taking into account the Stokes constants of arbitrary orders. Being based on this method, the task is to perform the research on the internal structure of the Earth. **Methodology**. The derivatives of the inhomogeneous mass distribution are presented by linear combinations of biorthogonal polynomials which coefficients are obtained from the system of equations. These equations follow from integral transformations of the Stokes constants, the calculation process is carried out by a sequential approximation and for the initial approximation we take a one-dimensional density model that is consistent with Stokes constants up to the second inclusive order. Next, the coefficients of expansion of the potential of higher orders are determined up to a predetermined order. In this case, the information on the power moments of the density of surface integrals makes it possible to analyze and control the iterative process. **Results.** The results of calculations using the software according to the described algorithm are obtained. A fairly high degree of approximation (sixth order) of three-dimensional mass distributions function is achieved. Carto diagrams were created being based on the values of deviations of the three-dimensional average distributions (“isodens”), which give a fairly detailed picture of the Earth’s internal structure. The presented maps of “inhomogeneity’s” at characteristic depths (2891 km core – mantle, 5150 km internal – external core) allow us to draw preliminary conclusions about global mass movements. At the same time, the information on derivatives is significant for interpretation. First of all, it should be noted that the gradient of “inhomogeneity’s” is directed toward the center of mass. The presented projections of this gradient on a plane perpendicular to the rotation axis (horizontal plane) show the tendency of spatial displacements. **Scientific novelty**. Vector diagrams of the gradient, in combination with carto diagrams, give a broad picture of the dynamics and possible mechanisms of mass movement within the planet. To a certain extent, these studies confirm the phenomenon of gravitational convection of masses. **Practical significance.** The proposed algorithm can be used in order to build regional models of the planet, and numerical results can be used to interpret global and local geodynamic processes inside and on the Earth’s surface.

Key words: potential; gradient; harmonic function; Earth; mass distribution model; Stokes constants.

Introduction

Spatial models of the Earth’s interior mass density form an idea about its three-dimensional external gravitational field. Therefore, the use of the gravitational field parameters during the study of the Earth’s internal structure is being justified.

The radial models for the planet masses interior distribution created in geophysics provide for the use of Stokes constants of zero and second orders (mass and moment of inertia) (Meshcheryakov, Fys, 1990). The addition of higher-orders gravitational field parameters generates a methodology for constructing three-dimensional models, where successive approximations are considered as their basis. In this case, a three-dimensional density model is taken as the initial value (Meshcheryakov, Fys, 1986), which is consistent with the Stokes constants up to the second inclusive order and corresponds to one of the standard models, for example, PRE (Dzewonski M. & Anderson, 1981). Further, the expansion coefficients of the potential with a certain order are calculated (Meshcheryakov, Fys, 1986). The model created in this way reflects in sufficient detail the internal structure of the planet (Meshcheryakov, Fys, 1990).

Considering the effectiveness of this approach (Meshcheryakov et al. 1986, Meshcheryakov et al. 1990), it should be applied in order to determine the gravitational field not only to the mass distribution function, but also to its derivatives. In this paper, an attempt of such an implementation has been made. The proposed method is also approximate, but the iterative process is partially reduced to controlled values (power density moments that are determined on the surface of the planet), which makes it possible to analyze the approximation process. The mass distribution function constructed by using the proposed method is more informative and describes the planet mass distribution in more detail, because using the data in the above-mentioned method increases the approximation order due to the possibility of restoring the mass distribution of the planet's bowels by its derivatives, in contrast to constructing only the density function (Meshcheryakov et al. 1986). Therefore, the obtained mass distribution function by the above-mentioned method and involving even Stokes constants up to the second inclusive order gives a detailed picture of the distribution of density anomalies (the deviation of the three-dimensional function from the averaged over the sphere – "isodens") compared with the "inhomogeneities" obtained using equality (1) based on the same data (Fys, Brydun & Yurkiv, 2018). Chart diagrams for derivatives provide the additional information about the possible mechanisms of the redistribution of masses within the planet.

Purpose

To create an algorithm for constructing a three-dimensional masses distribution function of the planet and its derivatives taking into account Stokes constants of arbitrary orders. Being based on this method, the task is to perform the research on the internal structure of the Earth.

Methodology

1. Theoretical basis of the approximate construction method of the gradient and the mass distribution function of the ellipsoidal planet.

An approximate method of constructing three-dimensional models of the Earth's density, taking into account the Stokes constants of a given order and features of the internal structure, has been proposed by A. Meshcheryakov (Meshcheryakov, 1975; Meshcheryakov and Fys, 1981). The essence of this method is to represent a piecewise continuous distribution function by the system of orthogonal (later biorthogonal $\{W_{mnk}\}$, $\{\omega_{mnk}\}$) polynomials of three variables in an ellipsoid $\tau:\left\{\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}\le 1\right\}$, that is:

$$\delta(x_1,x_2,x_3)=\delta^0(\rho)+\sum_{m+n+k=0}^{N} b_{mnk}W_{mnk}(x_1,x_2,x_3), \quad (1)$$

where $\delta^0(\rho)$ – spherically symmetrical radial model, for example, PREM (Dzewonski & Anderson, 1981).

The expansion of coefficients b_{mnk} in (1) are determined as follows:

$$b_{mnk}=\frac{\int_\tau \delta\omega_{mnk}d\tau}{\int_\tau \omega_{mnk}W_{mnk}d\tau} \quad (2)$$

and are a linear combination of the following quantities (power moments of the mass distribution function of the planet interior):

$$I_{pqs}=\frac{1}{Ma_e^t}\int_\tau x_1^p x_2^q x_3^s \delta d\tau \;\; (p+q+s=t), \quad (3)$$

where M – planet mass; a_e – equatorial radii of the Earth.

The construction of the density function of the planet interior using the Stokes constants of higher orders is possible only approximately and is detaily described in papers (Meshcheryakov, 1990; Meshcheryakov et al. 1986), in which the question of assessing the reliability degree of determining quantities was not considered. Therefore, there is a need for a technique that allows you to analyze the calculation process and objectively evaluate the reliability of the construction of the density distribution function. Also, along with the determination of the mass distribution function, there is a need to determine its derivatives, which in geophysics is mentioned as a study of the gradient, or the law of the growth rate of the mass distribution. An attempt of such construction was made in (Chernyaga, Fys, 2012), where the task reduces to controlled quantities (surface integrals). It has been done by moving to the construction of not one function, but its derivatives, which are presented as follows:

$$\frac{\partial\delta}{\partial x_i}=\frac{1}{a_e}\sum_{t-m+n+k=0}^{N} a_{mnk}^i W_{mnk}(x_1,x_2,x_3), \quad (4)$$

where $i=1,2,3$. The known coefficients a_{mnk}^i can be used to set the appearance of the function itself (Fys et al., 2018).

The coefficients of decomposition are calculated by the formula (2) according to the power moments of the derivatives of density δ :

$$I_{pqs}^i=\frac{1}{Ma_1^p a_2^q a_3^s}\int_\tau x_1^p x_2^q x_3^s \frac{\partial\delta}{\partial x_i}d\tau,\; p+q+s=t. \quad (5)$$

The quantities (5) are determined by the power moments of the density function δ using the Ostrogradsky formula:

$$I_{pqs}^i=\frac{1}{Ma_1^p a_2^q a_3^s}\left(-p\int_\tau x_1^{p-1}x_2^q x_3^s\delta d\tau + \iint_\sigma x_1^p x_2^q x_3^s \delta(x_1,x_2,x_3)\cos\alpha_i d\sigma\right), \quad (6)$$

where $(\cos\alpha_1,\cos\alpha_2,\cos\alpha_3)$ – normal vector to the surface, with next components

$$\cos\alpha_i, = \frac{x_i}{D(x_1,x_2)}, i = 1,2$$

$$\cos\alpha_3 = \frac{1}{D(x_1,x_2)},$$

$$x_3 = \pm a_3\sqrt{1-\frac{x_1^2}{a_1^2}-\frac{x_2^2}{a_2^2}} = f(x_1,x_2), \quad (7)$$

$$D(x_1,x_2) = \sqrt{1+\left(\frac{\partial f}{\partial x_1}\right)^2+\left(\frac{\partial f}{\partial x_2}\right)^2}.$$

The representation of the ellipsoid surface in formulas (7) can be considered in another form, for example, expressed x_1 through x_2, x_3. Therefore, the quantities (5) are determined through the power density moments (3), which degree is one less than in (6), and the expressions:

$$\sigma_{pqs} = \frac{1}{Ma_1^p a_2^q a_3^s}\iint_\sigma x_1^p x_2^q x_3^s \delta\cos\alpha_i d\sigma = \frac{1}{Ma_1^p a_2^q a_3^s}\iint_\sigma x_1^{p_1} x_2^{q_1} x_3^{s_1}\delta \frac{d\sigma}{D(x_2,x_3)}, \quad (8)$$

where $p_1 = p+\varepsilon(i-1)$, $q_1 = q+\varepsilon(i-2)$, $s_1 = s+\varepsilon(i-3)$, $\varepsilon(i) = \begin{cases}1, i=1\\ 0, i\neq 1\end{cases}$.

For σ_{pqs} there is an estimation, for example, for even degrees we have:

$$\frac{3\delta_{\min}(2p-1)!!(2q-1)!!(2s-1)!!}{\delta_c R(2(p+q+s)+1)!!} \le \sigma_{2p2q2s} \le \frac{3\delta_{\max}(2p-1)!!(2q-1)!!(2s-1)!!}{\delta_c R(2(p+q+s)+1)!!}, \quad (9)$$

where $\delta_{\min}, \delta_{\max}$ – minimum and maximum density on the planet's surface. Such an estimation makes it possible to control the process of calculating these quantities, which is an additional argument for the development of this approach.

The right side of equality (8) is the integral over the surface, and therefore we will call them surface power moments. These integrals can be calculated from the known values of the density on the surface, which, by the way, gives an impetus to the study of the behavior of the density function on the Earth's surface, in particular, to the construction of a "generalized" function of the surface mass distribution on the surface. However, it should be noted that the use of such information requires a separate study on the subject of error resistance, and therefore in the future we will look for values σ_{pqs} using the integral characteristics of the Earth's gravitational field:

$$C_{nk} = \frac{1}{Ma_e^n}\int_\tau \delta U_{nk} d\tau, \qquad S_{nk} = \frac{1}{Ma_e^n}\int_\tau \delta V_{nk} d\tau, \quad (10)$$

where U_{nk}, V_{nk} – harmonic polynomials inside a sphere. In a rectangular coordinate system, the formula for their representation is the following (Fys, Zazuliak, Zajats', 2004):

$$U_{nk} + iV_{nk} = \frac{RR(n-k)!}{2^k}\sum_{m=0}^{\left[\frac{n-k}{2}\right]}\frac{(-1)^m x_3^{n-k-2m}\left(x_1^2+x_2^2\right)^m (x_1+ix_2)^k}{2^{2m} m!(m+k)!(n-k-2m)!} =$$

$$= \sum_{p+q+s=n}\left(\alpha_{pqs}x_1^p x_2^q x_3^s + i\beta_{pqs}x_1^p x_2^q x_3^s\right). \quad (11)$$

Using the Ostrogradskii theorem, relation (10) can be represented as:

$$\begin{cases} C_{nk} = \frac{1}{Ma_e^n}\int_\tau \delta U_{nk} d\tau = \frac{1}{Ma_e^n}\left(-\int_\tau \frac{\partial\delta}{\partial x_i}U_{nk}^i d\tau + \iint_\sigma \delta U_{nk}^i \cos\alpha_i d\sigma\right), \\ S_{nk} = \frac{1}{Ma_e^n}\int_\tau \delta V_{nk} d\tau = \frac{1}{Ma_e^n}\left(-\int_\tau \frac{\partial\delta}{\partial x_i}V_{nk}^i d\tau + \iint_\sigma \delta V_{nk}^i \cos\alpha_i d\sigma\right), \quad i = 1,2,3, \end{cases} \quad (12)$$

where $U_{nk}^i = \int_0^{x_i} U_{nk}dx_i, V_{nk}^i = \int_0^{x_i} V_{nk}dx$ – polynomials of $n+1$ degree on variables x_1, x_2, x_3.

Thus, the system of equations (12) is linear with respect to the quantities (8) and given the relation (6) we have:

$$\begin{cases} C_{nk} = \dfrac{1}{M \cdot a_1^p a_2^q a_3^s}\left(C_{nk}^* + \dfrac{1}{p_1 q_1 s_1} \sum\limits_{p+q+s=t} \sigma_{p_2 q_2 s_2} \alpha_{pqs} \right), \\ S_{nk} = \dfrac{1}{M \cdot a_1^p a_2^q a_3^s}\left(S_{nk}^* + \dfrac{1}{p_1 q_1 s_1} \sum\limits_{p+q+s=t} \sigma_{p_2 q_2 s_2} \alpha_{pqs} \right), \end{cases} \tag{13}$$

here $p_1 = p + \varepsilon(i-1)$, $q_1 = q + \varepsilon(i-2)$, $s_1 = s + \varepsilon(i-3)$, $p_2 = p + 2\varepsilon(i-1)$, $q_2 = q + 2\varepsilon(i-2)$, $s_2 = s + 2\varepsilon(i-3)$, $C_{nk}^* = -\int\limits_{\tau} \frac{\partial \delta_N}{\partial x_i} U_{nk}^i d\tau$, $S_{nk}^* = -\int\limits_{\tau} \frac{\partial \delta_N}{\partial x_i} V_{nk}^i d\tau$ – Stokes constants are calculated by the known density function δ_N.

The add the identities of the form to relations (13)

$$\sigma_{p+2,q,s} + \sigma_{h,q+2,s} + \sigma_{p,q,s+2} = \sigma_{pqs}, \tag{14}$$

which can be symbolically represented as:

$$\left(\sigma_{p00} + \sigma_{0q0} + \sigma_{00s} \right)^{2t} = \sigma_{pqs},\ p,q,s = 0 \text{ або } 1 \tag{15}$$

Each elements of expansion (15) can be represented as follows:

$$\left(\sigma_{p00}\right)^i \left(\sigma_{0q0}\right)^j \left(\sigma_{00s}\right)^l = \sigma_{p+2i,q+2j,s+2l},$$

$$i + j + l = t.$$

Thus, the total number of equations which corresponds to the Stokes constant of the nth order and identities (15) is $3(2n+1)+3$ with $(n+1)(n+2)(n+3)/6+1$ unknowns σ_{pqs}.

Unknowns values from system (13), corresponding to Stokes constants:

$$C_{n0}, C_{n2}, \ldots, C_{nT}\ (I),\quad \left(T = n - 2 \times \left[\frac{n}{2} \right] \right);$$

$$C_{n1}, C_{n3}, \ldots, C_{n,n-T}\ (II);$$

$$S_{n1}, S_{n3}, \ldots, S_{n,n-T}\ (III);$$

$$S_{n2}, S_{n4}, \ldots, S_{n,T}\ (IV)$$

can be united into four groups:

$$\begin{array}{ll} \text{I.} & \sigma_{00n}, \sigma_{20n-2}, \sigma_{02n-2}, \sigma_{40n-4}, \sigma_{22n-4}, \sigma_{04n-4}, .., \sigma_{n-T0T}, .., \sigma_{Tn-T,T}, \\ \text{II.} & \sigma_{10n-1}, \sigma_{30n-3}, \sigma_{12n-2}, \sigma_{50n-5}, \sigma_{32n-5}, \sigma_{14n-5}, .., \sigma_{n-T0T}, .., \sigma_{Tn-T,T}, \\ \text{III.} & \sigma_{01n-1}, \sigma_{21n-3}, \sigma_{03n-3}, \sigma_{41n-5}, \sigma_{23n-5}, \sigma_{05n-5}, .., \sigma_{n-T,T,T}, .., \sigma_{Tn-T,T}, \\ \text{IV.} & \sigma_{01N-1}, \sigma_{21N-3}, \sigma_{03N-3}, \sigma_{40N-}, \sigma_{22N-4}, \sigma_{04N-4}, .., \sigma_{N-T0T}, .., \sigma_{TN-T,T}, \end{array} \tag{16}$$

with r_i equations and t_i unknowns, respectively:

I group: $r_1 = 3n/2 + 4$ – equations with $t_1 = n(n+1)/2$ unknowns,

II–IV групи: $r_2 = r_3 = r_4 = 3n/2 + 1$ – equations with $t_2 = t_3 = t_4 = \left[n(n+1)/2 \right]$ unknowns.

In the matrix form equations (13), (15) can be represented as follows:

$$B_i = A_i X_i,\quad i = 1,2,3,4, \tag{17}$$

where X_i – the column vector of the dimension t_i of unknowns σ_{pqs}, B_i – is the column vector of the dimension r_i of the right-hand sides of systems of equations (13) and (15), A_i – is the $r_i \times t_i$ matrix of the systems of equations (13) and (15).

For systems (17) at different values t_i and r_i, the following cases are possible: the system has a single solution, no solutions (overridden), multiple solutions (undefined). Numerical experiments in (Fys, Brydun, Yurkiv 2018; Fys et al. 2016) show that the only solution exists when $n = 3$ and $n = 4$ (except for I in (16)).

Let's write a generalized solution of system (17) looking for it with minimal deviation from some accepted value. Mathematically, the problem can be formulated as follows: *for a given density* δ_N, *we look for quantities*

$$\sigma_{pqs} = \sigma_{pqs}^{**} + \Delta\sigma_{pqs}, \tag{18}$$

where $\sigma_{pqs}^{**} = \frac{1}{M a_1^p a_2^q a_3^s} \int\limits_{\sigma} x^p y^q z_3^s \delta_N \frac{d\sigma}{D(x_2, x_3)}$ – *are the surface power moments according to a given model* δ_N.

Substitution (18) into relations (11) and (13) gives:

$$\begin{cases} C_{nk} = \dfrac{1}{M \cdot a_1^p a_2^q a_3^s}\left(C_{nk}^{**} + \dfrac{1}{p_1 q_1 s_1} \sum\limits_{p+q+s=t} \sigma_{p_2 q_2 s_2} \alpha_{pqs} \right) \\ S_{nk} = \dfrac{1}{M \cdot a_1^p a_2^q a_3^s}\left(S_{nk}^{**} + \dfrac{1}{p_1 q_1 s_1} \sum\limits_{p+q+s=t} \sigma_{p_2 q_2 s_2} \alpha_{pqs} \right) \end{cases}, \tag{19}$$

where

$$C_{nk}^{**} = -\int_\tau \frac{\partial \delta_N}{\partial x_i} U_{nk}^i d\tau + \iint_\sigma \delta_N U_{nk}^i d\sigma = \int_\tau \delta_N U_{nk} d\tau$$

$$S_{nk}^{**} = -\int_\tau \frac{\partial \delta_N}{\partial x_i} V_{nk}^i d\tau + \iint_\sigma \delta_N V_{nk}^i d\sigma = \int_\tau \delta_N V_{nk} d\tau .$$

Then the identity (15) for the corrections $\Delta\sigma_{mnk}$ of the fixed density function will look like

$$\left(\Delta\sigma_{p00} + \Delta\sigma_{0q0} + \Delta\sigma_{00s}\right)^n = 0. \qquad (20)$$

The solution of system (17), (20) under condition $\sum \Delta^2\sigma_{pqs} \to \min$ can be given as follows

$$\Delta\sigma_i = A_i \left(A_i A_i^T\right)^{-1} \left(C_i - C_i^{**}\right). \qquad (21)$$

The power moments for the 1-st approximation of the derivatives of the density function δ with respect to (6) have the form:

$$I_{pqs}^i = \frac{1}{M a_1^p a_2^q a_3^s} \int_\tau x_1^p x_2^q x_3^s \frac{\partial \delta}{\partial x_i} d\tau$$

$$= I_{pqs}^* + \sigma_{pqs} = I_{pqs}^* + \sigma_{pqs}^{**} + \Delta\sigma_{pqs}$$

$$= \left(I_{pqs}^*\right)^i + \Delta\sigma_{pqs},$$

$$i = 1,2,3, \quad p+q+s = t.$$

All expansion coefficients (4) when $m+n+k = t > N-1$ for derivatives $\frac{\partial \delta_{N-1}}{\partial x_i}$ are equal zero,

$$\left(a_{mnk}^i\right)^* = \sum_{p+q+s\le t} d_{pqs}^i \left(I_{pqs}^i\right)^* = 0.$$

As a result, the elements of the l-st approximation are defined as follows:

$$a_{mnk}^i = \sum_{p+q+s\le t} d_{pqs}^i I_{pqs}^i$$

$$= \sum_{p+q+s\le t} d_{pqs}^i I_{pqs}^{i*} + \sum_{p+q+s=t} d_{pqs}^i \Delta\sigma_{pqs}$$

$$= \frac{(2t+1)!!}{t!} \Delta\sigma_{pqs},$$

and the required approximation, respectively

$$\frac{\partial\left(\Delta\delta_N\right)}{\partial x_i} = \sum_{m+n+k=N}^{N+1} \frac{(2t+1)!!}{t!} W_{mnk},$$

$$\frac{\partial \delta_{N+1}}{\partial x_i} = \frac{\partial \delta_{N-1}}{\partial x_i} + \frac{\partial\left(\Delta\delta_N\right)}{\partial x_i}, \; N = 4,6,8, \qquad (22)$$

where $\frac{\partial \delta_{N-1}}{\partial x_i}$ – the derivatives of the mass distribution function set in the previous step (Fys, Brydun, Yurkiv 2018; Fys et al. 2016).

Note that each l-st step of iterations involves the use of two adjacent orders $2l, 2l-1$ of Stokes constants with corresponding approximation numbers $N = 2l,\, N = 2l+1$. The coordination of the function $\Delta\delta_N(x_1, x_2, x_3)$ with the Stokes constant of $n-1, n$ orders is determined by polynomials W_{mnk} of N, N+1 order, and therefore does not affect the value of the Stokes constants of the lower orders t already involved. Indeed, identities (12) can be represented as:

$$C_{nk} + iS_{nk} = -p_2 q_2 s_2 \left[\sum_{p+q+s=t} \left(\alpha_{pqs} + i\beta_{pqs}\right) \int_\tau x_1^{p_1} x_2^{q_1} x_4^{s_1} \frac{\partial\left(\delta_N + \Delta\delta_N\right)}{\partial x_i} d\tau + \sum_{p+q+s\ne t+2} \sigma_{pqs} \left(\left(\alpha_{pqs} + i\beta_{pqs}\right)\right) \right]$$

$$- p_2 q_2 s_2 \left[\sum_{p+q+s=t} \left(\alpha_{pqs} + i\beta_{pqs}\right) \int_\tau x_1^{p_1} x_2^{q_1} x_4^{s_1} \frac{\partial \delta_N}{\partial x_i} d\tau + \sum_{p+q+s\ne t+2} \sigma_{pqs} \left(\left(\alpha_{pqs} + i\beta_{pqs}\right)\right) \right]$$

$$- p_2 q_2 s_2 \sum_{p+q+s=t} \left(\alpha_{pqs} + i\beta_{pqs}\right) \int_\tau x_1^{p_1} x_2^{q_1} x_4^{s_1} \frac{\partial \Delta\delta_N}{\partial x_i} d\tau.$$

Their right-hand sides do not change because, for $t < N$ $\int_\tau x_1^{p_1} x_2^{q_1} x_3^{s_1} \frac{\partial \Delta\delta_N}{\partial x_i} d\tau = 0$ (the combination of derivatives of N, N+1 orders), and σ_{pqs} are previously determined from equality (6).

Therefore, according to the known approximation of the mass distribution function δ_N, we obtain expressions for derivatives that are corresponding with the Stokes constants of an already higher order $2l-1, 2l$, and then the appearance of the mass distribution function δ_{N+2} itself. The iterative process is carried out to a predetermined approximation order.

2. The implementation of the approximate construction method algorithm.

For the basic function in the construction we take the function of mass distribution in the form (Meshcheryakov, 1991):

$$\delta_2(x_1, x_2, x_3) = \delta^0(\rho) +$$

$$+ \sum_{m+n+k=0}^{2} b_{mnk} W_{mnk}(x_1, x_2, x_3)$$

where $\delta^0(\rho)$ – one-dimensional spherical mass distribution model (for the Earth, this is the reference PREM model (Dzewonski & Anderson, 1981)).

Coefficients b_{mnk} in (1) are determined by the Stokes constants and dynamic compression (Meshcheryakov, Fys, 1986):

$$b_{000}=\delta_c\left(1-\frac{3}{\delta_c}\int_0^1\delta^0(\rho)\rho^2d\rho\right),\ b_{110}=35\delta_cS_{21},\ b_{101}=35\delta_cC_{21},\ b_{011}=35\delta_cS_{21},$$

$$b_{200}=\frac{7}{2}\delta_c\left[5\left(\frac{-C_{20}}{2H}+2C_{22}\right)-1-\frac{5}{\delta_c}\int_0^1\delta^0(\rho)\rho^4d\rho+\frac{3}{\delta_c}\int_0^1\delta^0(\rho)\rho^2d\rho\right],$$

$$b_{020}=\frac{7}{2}\delta_c\left[5\left(\frac{-C_{20}}{2H}-2C_{22}\right)-1-\frac{5}{\delta_c}\int_0^1\delta^0(\rho)\rho^4d\rho+\frac{3}{\delta_c}\int_0^1\delta^0(\rho)\rho^2d\rho\right], \quad (23)$$

$$b_{002}=\frac{7}{2}\delta_c\left[5\left(1-\frac{1}{2H}\right)C_{20}-1-\frac{5}{\delta_c}\int_0^1\delta^0(\rho)\rho^4d\rho+3\int_0^1\delta^0(\rho)\rho^2d\rho\right]$$

The values of power moments (3) are defined [15] as follows

$$I_{200}=\frac{-C_{20}}{2H}+2C_{22},\ I_{020}=\frac{-C_{20}}{2H}-2C_{22},$$

$$I_{002}=C_{20}\left(1-\frac{1}{2H}\right),\quad I_{101}=C_{21},$$

$$I_{011}=S_{21},\quad I_{110}=\frac{1}{2}S_{22}. \quad (24)$$

The approximate surface moments $\sigma_{pqs}(p+q+s\le 4)$ are determined by involving Stokes constants in the second order, inclusively, and by known coefficients b_{000}, b_{200}, b_{020}, b_{002}, b_{110}, b_{101}, b_{011}, namely (Fys et al., 2016):

$$\sigma_{200}=\frac{5\delta_cC_{00}+(3b_{200}+b_{020}+b_{002})}{5},$$

$$\sigma_{020}=\frac{5\delta_cC_{00}+(b_{200}+3b_{020}+b_{002})}{5},$$

$$\sigma_{002}=\frac{5\delta_cC_{00}+(b_{200}+b_{020}+3b_{002})}{5}.$$

Using identity (14) for $t=1$, we determine σ_{000}:

$$\sigma_{000}=\sigma_{200}+\sigma_{020}+\sigma_{002}.$$

According to the Stokes constants of the first order, we establish the value of $\sigma_{pqs}(p+q+s=3)$:

$$\sigma_{201}=\sigma_{021}=C_{10},\sigma_{003}=2C_{10},$$

$$\sigma_{300}=2C_{11},\sigma_{129}=\sigma_{192}=C_{11},$$

$$\sigma_{030}=S_{11},\sigma_{219}=\sigma_{012}=\frac{S_{11}}{2}.$$

When placing the origin of the coordinate system in the center of mass from the above relations we obtain that

$$\sigma_{pqs}=0,\quad (p+q+s=3),$$

then we have:

$$\sigma_{100}=\sigma_{300}+\sigma_{120}+\sigma_{102}=0,$$

$$\sigma_{010}=\sigma_{030}+\sigma_{210}+\sigma_{012}=0,$$

$$\sigma_{001}=\sigma_{003}+\sigma_{201}+\sigma_{021}=0.$$

Odd values σ_{pqs} of the fourth order (*p*, or *q*, or *s* are odd) are determined by the Stokes constants C_{21}, S_{22}:

$$\sigma_{103}=4C_{21},\sigma_{121}=C_{21},$$

$$\sigma_{310}=\sigma_{130}=4S_{22},\sigma_{112}=S_{22},$$

from identity (14) and using these relations, we have

$$\sigma_{101}=\sigma_{301}+\sigma_{121}+\sigma_{103},\sigma_{110}=\sigma_{310}+\sigma_{130}+\sigma_{112}.$$

The calculation of fourth-order values σ_{pqs} with paired indices is carried out using the Stokes constants C_{20}, C_{22}.

In the article (Fys et al., 2016), a method for determining these elements by a phased solution of systems of equations with one free unknown and its further determination from condition (14) for $t=2, p=q=s=0$ has been proposed To do this, we form a matrix of equations according to formulas (1) and free terms according to (2), whence we obtain the necessary surface moments σ_{pqs}, $p+q+s=4$, $p,q,s-$pairwise, according to which with (24) we calculate the power moments (6), then using the power moments and formula (2) – coefficients a^i_{mnk} $(m+n+k\le 3)$. As a result, we obtain (Fys et al., 2016):

$$\delta_4(x_1x_2,x_3)=\delta^0(\rho)+$$
$$+\sum_{m+n+k=0}^{3} a^1_{mnk}\int_0^{x_1} W_{mnk}(x_1,x_2,x_3)\,dx_1+$$
$$+\sum_{m+n+k=0}^{3} a^2_{mnk}\int_0^{x_2} W_{mnk}(0,x_2,x_3)\,dx_2+$$
$$+\sum_{m+n+k=0}^{3} a^3_{mnk}\int_0^{x_3} W_{mnk}(0,0,x_3)\,dx_3.$$

We construct the next approximation by the Stokes constants of the third and fourth orders. According to an unknown scheme, we find δ_6, taking for the initial approximation δ_4. Moreover, the approximate moments are determined by formulas (21). An exception is one case when system (19) contains only one equation:

$$S_{32}=\frac{1}{2Ma_e^3}\int_\tau \delta x_1x_2x_3d\tau.$$

Then some of the quantities σ_{pqs}, are determined directly:

$$\sigma_{311}=4s_{32}+\frac{4}{630}b_{011},$$
$$\sigma_{131}=4s_{32}+\frac{4}{630}b_{101},$$
$$\sigma_{113}=4s_{32}+\frac{4}{630}b_{110}.$$

The found values by the given algorithm determine the coefficients $a^i_{mnk}\,(4\le m+n+k\le 5)$, and therefore δ_8.

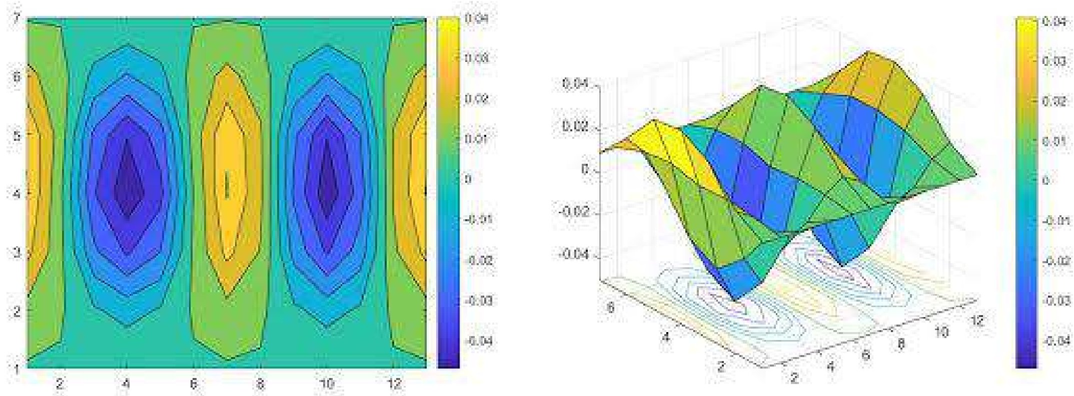
Taking the found approximation δ_6 and the Stokes constants of the 5–6th orders we obtain respectively δ_8 and so on to the order N_k established in advance.

Results

The described method is more general in comparison with those described and implemented in (Fys et al., 2016). The generalizations relate primarily to the construction of matrix relations A_i with (17) and obtaining an approximate solution of (17) regardless of the conditions of its existence. The comparison of the calculation results in different ways gives similar results, so the application of the above algorithm is justified.

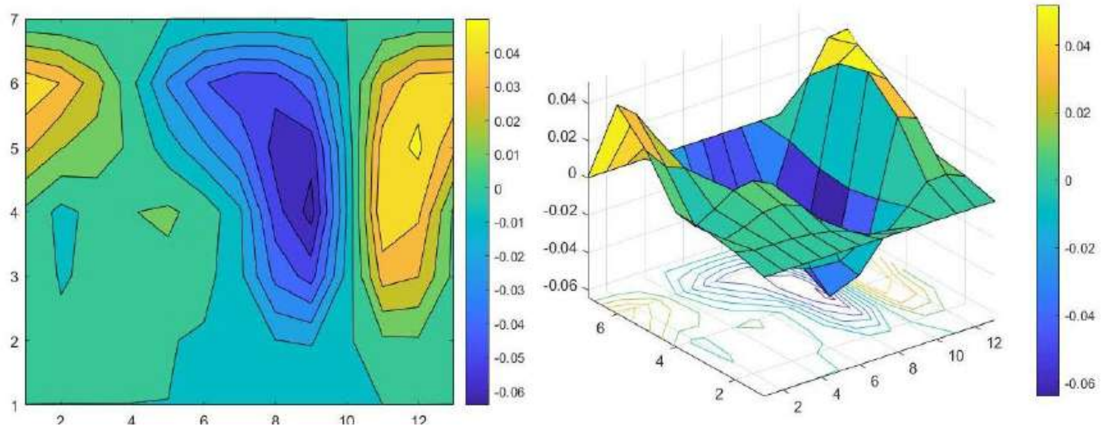
Based on the above technique, a three-dimensional density model was constructed with the use of Stokes constants up to fourth order inclusively, which retains all the basic properties of the PREM reference model: the magnitude of jumps and their depth, the nature of the density change along the relative radius. Moreover, in contrast to the model, density anomalies are more structured, that is, they give a more detailed picture of the mass distribution. So, at different depths, a redistribution of masses is observed. However, the property of moving masses toward the surface follows from the obtained maps, the reason for which is the planetary rotational motion. It is characteristic that such clusters are inherent throughout the radius of the Earth. On the contrary, near the axis of rotation there is a dilution of the masses in depth. This is illustrated by the isodens map in Fig. 2 (depth 5150 km, boundary inner-outer core) and in Fig. 3 (depth 2891 km, core-mantle boundary). So, based on the same information, we get a density model that gives a more detailed picture of the distribution of masses inside the planet.

For clarity, the density anomalies (Fig. 1) and their derivatives (Fig. 2–4) were calculated at the "core-mantle" boundary, which are illustrated by the corresponding cartodiagrams.



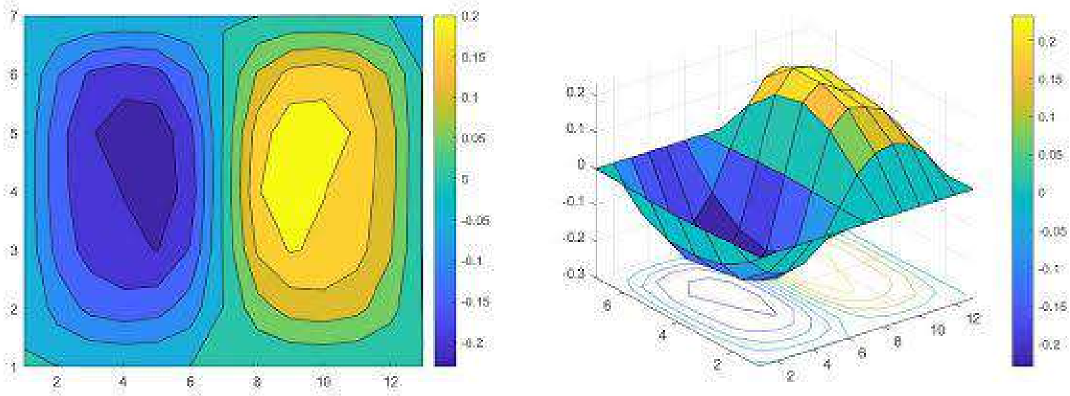
a *b*

Fig. 1. Map of isolines of the Earth's interior density anomalies on the "core-mantle" (depth 2891 km) (*a*) and their spatial image (*b*)



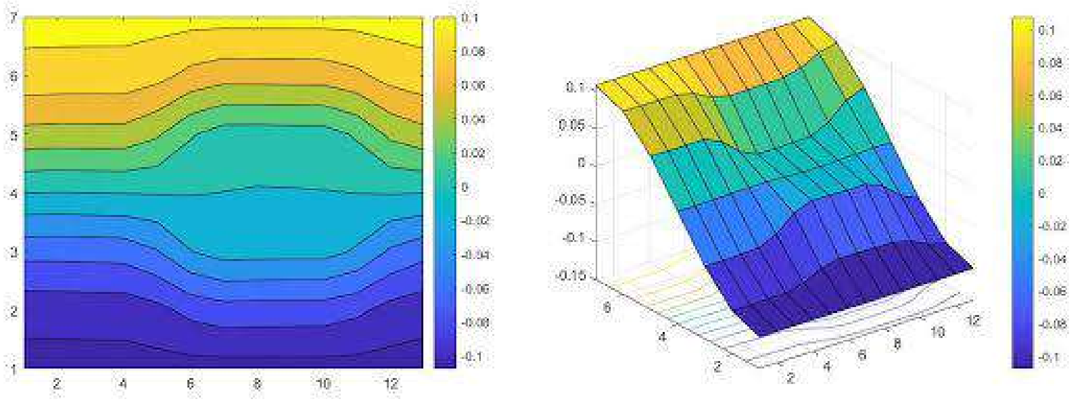
a *b*

Fig. 2. Map of isolines of the derivative with respect to variable x_1 anomalies of the Earth's interior density on the core-mantle (depth 2891 km) (*a*) and their spatial image (*b*)



a *b*

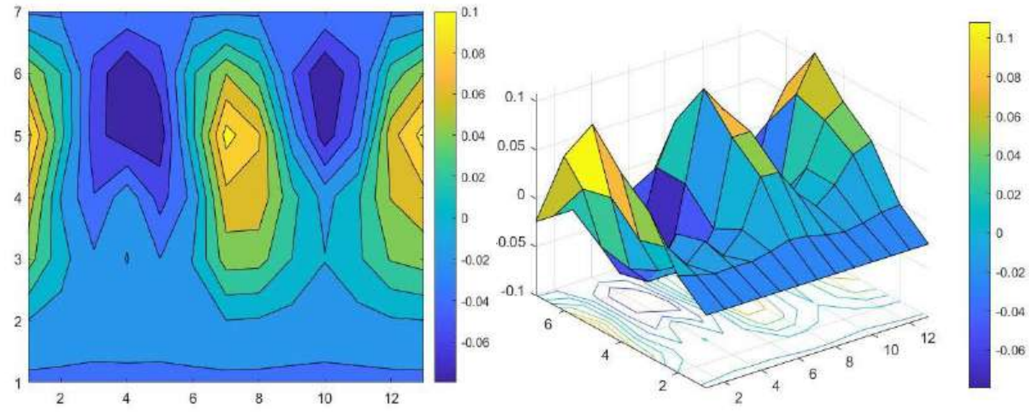
Fig. 3. Map of isolines of the derivative with respect to variable x_2 anomalies of the Earth's interior density on the core-mantle (depth 2891 km) (*a*) and their spatial image (*b*)



a *b*

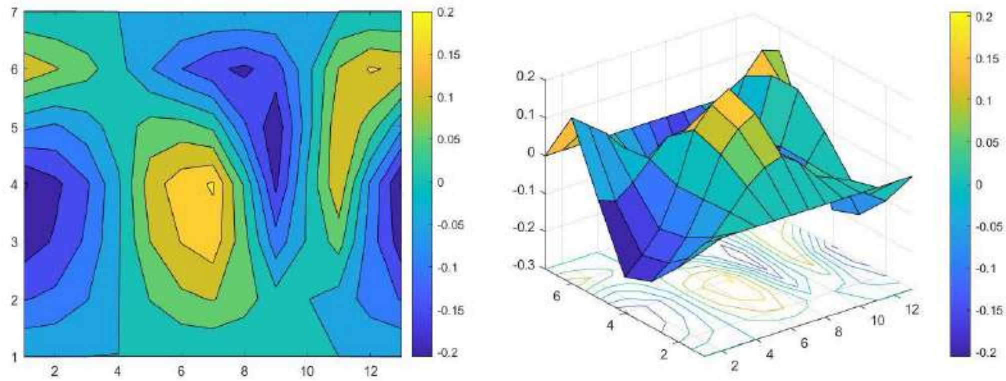
Fig. 4. Map of isolines of the derivative with respect to variable x_3 (rotation axis) anomalies of the Earth's interior density on the core-mantle (depth 2891 km) (*a*) and their spatial image (*b*)

Similarly, we will carry out calculations and make corresponding illustrations for a depth of 200 km (Fig. 5–8). The choice of this value is due to the location of the objects of study in the middle of the mantle and their possible influence on the geodynamic processes of the Earth.



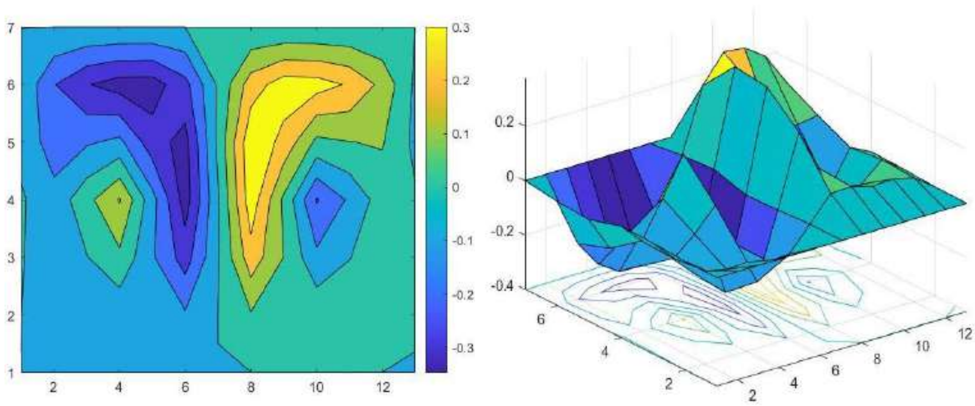
a *b*

Fig. 5. Map of isolines of the Earth's interior density anomalies at a depth of 200 km (*a*) and their spatial image (*b*)



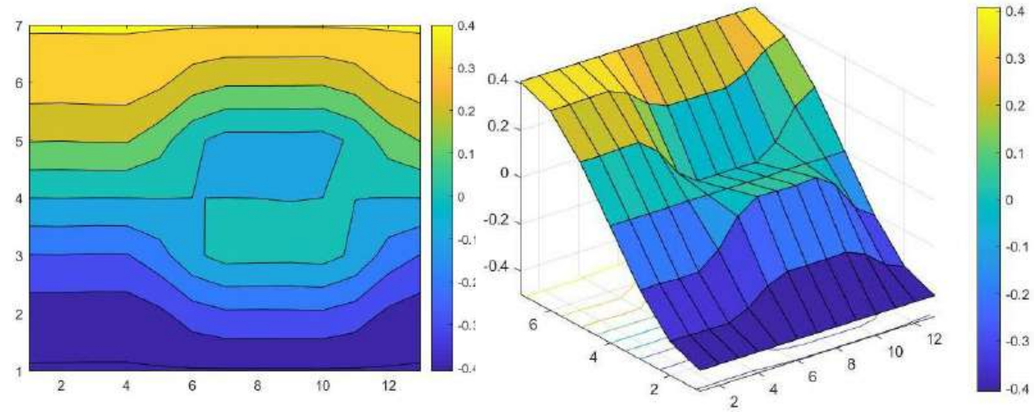
a *b*

Fig. 6. Map of isolines of the derivative with respect to variable x_1 anomalies of the Earth's interior density at a depth of 200 km (*a*) and their spatial image (*b*)



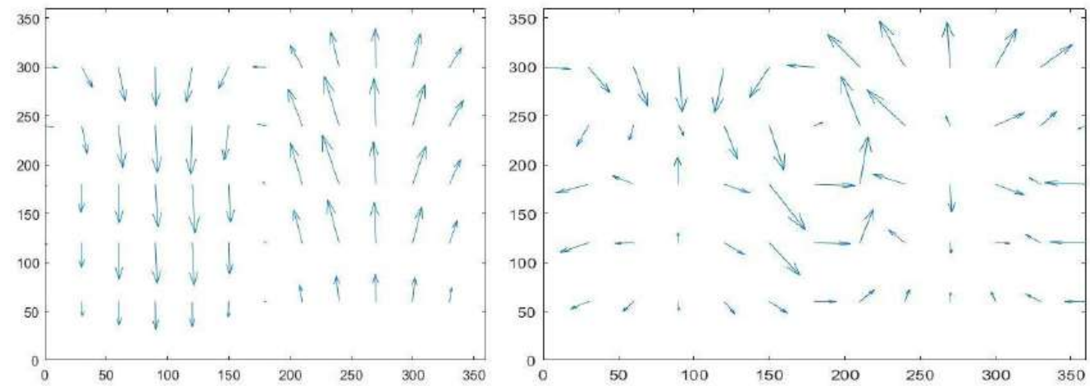
a *b*

Fig. 7. Map of isolines of the derivative with respect to variable x_2 anomalies of the Earth's interior density at a depth of 200 km (*a*) and their spatial image (*b*)



a *b*

Fig. 8. Map of isolines of the derivative with respect to variable x_3 anomalies of the Earth's interior density at a depth of 200 km (*a*) and their spatial image (*b*)



a *b*

Fig. 9. Map of the projections of the density anomalies gradient at the points of the ellipsoidal surface onto the xOy plane (horizontal plane) at the core-mantle boundary (depth 2900 km) (*a*) and at a depth of 200 km (*b*)

We note the most significant points of the above results. Fig. 4, 8 reflect the distribution of density derivatives anomalies along the axis Ox_3 (in a certain sense, the vertical derivative). From the figures it is clear that the density anomalies gradient is directed towards the center of mass, because for two depths (Fig. 4, 8) for $0 \le \vartheta \le \frac{\pi}{2}$, the angle between it and the axis is obtuse, and for $\frac{\pi}{2} \le \vartheta \le \pi$ – it is sharp. The nodal point of Fig. 2 with approximate coordinates $\vartheta = 120°, \lambda = 35°$, which can be interpreted as a point of compression and tension in different directions. Interestingly, it falls into the area of interaction between the Arabian and African tectonic plates.

Obviously, a more detailed interpretation requires other methods for the integrated presentation of information, for example, illustrating the total effect of derivatives with respect to variables x_1, x_2.An attempt of such an approach was performed in Fig. 9. Even the first step in this approach reveals features (Fig. 8a), namely: the redistribution of masses on the edge of the "core-mantle" is carried out from the south to the north pole, which coincides with the action of the magnetic field. It is characteristic that in the second case (Fig. 8b) the picture of the movement is completely different, and it can be associated with tectonic movements. Hence, we can conclude that the maps of density anomalies and its gradient complement the studies of the inner structure of the upper mantle and the earth's crust using seismological methods, and a comprehensive interpretation of tomography data and data obtained on the basis of the above technique makes it possible to establish the sources of mass redistribution in the upper layers of the Earth.

Conclusions

1. The proposed method for the approximate construction of the mass distribution of the Earth and its derivatives makes it possible to use the information about the planet's gravitational field more thoroughly.

2. Unlike the traditional method of determining the density, the proposed method allows to control partially the calculation process, therefore, to evaluate the degree of reliability of a such construction.

3. The constructed maps of density anomalies at certain depths make it possible to draw preliminary conclusions about their accumulations inside the Earth, and the vector schemes reflect the possible movements of the planet's masses, due to the rotating component of gravity.

4. The above technique makes it possible to construct a model of the planet's interior masses distribution and its gradient of an arbitrary order, and makes it possible to begin a more detailed study of the Earth's internal structure.

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МЕТОДИКА НАБЛИЖЕНОЇ ПОБУДОВИ ТРИВИМІРНОЇ ФУНКЦІЇ РОЗПОДІЛУ МАС ТА ЇЇ ГРАДІЄНТА ДЛЯ НАДР ЕЛІПСОЇДАЛЬНОЇ ПЛАНЕТИ

Мета. Створити алгоритм побудови тривимірної функції розподілу мас планети та її похідних з урахуванням стоксових сталих довільних порядків. Спираючись на цей алгоритм, виконати дослідження внутрішньої будови Землі. **Методика**. Похідні неоднорідного розподілу мас подають лінійними комбінаціями біортогональних многочленів, коефіцієнти яких отримують із системи рівнянь. Ці рівняння одержують інтегральними перетвореннями стоксових сталих, а процес обчислень здійснюється послідовним наближенням і за початкове наближення беруть одновимірну модель густини, узгоджену зі стоксовими сталими до другого порядку включно. Далі визначають коефіцієнти розкладу потенціалу до третього, четвертого і т. д. порядків, аж до наперед заданого порядку. Зведення степеневих моментів густини до поверхневих інтегралів дає можливість аналізувати та контролювати ітераційний процес. **Результати**. Результати обчислень отримано з використанням програмного продукту за описаним алгоритмом. Досягнуто достатньо високого степеня апроксимації (шостого порядку) тривимірних розподілів та створено картосхеми за врахованими значеннями відхилень тривимірних розподілів від середнього (“ізоденси”), які дають доволі детальну картину внутрішньої будови Землі. Наведені карти “неоднорідностей” на характерних глибинах (2891 км ядро–мантія, 5150 км внутрішнє–зовнішнє ядро) дають підстави зробити попередні висновки про глобальні переміщення мас. Значущою для інтерпретації є інформація про похідні. Насамперед можна наголосити, що ґрадієнт “неоднорідностей” спрямований до центра мас. Подані проєкції цього ґрадієнта на площину, перпендикулярно до осі обертання (горизонтальної площини), відображають тенденцію просторових переміщень. **Наукова новизна**. Векторграми ґрадієнта в сукупності із картосхемами дають ширше уявлення про динаміку ймовірного переміщення мас всередині планети та можливі механізми, що їх спричиняють. Певною мірою ці дослідження підтверджують явище гравітаційної конвекції мас. **Практична значущість.** Запропонований алгоритм можна використовувати для побудови регіональних моделей планети, а числові результати – для інтерпретації глобальних та локальних геодинамічних процесів всередині та на поверхні Землі.

Ключові слова: потенціал; ґрадієнт; гармонічна функція; Земля; модель розподілу мас; стоксові сталі.

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